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Abstract

This paper addresses the challenges associated with pricing exotic options, specifically path-dependent ones, with a focus on the limitations of standard Monte Carlo simulations and the advantages provided by Conditional Monte Carlo methods, introduced by Babsiri and Noel in 1998. Path dependent options, such as first and second-generation barrier and lookback options, require continuous monitoring of asset prices throughout their lifetime, making accurate pricing computationally demanding and prone to errors when using traditional Monte Carlo methods.
This work begins by presenting different exotic options, offering a detailed comparison between the exact pricing formulas and the results obtained from Crude Monte Carlo simulations. The Conditional Monte Carlo method is then applied to address the bias introduced by discrete monitoring intervals in the simulations, a critical issue in path-dependent options. A market case based on the valuation of a Bonus Cap certificate has also been shown.

Key Words: Exotic options, Path-dependent options, continuous monitoring, Brownian Bridge, Conditional Monte Carlo, Barrier Options, Lookback Options, Bonus Cap certificate.

JEL codes: C53, C63, G12, G32

1) Introduction

The valuation of many complex financial instruments, for which there are no analytical pricing formulas, is done through the Monte Carlo technique, which involves the integration of the stochastic differential equation governing the dynamics of the underlying, with the aim of deriving the financial variables that constitute the pay-off of the derivative (Giribone, 2024).
Generally, such an approach, called Crude Monte Carlo, does not introduce any numerical errors that would be critical enough to compromise weak convergence to the fair-value of the financial instrument (Giribone and Ligato, 2012).
However, there is a category of path-dependent options that exhibit significant divergence from the expected value, so that the reliability of the approach is invalidated (Tropiano, 2024).
To reduce the error committed in critical cases to reach acceptable values, the literature proposes the Conditional Monte Carlo variant, which is based on the probabilistic method known in literature as Brownian Bridge (Huynh, Lai and Soumarè, 2008).
In fact, this arrangement allows to drastically reduce the discretization error introduced by classical stochastic integration in first- and second-generation barrier and lookback options, which involve continuous monitoring of the underlying asset (Babsiri, Noel, 1998).
Such a simulation bias correction is therefore crucial to reach an accurate estimation of the fair-value of any derivative whose payoff depends on the extreme values reached by the underlying equity during the life of the contract.
The valuation of financial options has long been a cornerstone of quantitative finance, with particular interest in the complex pricing of exotic options like barrier and lookback options.
Barrier options are one of the most common types of exotic option, in which the payoff depends on whether the underlying asset price reaches a certain barrier level. Various methodologies have been proposed to address the challenges in pricing these instruments.
Reiner and Rubinstein (1991) provided a closed formula for the valuation of the standard type of Barrier options. Carr (1995) introduced two modifications to the valuation of barrier options: the first allows for an initial protection period during which the option cannot be knocked out, while the second considers an option which is only knocked out if a second asset reaches an upper barrier.
Other notable works include Metwally and Atiya (2003), who developed fast Monte Carlo methods for pricing barrier options in jump diffusion processes. This approach became a crucial tool for a more accurate and computationally efficient pricing of options. In a similar vein, Wang et al. (2009) proposed a hybrid approach combining binomial models and Monte Carlo simulations, which allowed for flexibility in modeling complex boundary conditions, further enhancing the accuracy of barrier option pricing.
Additionally, Sudding and Kalla (2021) introduced a method combining Monte Carlo simulations and binomial lattice models to estimate the price of lookback options, demonstrating the utility of lattice-based methods in the valuation of barrier options as well. Their approach allows for precise computation while maintaining reasonable computational complexity, a common challenge in the field.
Geman and Yor (1996) presented a probabilistic approach for pricing and hedging double-barrier options, considering a continuous-time framework.
Ikeda and Kunitomo (1992) studied the valuation of the second generation type double-barrier option, in which both a Lower barrier and an Upper barrier are present; while another second-generation type of the instrument, the soft-barrier option, has been valued with a closed formula from Hart and Ross (1994).
Lookback options, which allow the holder to "look back" at the underlying asset price during the life of the option and base the payoff on either the maximum or the minimum price reached, have drawn significant attention in recent years. Pricing these options is particularly challenging due to the need to track the path of the underlying asset over time, requiring advanced numerical methods.
As previously said, Sudding and Kalla implemented binomial lattice models along Monte Carlo simulations to estimate their price; Singirankabo (2020) addressed the pricing of lookback options using multinomial lattice methods, offering a computationally efficient

solution for these path-dependent options. Their method incorporates both the dynamics of the asset price and the boundary conditions necessary for accurate pricing. This approach, though rooted in classical lattice models, provides a modern solution to a problem that has historically been treated with more computationally demanding methods.
The work by Kudryavtsev et al. (2024) extended the Monte Carlo approach for pricing lookback options under Lévy processes, offering a comprehensive solution to pricing these instruments in markets with jumps and stochastic volatility. This work represents a step toward understanding the behavior of lookback options in more realistic financial models, especially those incorporating heavy tails and discontinuities in asset prices.
Further contributions include the study by Chen et al. (2019), which explored the pricing of lookback options using mixed fractional Brownian motion. Their research provides insights into the pricing of lookback options in markets with long memory, a phenomenon that is increasingly recognized as important in the modeling of financial markets.
Numerous studies have also explored hybrid methods and approximation techniques to improve the efficiency and accuracy of option pricing models. Babbs (2000) examined the binomial valuation of lookback options, proposing an approximation that simplifies the path-dependent nature of these options while maintaining reasonable accuracy. Grosse-Erdmann and Heuwelyckx (2016) further investigated binomial approximations for lookback options, focusing on improving the numerical stability and convergence properties of these methods.
In a more recent approach, Febrianti (2022) applied adaptive differential evolution methods with learning parameters to approximate the pricing of barrier options. This adaptive technique allows for fine-tuning of the parameters during the optimization process, leading to more accurate pricing results and offering an alternative to traditional Monte Carlo and lattice-based methods.
The literature on the pricing of barrier and lookback options is extensive and multifaceted: the development of Monte Carlo methods, hybrid models, and lattice-based approaches has significantly advanced the field, providing both computational efficiency and accuracy in the pricing of these complex options. As financial markets continue to evolve, further research into more accurate models incorporating jumps, volatility clustering, and fractional Brownian motion will be crucial for improving the pricing of exotic options.
It is also worth to note that the option theory based on barrier monitoring can also be extended in the credit risk context, as shown in Agosto and Moretto, 2012.
This study can be conceptually divided into three parts: the first one explains in detail the continuous monitoring problem in a Monte Carlo engine based on the Black-Scholes-Merton pricing framework and how this can be solved through the Babsiri and Noel approach. The second part of the paper validates the methodology with first (standard lookback and barrier option) and second (soft barrier and double barrier option) generation exotic path dependent options. The bias introduced by standard Monte Carlo will be quantified and we will show that it can be zeroed through the implementation of Conditional Monte Carlo. The last part of the study is devoted to a concrete market case: an investment certificate of the Bonus Cap type (ACEPI certificates map, 2024) characterized by continuous barrier monitoring will be evaluated.

2) Methodology

The Brownian Bridge, also known in the literature as Tied Down Brownian Motion, is described for deriving a Monte Carlo method suitable for valuing path-dependent options, i.e. the Conditional Monte Carlo. The key idea of this approach, which avoids simulation bias distortion, is to directly derive the extreme value (S_{min} o S_{max}) from a probability distribution valid for the type of simulation performed. This methodology, based on the application of the Brownian Bridge and the reflection principle of Brownian motion, eliminates the need for a very fine partition for discretization. It proves to be effective both in terms of the accuracy of the fair value of the derivative, and for computational time performance. To keep the discussion manageable, only the fundamental steps needed for this characterization are presented, with the formal proof omitted but referenced in the bibliography (Kloeden and Platen, 1992). The next step is to determine the probability distribution governing the simulation of the maximum value potentially achieved by the underlying asset.

$$dZ_t = d\ln[S(t)] = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW_t \rightarrow S_t = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t\right] \quad (1)$$

Where $S(t)$ is the price of the underlying asset at time t, S_0 is the initial price of the underlying asset at time $t = 0$, r is the risk-free interest rate, σ is the volatility of the underlying asset, dW_t is the increment of a Wiener process (standard Brownian motion), dZ_t is the change in the logarithm of the asset price, representing the stochastic component of the process, and dt is the increment of time.
Considering equation (1) the probability that a path of Z starts at Z_i at time t_i and ends at Z_{i+1} at time t_{i+1} is given by the probability density function of the transition in (2) where $a = r - \frac{\sigma^2}{2}$ is the drift of the stochastic process.

$$p\{Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1}\} = \frac{1}{\sigma\sqrt{2\pi\Delta t}} \exp\left[-\frac{(Z_{i+1} - Z_i - a\Delta t)^2}{2\sigma^2\Delta t}\right] \quad (2)$$

Given the initial and final values of Z, the probability that this path crosses a certain level b, which is the barrier $b = \ln(B)$ in the case of Barrier Options or is equivalent to the extreme variable value of the asset $b = \ln(S_{max})$ in the case of Lookback Options, within a time interval $t_i < \tau_b < t_{i+1}$ is as shown in (3).

$$p\{t_i < \tau_b < t_{i+1} | Z_i, Z_{i+1}\} = \frac{p\{t_i < \tau_b < t_{i+1}, Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1}\}}{p\{Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1}\}} \quad (3)$$

The denominator of this fraction is given by the equation (2). The numerator is computed as in (4)

$p\{t_i < \tau_b < t_{i+1} | Z_i, Z_{i+1}\} =$

$$p\{t_i < \tau_b < t_{i+1}\} \cdot p\{Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1} | t_i < \tau_b < t_{i+1}\} =$$

$$p\{t_i < \tau_b < t_{i+1}\} \cdot p\{Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1}^R | t_i < \tau_b < t_{i+1}\} =$$

$$= p\{Z(t_i) = Z_i, Z(t_{i+1} = Z_{i+1}^R, t_i < \tau_b < t_{i+1}\} \quad (4)$$

In equation (4), the reflection principle of Brownian motion is applied. According to this principle, the probability that a path Z starts at (t_i, Z_i) and ends at (t_{i+1}, Z_{i+1}) while crossing the level b is the same as that of starting from the same initial point and ending at (t_{i+1}, Z_{i+1}^R), where Z^R is the level of reflection of Z along b.

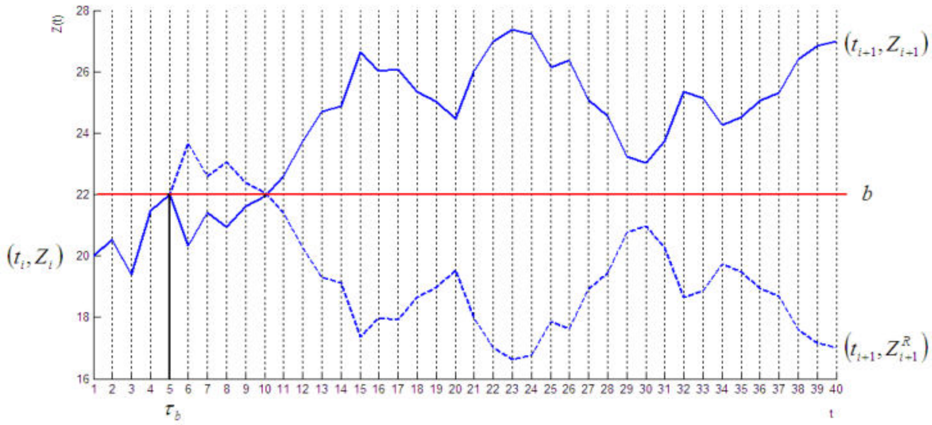


Figure 1: The reflection principle of a Brownian motion

$$p\{Z(t_i) = Z_i, Z(t_{i+1} = Z_{i+1}^R, t_i < \tau_b < t_{i+1}\} \quad (5)$$

The probability shown in (5) can be expressed by the following probability density function in (6) where $N(x)$ is the normal density distribution.

$$p\{t_i < \tau_b < t_{i+1}, Z_i, Z_{i+1}\} = \exp\left[\frac{2a(b - Z_i)}{\sigma^2}\right] \frac{1}{\sigma\sqrt{\Delta t}} N\left(-\frac{2b - Z_{i+1} - Z_i + a\Delta t}{\sigma\sqrt{\Delta t}}\right) \quad (6)$$

$$p\{Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1}\} = \frac{1}{\sigma\sqrt{2\pi\Delta t}} \exp\left[-\frac{(Z_{i+1} - Z_i - a\Delta t)^2}{2\sigma^2\Delta t}\right] \quad (7)$$

By substituting equations and inserting (5) into (7), we obtain the desired conditional probability.

$$p\{t_i < \tau_b < t_{i+1} | Z_i, Z_{i+1}\} = \frac{\exp\left[\frac{2a(b - Z_i)}{\sigma^2}\right] \frac{1}{\sigma\sqrt{2\pi\Delta t}} \exp\left[-\frac{(2b - Z_{i+1} - Z_i + a\Delta t)^2}{2\sigma^2\Delta t}\right]}{\frac{1}{\sigma\sqrt{2\pi\Delta t}} \exp\left[-\frac{(Z_{i+1} - Z_i - a\Delta t)^2}{2\sigma^2\Delta t}\right]} \quad (8)$$

Performing the appropriate simplifications and calculations, we obtain equation (9).

$$p\{t_i < \tau_b < t_{i+1} | Z_i, Z_{i+1}\} = \exp\left[-\frac{2(b - Z_{i+1})(b - Z_i)}{\sigma^2\Delta t}\right] \quad (9)$$

$$p\{t_i < \tau_b < t_{i+1} | S_i, S_{i+1}\} = \exp\left(-\frac{2[\ln(B) - \ln(S_i)][\ln(B) - \ln(S_{i+1})]}{\sigma^2\Delta t}\right) = \exp\left(\frac{1}{\sigma^2\Delta t}\left[2\ln\left(\frac{B}{S_i}\right)\ln\left(\frac{S_{i+1}}{B}\right)\right]\right) \quad (10)$$

Rewriting the equation in terms of the underlying asset S, substituting $Z_i = \ln(S_i)$, $Z_{i+1} = \ln(S_{i+1})$, and $b = \ln(B)$, we obtain equation (10).
Therefore, to simulate the maximum value achieved by the asset in the interval $[t_i, t_{i+1}]$ given the Brownian Bridge endpoints S_i and S_{i+1}, it is sufficient to generate a uniformly distributed random variable $u \in U[0,1]$ and set it equal to the expression in (8).

$$u = \exp\left(\frac{1}{\sigma^2\Delta t}\left[2\ln\left(\frac{S_{max}}{S_i}\right)\ln\left(\frac{S_{i+1}}{S_{max}}\right)\right]\right) \quad (11)$$

Analytically solving equation (11) yields a direct expression for simulating the price of S_{max} as shown in (12).

$$S_{max} = \exp\left[\frac{\ln(S_{i+1} \cdot S_i) + \sqrt{\left[\ln\left(\frac{S_{i+1}}{S_i}\right)\right]^2 - 2\left(\frac{\sigma^2}{S_i}\right)^2 \Delta t \ln(u)}}{2}\right] \quad (12)$$

To streamline this into a computationally more efficient distribution for programming environments, Babsiri and Noel (1998) suggested focusing on the log ratio between the simulated right endpoint of the Brownian Bridge $S_{i+1} = S(T)$ and the known left endpoint $S_i = S(0)$, defining this quantity as x in (13).

$$x = \ln\left(\frac{S(T)}{S(0)}\right) \quad (13)$$

Given the known distribution of this log ratio, $N(r\Delta t, \sigma\sqrt{\Delta t})$, it is possible to estimate the conditional probability of the maximum value x using similar logical steps as before (see Equation (14)).

$$p\left\{\max\ln\left(\frac{S(t)}{S(0)}\right) \leq y : \ln\left(\frac{S(T)}{S(0)}\right) = x, t \in [0,T]\right\} = 1 - \exp\left[\frac{2y(x-y)}{\sigma^2 T}\right] \quad (14)$$

Thus, to simulate the maximum value x in the interval $t \in [0,T]$ it is sufficient to generate a uniformly distributed random variable $u \in U[0,1]$and set it equal to the one in (15).

$$u = 1 - \exp\left[\frac{2y(x-y)}{\sigma^2 T}\right] \rightarrow y_{MAX} = \frac{x + \sqrt{x^2 - 2\sigma^2 T \ln(1-u)}}{2} \quad (15)$$

The introduced transformation provides a distribution expression for the maximum value (15) that is more efficient to process compared to equation (12), making it preferable, especially when many simulations are needed.
To determine the probability distribution of the minimum value of x, we simply compute the complementary probability of (11) and set the probability to $u \in U[0,1]$.

$$p\left\{\min\ln\left(\frac{S(t)}{S(0)}\right) \leq y : \ln\left(\frac{S(T)}{S(0)}\right) = x\right\} = 1 - p\left\{\min\ln\left(\frac{S(t)}{S(0)}\right) \leq y : \ln\left(\frac{S(T)}{S(0)}\right) = x\right\} = \exp\left[\frac{2y(x-y)}{\sigma^2 T}\right] \quad (16)$$

$$u = \exp\left[\frac{2y(x-y)}{\sigma^2 T}\right] \rightarrow y_{MIN} = \frac{\left(x - \sqrt{x^2 - 2\sigma^2 T \ln(u)}\right)}{2} \quad (17)$$

By inverting this expression, we obtain (17), the simulation for the desired minimum value.

3) Empirical Results and Discussion

In this section, we present our analysis, applying both traditional and Conditional Monte Carlo methods to four types of exotic, path-dependent options: standard barrier, lookback, soft barrier, and double barrier options (Haug, 2007). For each option type, we evaluate and compare the pricing performance of both approaches, focusing on key metrics such as accuracy, convergence rate, and computational efficiency. It is important to highlight that the proposed methodology is valid when the dynamics which rules the underlying process is a Geometric Brownian Motion. In fact, this also constitutes a fundamental hypotheses of the Black-Scholes-Merton pricing framewok. We also discuss the implications of these findings in terms of their practical applicability, robustness in different market scenarios, and relevance to risk management strategies. Each subsection (3.1, 3.2, 3.3, and 3.4) provides a detailed assessment of its unique characteristics and the results obtained through the Monte Carlo simulations.

3.1) Standard Barrier Options

Barrier options are the first class of exotic options we studied. They are of course part of the path dependent options, and their payoff depends on whether the underlying asset price reaches a predetermined barrier level during the life of the option. This feature makes barrier options more complex than vanilla options, as the path of the underlying asset price - not just its final price - determines the option payoff. The barrier can either activate the option (knock-in) or terminate it (knock-out).

Barrier options are widely traded in the over-the-counter (OTC) market and they are popular because they often have lower premiums compared to standard options. This lower cost reflects the reduced likelihood of the option paying out, given the presence of the barrier. The level of the barrier has a significant impact on the option value, making barrier options highly sensitive to the volatility of the underlying asset.

Barrier options can be broadly categorized into two main types:

Knock-Out Options: These options cease to exist if the underlying asset price breaches the barrier level, H.
Knock-out options can further be classified into:

- Down-and-Out Call/Put: The option is knocked out (terminated) if the asset price falls to or below a certain barrier level, H, with $H < S_0$.
- Up-and-Out Call/Put: The option is knocked out if the asset price rises to or above a certain barrier level.

Knock-In Options: These options only come into existence if the underlying asset price breaches the barrier level. Knock-in options are categorized as:

- Down-and-In Call/Put: The option becomes active if the asset price falls to or below a certain barrier level, $H < S_0$.
- Up-and-In Call/Put: The option becomes active if the asset price rises to or above a certain barrier level.

Valuing barrier options involves more complexity than valuing standard options due to the path-dependency of the payoff. The valuation must account for the probability that the barrier will be breached. The Black-Scholes model, often used for vanilla options, can be adapted for barrier options with the addition of specific adjustments to account for the barrier feature.
If $H \leq K$, the current value c_{DI} for a Down-and-in call is:

$$c_{DI} = S_0 \cdot \exp(-qt) \cdot \left(\frac{H}{S_0}\right)^{2\lambda} N(y) - K\exp(-rt) \cdot \left(\frac{H}{S_0}\right)^{2\lambda-2} N\left(y - \sigma\sqrt{T}\right) \ (18)$$

$$\lambda = \frac{r-q+\frac{\sigma^2}{2}}{\sigma^2} \ (19), y = \frac{\ln\left(\frac{H^2}{S_0 K}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{(T)} \ (20)$$

An ordinary call (c) equals the sum of the corresponding down-and-in and down-and-out calls. A down-and-out call option is a standard call option that ceases to exist if the underlying asset price falls to a barrier level H (where $H < S_0$, and S_0 is the initial price of the underlying asset). The value of this option, c_{DO}, can be derived as: $c_{DO} = c - c_{DI}$

Where:
c is the value of a standard European call option.

c_{DI} is the value of the corresponding down-and-in call option.

If the barrier is never breached, the down-and-out call option has a value at maturity. If the barrier is breached, the option ceases to exist, and the payoff is zero.
If $H > K$, the value of a down-and-out call is as shown in (23).

$$c_{DO} = S_0 \cdot \exp(-qt) \cdot N(x_1) - K\exp(-rt) \cdot N\left(x_1 - \sigma\sqrt{T}\right) - S_0 \cdot \exp(-qt)\left(\frac{H}{S_0}\right)^{2\lambda} N(y_1) + K \cdot \exp(-rt)\left(\frac{H}{S_0}\right)^{2\lambda-2} N\left(y_1 - \sigma\sqrt{T}\right) \ (21)$$

$$x_1 = \frac{\ln\left(\frac{S_0}{H}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \ (22), y_1 = \frac{\ln\left(\frac{H}{S_0}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \ (23)$$

Similarly to the previous case, the value of a down-and-in call is given by $c_{DI} = c - c_{DO}$
Up-and-out calls are also knock-out options. They are ordinary calls that cease to exist when the price of the underlying asset rises to H, with $H > S_0$.
If $H \leq K$, the current value of an up-and-out call, c_{UO} is null and the current value of an up-and-in call, c_{UI}, is equal to vanilla call c.
In case $H > K$, the value for an up-and-in call is shown in equation (24).

$$c_{UI} = S_0 \cdot \exp(-qt) \cdot N(x_1) - K\exp(-rt) \cdot N(x_1 - \sigma\sqrt{T}) - S_0 \cdot \exp(-qt)\left(\frac{H}{S_0}\right)^{2\lambda}[N(-y) - N(-y_1)] + K \cdot \exp(-rt)\left(\frac{H}{S_0}\right)^{2\lambda-2}\left[N\left(-y + \sigma\sqrt{T}\right) - N\left(-y_1 + \sigma\sqrt{T}\right)\right] \ (24)$$

while the current value of an up-and-out call is $c_{UO} = c - c_{UI}$.
Barrier puts work similarly to barrier calls but the direction of the price movement is reversed. Standard up-and-out puts cease to exist when the price of the underlying asset rises to H, with $H > S_0$. Standard up-and-in puts only start to exist when the price of the underlying asset rises to H, with $H > S_0$.
If $H > K$, the current value of an up-and-in put, p_{UI} is shown in (25).

$$p_{UI} = -S_0 \cdot \exp(-qT)\left(\frac{H}{S_0}\right)^{2\lambda} N(-y) + K \cdot \exp(-rT)\left(\frac{H}{S_0}\right)^{2\lambda-2} N\left(-y + \sigma\sqrt{T}\right) \ (25)$$

While the current value of an up-and-out put is $p_{UO} = p - p_{UI}$.
In case $H \leq K$, the current value of an up-and-out put is equal to (26).

$$p_{UO} = -S_0 \cdot \exp(-qT)\, N(-x_1) + K \cdot \exp(-rT)\, N\left(-x_1 + \sigma\sqrt{T}\right) + S_0 \cdot \exp(-qT)\left(\frac{H}{S_0}\right)^{2\lambda} N(-y_1) - K \cdot \exp(-rT)\, N\left(-y_1 + \sigma\sqrt{T}\right)\left(\frac{H}{S_0}\right)^{2\lambda-2} \ (26)$$

while the current value of an up-and-in put is $p_{UI} = p - p_{UO}$.
Down-and-out puts cease to exist when the price of the underlying asset falls to H, with $H < S_0$. Down-and-in puts only start to exist when the price of the underlying asset falls to H, with $H < S_0$.
If $H \geq K$, the current value, p_{DO}, of a down-and-out put is zero and the current value, p_{DI}, of a down-and-in put is equal to p. If $H < K$, the current value of a down-and-in put, p_{DI} is as shown in (27) and the current value of a down-and-out put is $p_{DO} = p - p_{DI}$.

$$p_{DI} = -S_0 \cdot \exp(-qT)\, N(-x_1) + K \cdot \exp(-rT)\, N\left(-x_1 + \sigma\sqrt{T}\right) + S_0 \cdot \exp(-qT)\left(\frac{H}{S_0}\right)^{2\lambda}[N(y) - N(y_1)] - K \cdot \exp(-rT)\left(\frac{H}{S_0}\right)^{2\lambda-2}\left[N\left(y - \sigma\sqrt{T}\right) - N\left(y_1 - \sigma\sqrt{T}\right)\right] \ (27)$$

All the valuation formulas for the barrier options presented are based on the assumption that the probabilistic distribution of the share price in a future instant of time is log-normal (Di Franco, Polimeni and Proietti, 2002).
A crucial aspect of barrier options is how frequently the underlying asset price is observed. The formulas presented earlier assume continuous observation, which is an idealization. In reality, the price is often observed discretely - daily, weekly, or at other intervals. This introduces the need for adjustments in valuation to account for the possibility of the barrier being breached between observation points.

Broadie, Glasserman and Kou (1999) have developed an approximation of the formula to take into account the discretization of the observation frequency. The correction factor proposed by these researchers is based on the modification to be made, for each observation, on the level of the barrier with: $H_U = H \cdot \exp(\beta\sigma\sqrt{\Delta t})$ if the barrier is an upper-bound for the asset underlying the option. If the barrier represents a lower-bound, the adjustment is $H_D = H \cdot \exp(-\beta\sigma\sqrt{\Delta t})$. Δt is the time that elapses between the instants of observation of the barrier. $\beta = \frac{\zeta(0.5)}{\sqrt{2\pi}} \approx 0.5826$, where $\zeta(\cdot)$ is the Riemann zeta-function.
In order to implement this in a programming environment, it is useful to rearrange the previous formulas of Reiner and Rubinstein (1991) according to the classification proposed by Rich (1994).
This pricing procedure provides for the use of the cost-of-carry, $b = r - q$, and the rebate feature, (R), where the option holder receives a fixed amount if the barrier is breached, and the option is knocked out. This rebate can be structured as either a cash payment or a payment in the form of an asset. The inclusion of a rebate alters the valuation, as it provides a fallback payoff, reducing the risk for the option holder.

$$A = \phi S \cdot \exp[(b-r)T]\, N(\phi x_1) - \phi K \cdot \exp(-rT)\, N(\phi x_1 - \phi\sigma\sqrt{T}) \quad (28)$$

$$B = \phi S \cdot \exp[(b-r)T]\, N(\phi x_2) - \phi K \cdot \exp(-rT)\, N(\phi x_2 - \phi\sigma\sqrt{T}) \quad (29)$$

$$C = \phi S \cdot \exp[(b-r)T] \left(\frac{H}{S}\right)^{2(\mu+1)} N(\eta y_1) - \phi K \cdot \exp(-rT) \left(\frac{H}{S}\right)^{2\mu} N(\eta y_1 - \eta\sigma\sqrt{T}) \quad (30)$$

$$D = \phi S \cdot \exp[(b-r)T] \left(\frac{H}{S}\right)^{2(\mu+1)} N(\eta y_2) - \phi K \cdot \exp(-rT) \left(\frac{H}{S}\right)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T}) \quad (31)$$

$$E = R \cdot \exp(-rT) \left[N(\eta x_2 - \eta\sigma\sqrt{T}) - \left(\frac{H}{S}\right)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T})\right] \quad (32)$$

$$F = R \cdot \left[\left(\frac{H}{S}\right)^{\mu+\lambda} N(\eta z) + \left(\frac{H}{S}\right)^{\mu-\lambda} N(\eta z - 2\eta\lambda\sigma\sqrt{T})\right] \quad (33)$$

Where:

$$x_1 = \frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \ (34), \quad x_2 = \frac{\ln\left(\frac{S}{H}\right)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \ (35), \quad y_1 = \frac{\ln\left(\frac{H^2}{SK}\right)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \ (36)$$

$$y_2 = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \ (37), \quad z = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} (38),$$

$$\mu = \frac{b - \frac{\sigma^2}{2}}{\sigma^2} \ (39), \quad \lambda = \sqrt{\mu^2 + \frac{2r}{\sigma^2}} \ (40)$$

Down-and-in call $S > H$
Pay-off: max $(S - K; 0)$ if $S \leq H$ before T otherwise R at maturity.

$$c_{K>H}^{DI} = C + E \ (41), \ \eta = +1, \ \phi = +1 \ (42)$$

$$c_{K<H}^{DI} = A - B + D + E \ (43), \ \eta = +1, \ \phi = +1 \ (44)$$

Up-and-in call $S < H$
Pay-off: max $(S - K; 0)$ if $S \geq H$ before T otherwise R at maturity.

$$c_{K>H}^{UI} = A + E \ (45), \ \eta = -1, \phi = +1 \ (46)$$

$$c_{K<H}^{UI} = B - C + D + E \ (47) \qquad \eta = -1, \phi = +1 \ (48)$$

Down-and-in put $S > H$
Pay-off: $\max(K - S; 0)$ if $S \leq H$ before T otherwise R at maturity.

$$p_{K>H}^{DI} = B - C + D + E \ (49) \qquad \eta = +1, \phi = -1 \ (50)$$
$$p_{K<H}^{DI} = A + E \ (51) \qquad \eta = +1, \phi = -1 \ (52)$$

Up-and-in put $S < H$
Pay-off: $\max(K - S; 0)$ if $S \geq H$ before T otherwise R at maturity.

$$p_{K>H}^{UI} = A - B + D + E \ (53) \qquad \eta = -1, \phi = -1 \ (54)$$
$$p_{K<H}^{UI} = C + E \ (55) \qquad \eta = -1, \phi = -1 \ (56)$$

Down-and-out call S > H
Pay-off: $\max(S - K; 0)$ if S > H before T otherwise R at the hit.

$$c_{K>H}^{DO} = A - C + F \quad (57) \qquad \eta = +1, \phi = +1 \quad (58)$$
$$c_{K<H}^{DO} = B - D + F \quad (59) \qquad \eta = +1, \phi = +1 \quad (60)$$

Up-and-out call S < H
Pay-off: $\max(S - K; 0)$ if S < H before T otherwise R at the hit.

$$c_{K>H}^{UO} = F \quad (61) \qquad \eta = -1, \phi = +1 \quad (62)$$
$$c_{K<H}^{UO} = A - B + C - D + F \quad (63) \qquad \eta = -1, \phi = +1 \quad (64)$$

Down-and-out put $S > H$
Pay-off: $\max(K - S; 0)$ if $S > H$ before T otherwise R at the hit.

$$p_{K>H}^{DO} = A - B + C - D + F \quad (65) \qquad \eta = +1, \phi = -1 \quad (66)$$
$$p_{K<H}^{DO} = F \quad (67) \qquad \eta = +1, \phi = -1 \quad (68)$$

Up-and-out put S < H
Pay-off: $\max(K - S; 0)$ if S < H before T otherwise R at the hit.

$$p_{K>H}^{UO} = B - D + F \quad (69) \qquad \eta = -1, \phi = -1 \quad (70)$$
$$p_{K<H}^{UO} = A - C + F \quad (71) \qquad \eta = -1, \phi = -1 \quad (72)$$

3.1.1) Crude Monte Carlo application

Barrier options are sensitive to the path taken by the underlying asset, especially in relation to the barrier level. The assumption of continuous monitoring - where the asset price is constantly observed - simplifies the theoretical valuation of these options but it is impractical in real-world applications. Instead, the asset price is typically observed at discrete intervals, such as daily or weekly. This discrete monitoring can lead to different outcomes compared to continuous monitoring, thus influencing the estimated option price.
To implement a Crude Monte Carlo simulation for pricing a barrier option, we need to follow these steps:

1. **Model the Asset Price Path**: The asset price is typically modeled using a Geometric Brownian Motion (GBM), which follows the stochastic differential equation (73).

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (73)$$

2. **Simulate Asset Paths**: We need to generate multiple simulations of the asset price path over the life of the option, taking into account the discrete monitoring points. The time steps Δt between monitoring points are crucial. For example, if we observe the price daily over a year, we have 252 steps (assuming 252 trading days).

3. **Check the Barrier Condition**: For each simulated path, we need to check if the barrier level is breached at any monitoring point. Depending on the type of barrier option (knock-in or knock-out), this will determine whether the option is activated or deactivated.

4. **Calculate the Payoff for Each Path**: After checking the barrier, we calculate the payoff for each simulated path. For a knock-out option, the payoff is zero if the barrier is breached. For a knock-in option, the payoff is calculated only if the barrier is breached.

5. **Average Payoffs Across Simulations**: The option price is then estimated as the discounted average of the payoffs from all simulations.

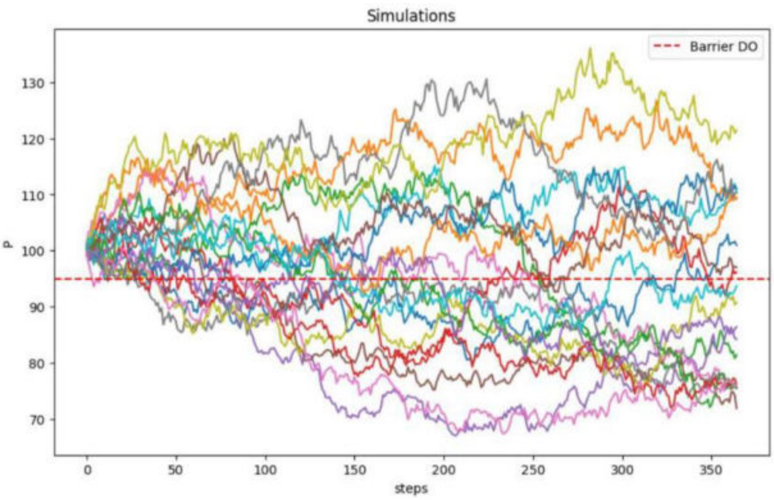


Figure 2: Different paths simulated through the Monte Carlo model

The key aspect to explore using Monte Carlo simulation is how different monitoring frequencies affect the barrier option price. A more frequent monitoring (e.g., daily) better approximates the continuous monitoring assumption, while a less frequent monitoring (e.g., weekly or monthly) may lead to different outcomes.

To explore this bias, a Crude Monte Carlo simulation has been implemented, to price a down-and-out call option under varying monitoring frequencies: 24 hours (daily), 1 hour, 30 minutes, and 15 minutes. The purpose of these experiments was to investigate how the choice of a monitoring interval impacts the estimated option price and to highlight the extent of the bias introduced by less frequent observations.

For the simulation, a Geometric Brownian Motion model has been used to generate the price paths of the underlying asset. The key parameters - initial stock price (S_0), strike price (K), barrier level (H), volatility (σ), risk-free rate (r), and time to maturity (T) - were kept constant across all trials to ensure consistency in the results. The only variable that was adjusted was the frequency at which the asset price was monitored to determine whether it breached the barrier.

The monitoring frequencies have been set at twenty-four hours, one hour, thirty minutes and fifteen minutes. The number of simulations at each iteration are set to 10.000; the loop went for 200 iterations. Each scenario was run through the Monte Carlo simulation to estimate the down-and-out barrier option price. The results were then aggregated and compared to understand the impact of different monitoring intervals on the option estimated value. The settings used to conduct the study are as follows:

$$S_0 = 100, K = 100, T = 1.0, r = 0.05, q = 0.02, \sigma = 0.2, H = 95$$

The exact price of this Down-and-Out call option is 4.8835.

The simulation results reveal a clear trend: as the monitoring frequency increases, the estimated price of the down-and-out option decreases, converging toward the theoretical value expected under continuous monitoring. This is due to the increased likelihood of the barrier being breached when the asset price is observed more frequently.

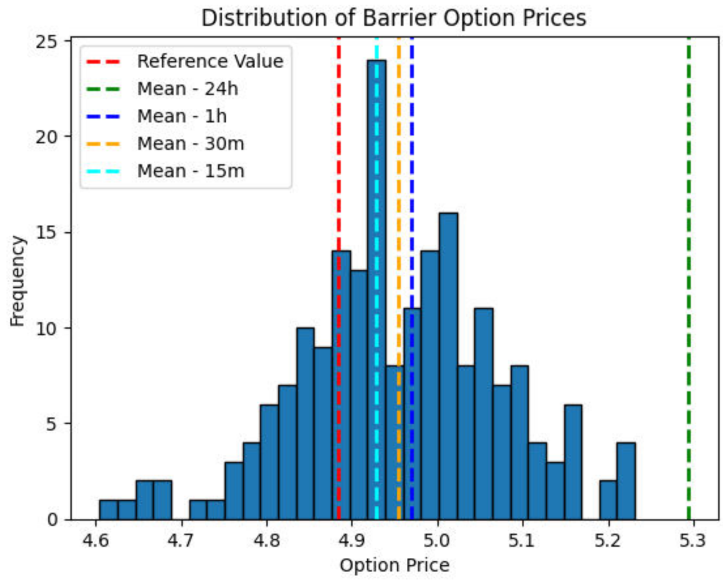


Figure 3: Different monitoring frequencies of Monte Carlo for Barrier Option Prices

Daily Monitoring (24 hours): This scenario produced the highest estimated option price. With only 24 observations across the life of the option, there were fewer opportunities for the price to hit the barrier, resulting in a lower probability of the option being knocked out and, therefore, a higher price.

Hourly Monitoring (1 hour): the option price was lower than the one computed with daily monitoring, reflecting the higher chance of the barrier being breached.

30-Minute Monitoring: As expected, the price continued to decrease with a more frequent monitoring, showing greater alignment with the continuous monitoring assumption.

15-Minute Monitoring: This scenario yielded the lowest estimated option price, most closely approximating the theoretical value, as the frequent checks made it more likely for the asset price to breach the barrier.

The results underscore the importance of accounting for a discretization bias in barrier option pricing. Traders and risk managers relying on less frequent monitoring may overestimate the value of a down-and-out barrier option, leading to potential mispricing and exposure to unanticipated risks. Conversely, more frequent monitoring, while computationally intensive, provides a more accurate estimate that better reflects the true risk profile of the option.

This bias is particularly relevant in markets where high-frequency trading and rapid price fluctuations are common. In such environments, the likelihood of the barrier being breached increases, making it crucial to adopt a monitoring strategy that closely approximates continuous observation.

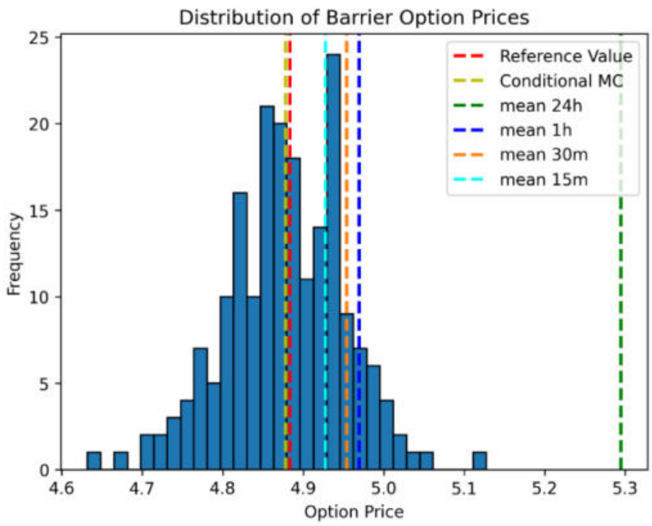
The Crude Monte Carlo simulation results demonstrate the significant impact of monitoring frequency on the estimated price of down-and-out barrier options. By systematically reducing the time interval between observations - from 24 hours to 15 minutes - it becomes evident that discretization bias can lead to overvaluation when the barrier is monitored less frequently. These findings highlight the necessity for market participants to carefully consider the frequency of monitoring when pricing and managing barrier options, especially in fast-moving markets.

3.1.2) Conditional Monte Carlo application

In the context of barrier options, the Conditional Monte Carlo method is particularly suitable. It leverages the conditional expectation of the payoff given that the barrier has not been breached up to the current time. This approach reduces the noise in the simulation, as it only focuses on paths that are relevant to the option final payoff, thereby accelerating the convergence to the true option price. To demonstrate this, a Conditional Monte Carlo method has been implemented as well, on the same options analysed in the previous sections. The parameters used - initial stock price (S_0), strike price (K), barrier level (H), volatility (σ), risk-free rate (r), and time to maturity (T) - remained consistent with those used in the Crude Monte Carlo simulations. The goal is to highlight the improvements in both speed and precision when using the Conditional Monte Carlo method. The key difference between the two methods lies in how they simulate the price paths of the underlying asset:

Crude Monte Carlo: This method simulates numerous independent price paths of the underlying asset, checking whether the barrier has been breached at each time step. If the barrier is breached, the option becomes worthless for that path. This process, while straightforward, often requires many simulations to achieve a high degree of accuracy, as it does not account for any prior knowledge about the probability of the barrier being breached.
Conditional Monte Carlo: In contrast, the CMC method is conditional on the event that the barrier has not been breached by a certain time. This approach allows for the direct calculation of the expected payoff of the option, given that the price path is still valid (i.e., it has not hit the barrier). By focusing on these relevant paths, the CMC method reduces the variance of the estimated option price, leading to faster convergence and more accurate results with fewer simulations.

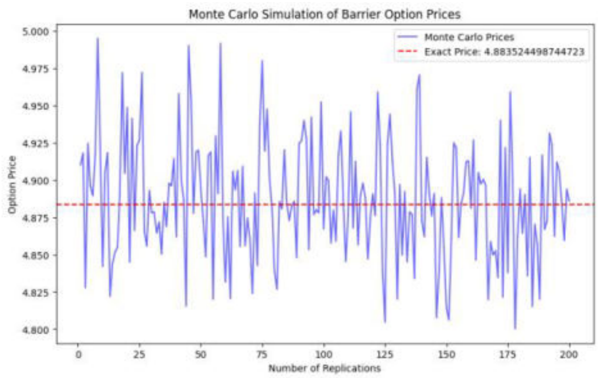
The Conditional method demonstrated a clear advantage in computational speed. The crude Monte Carlo method took increasingly longer with each reduction of the monitoring time window, reaching up to 870 minutes for the "15m" monitoring period. On the other hand, the Conditional Monte Carlo method only took one minute. By reducing the number of irrelevant paths (those where the barrier is breached early), the Conditional method required significantly fewer simulations to reach a given level of accuracy. This reduction in computational effort translates directly into faster runtimes, making the CMC method more suitable for real-time pricing and risk management applications where speed is critical.
In terms of precision, the Conditional method consistently produced more accurate estimates of the down-and-out barrier option price. The reduction in variance achieved by being conditional on the relevant paths meant that the option prices estimated by the Conditional method had a much narrower confidence interval compared to those produced by the Crude Monte Carlo method. This precision is particularly valuable when pricing options in volatile markets, where small errors can lead to significant financial consequences.

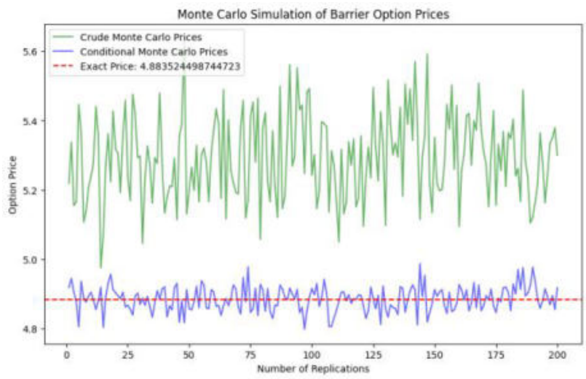


	Mean	Std. Dev
24h	5.2945	0.1230
1h	4.9690	0.1187
30m	4.9537	0.1180
15m	4.9277	0.1257
Cond.	4.8828	0.0413

Figure 4: Conditional Monte Carlo compared with the Crude Monte Carlo for Barrier Option Prices

A plot with the Conditional Monte Carlo results is shown in Figure 5. It has been run with two hundred replications loop, each with a hundred thousand simulations.





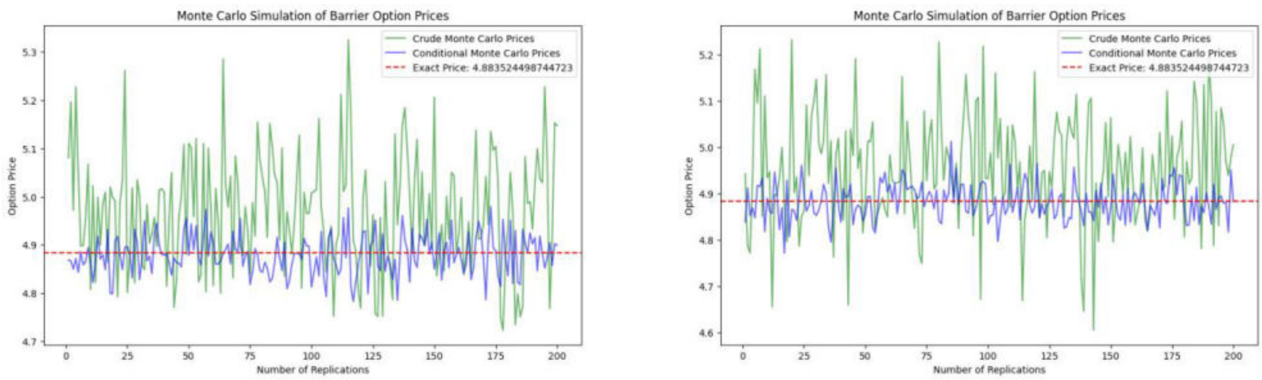


Figure 5A: 200 replications of Conditional Monte Carlo simulations for Barrier Option Prices
Figure 5B: 200 replications of Crude Monte Carlo simulations for Barrier Option Prices, 24 hours monitoring frequency
Figure 5C: 200 replications of Crude Monte Carlo simulations for Barrier Option Prices, 1 hour monitoring frequency
Figure 5D: 200 replications of Crude Monte Carlo simulations for Barrier Option Prices, 30 minutes monitoring frequency

3.2) Lookback Options

Lookback Options are sophisticated financial derivatives whose value depends on the minimum or maximum price reached by the underlying asset during the entire lifespan of the option. Unlike traditional options, where the strike price is fixed at the time of contract initiation, lookback options allow the holder to "look back" at the underlying asset price history to determine the optimal exercise price. There are two main types of lookback options: floating-strike lookback options and fixed-strike lookback options, each with its unique valuation method and payout structure.

In some cases, the observation period for the extreme values (maximum or minimum) of the underlying asset might be shorter than the full life of the option. These derivatives are known as Partial-Time Lookback Options, and they can be further categorized into partial-time fixed-strike and partial-time floating-strike lookback options. Given the complexity of these instruments, numerical methods are often required to accurately value them.

Lookback options are powerful tools for investors looking to hedge against or capitalize on significant price movements in the underlying asset. Their value derives from the most favorable price movements observed during the life of the option, making them particularly useful in volatile markets. However, the complexity of their valuation requires a deep understanding of the underlying models and assumptions, as well as a consideration of market conditions and the specific terms of the option contract.

In floating-strike lookback options, the strike price is not set in advance, but it is determined retrospectively, based on the minimum or maximum price reached by the underlying asset during the life of the option.

The final value of a floating-strike lookback call option is determined by the difference between the final price of the underlying asset S_T and the minimum price S_{min} recorded during the lifespan of the option. Mathematically, the payoff is expressed in (74).

$$c(S, S_{min}, T) = \max(S - S_{min}; 0) = S_T - S_{min} \quad (74)$$

Conversely, the final value of a floating-strike lookback put option depends on the difference between the maximum price S_{max} reached by the underlying asset during the life of the option and its final price S_T.

$$p(S, S_{max}, T) = \max(S_{max} - S; 0) = S_{max} - S_T \quad (75)$$

The payoff is given by equation (75). The valuation of these options can be complex and is often calculated using models like the Goldman-Sosin-Gatto (1979) and the Garman (1989) formulas. These models incorporate factors such as the cumulative normal distribution $N(\cdot)$ and the standard normal distribution $n(\cdot)$ to account for the stochastic behavior of asset prices.

The closed formula for the valuation of a Floating-Strike Lookback call option is shown in (76) and (77).

If $b \neq 0$

$$c = S \cdot \exp[(b-r)T]\, N(a_1) - S_{min} \cdot \exp(-rT)\, N(a_2) + \\ +S \cdot \exp(-rT) \frac{\sigma^2}{2b} \left[\left(\frac{S}{S_{min}} \right)^{-\frac{2b}{\sigma^2}} N\left(-a_1 + \frac{2b}{\sigma}\sqrt{T}\right) - \exp(bT)\, N(-a_1) \right] \quad (76)$$

If $b = 0$

$$c = S \cdot \exp(-rT)\, N(a_1) - S_{min} \cdot \exp(-rT)\, N(a_2) + \\ +S \cdot \exp(-rT)\, \sigma\sqrt{T}\{n(a_1) + a_1[N(a_1) - 1]\} \quad (77)$$

Where:

$$a_1 = \left(\frac{\ln\left(\frac{S}{S_{min}}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{(T)}} \right) \quad (78), \quad a_2 = a_1 - \sigma\sqrt{T} \quad (79)$$

Conversely, the exact formula for the put version of the option is in (80) and (81).

If $b \neq 0$

$$p = S_{max} \cdot \exp(-rT)\, N(-b_2) - S \cdot \exp[(b-r)T]\, N(-b_1) + S \cdot \exp(-rT) \frac{\sigma^2}{2b} \left[-\left(\frac{S}{S_{max}}\right)^{-\frac{2b}{\sigma^2}} N\left(b_1 - \frac{2b}{\sigma}\sqrt{T}\right) + \exp(bT)\, N(b_1) \right] \quad (80)$$

If $b = 0$

$$p = S_{max} \cdot \exp(-rT)\, N(-b_2) - S \cdot \exp[(b-r)T]\, N(-b_1) + S \cdot \exp(-rT)\, \sigma\sqrt{T}\{n(b_1) + N(b_1)b_1\} \quad (81)$$

$$b_1 = \frac{\ln\left(\frac{S}{S_{max}}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}} \quad (82), \quad b_2 = b_1 - \sigma\sqrt{T} \quad (83)$$

S: Current underlying price
r: Risk-free rate
q: Dividend yield
σ: volatility of underlying
T: Time to maturity
S_{min}: Minimum underlying price observed since the beginning of the contract
S_{max}: Maximum underlying price observed since the beginning of the contract

In contrast to floating-strike options, fixed-strike lookback options have a pre-determined strike price K that is set at the time the contract is entered into. The value of these options at expiration depends on the highest or lowest price reached by the underlying asset during the life of the option, relative to this fixed strike price.
The call Fixed-Strike Lookback option pays the maximum between the difference of the highest price observed during the life of the option S_{max} and the strike price K, or zero.

$$c(S, S_{max}, T) = \max(S_{max} - K; 0) \quad (84)$$

The payout for a fixed-strike lookback put is the maximum between the difference of the strike price K and the lowest price observed S_{min}, or zero.

$$p(S, S_{min}, T) = \max\,(K - S_{min}; 0) \quad (85)$$

Valuing fixed-strike lookback options often involves formulas developed by Conze and Viswanathan (1991), which account for the potential variance in outcomes depending on whether the strike price is greater than or less than the observed maximum or minimum prices.
For the call option:

If $K > S_{max}$

$$c = S \cdot \exp[(b-r)T]\, N(d_1) - K \cdot \exp(-rT) N(d_2) + \\ + S \cdot \exp(-rT)\ \frac{\sigma^2}{2b} \left[-\left(\frac{S}{K}\right)^{-\frac{2b}{\sigma^2}} N\left(d_1 - \frac{2b}{\sigma}\sqrt{T}\right) - \exp(bT)\, N(d_1) \right] \quad (86)$$

Where:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}} \quad (87), \quad d_2 = d_1 - \sigma\sqrt{T} \quad (88)$$

If $K \leq S_{max}$

$$c = \exp(-rT) \cdot (S_{max} - K) + S \cdot \exp[(b-r)T]\, N(e_1) - S_{max} \cdot \exp(-rT)\, N(e_2) + \\ + S \cdot \exp(-rT)\ \frac{\sigma^2}{2b} \left[-\left(\frac{S}{S_{max}}\right)^{-\frac{2b}{\sigma^2}} N\left(e_1 - \frac{2b}{\sigma}\sqrt{T}\right) + \exp(bT)\, N(e_1) \right] \quad (89)$$

Where:

$$e_1 = \frac{\ln\left(\frac{S}{S_{max}}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}} \quad (90), \quad e_2 = e_1 - \sigma\sqrt{T} \quad (91)$$

The valuation for a Put option, on the other hand, is as follows:

If $K < S_{min}$

$$p = K \cdot \exp(-rT)\, N(-d_2) - S \cdot \exp[(b-r)T]\, N(-d_1) + \\ + S \cdot \exp(-rT)\ \frac{\sigma^2}{2b} \left[\left(\frac{S}{K}\right)^{-\frac{2b}{\sigma^2}} N\left(-d_1 + \frac{2b}{\sigma}\sqrt{T}\right) - \exp(bT)\, N(-d_1) \right] \quad (92)$$

If $K \geq S_{min}$

$$p = \exp(-rT) \cdot (K - S_{min}) - S \cdot \exp[(b-r)T]\, N(-f_1) - S_{min} \cdot \exp(-rT)\, N(-f_2)$$
$$+S \cdot \exp(-rT)\ \frac{\sigma^2}{2b}\left[\left(\frac{S}{S_{min}}\right)^{-\frac{2b}{\sigma^2}} N\left(-f_1 + \frac{2b}{\sigma}\sqrt{T}\right) - \exp(bT)\, N(-f_1)\right] \quad (93)$$

Where $f_1 = \frac{\ln\left(\frac{S}{S_{min}}\right) + (b+\sigma^2/2)T}{\sigma\sqrt{T}}$ (94), $f_2 = f_1 - \sigma\sqrt{T}$ (95).

3.2.1) Crude Monte Carlo application

The Crude Monte Carlo method directly replicates the logic of a lookback option. In this approach, the daily prices of the underlying asset are simulated and stored in a vector. At the end of the simulation, the minimum or maximum value—depending on whether it is a call or a put option—is selected from the vector for use in calculating the payoff.
For the simulation, a Geometric Brownian Motion model has been used to generate the price paths of the underlying asset. The key parameters—initial stock price (S), strike price (K), volatility (σ), risk-free rate (r), and time to maturity (T)—were kept constant across all trials to ensure consistency in the results. The starting parameter of the function, M_0, is S_{max} in case of a Put option, S_{min} if the option valued is a Call. The only variable that was adjusted was the frequency at which the asset price was monitored.
The monitoring frequencies have been set at twenty-four hours, one hour, thirty minutes and fifteen minutes. As before, the number of simulations at each iteration are set to 10.000; the loop went for 200 iterations. Each scenario was run through the Monte Carlo simulation to estimate a Floating-Strike Lookback Put option price first, and then the method has been applied on the valuation of a Fixed-Strike Lookback Call. The results have been aggregated and compared to understand the impact of different monitoring intervals on the option estimated value. The settings used to conduct the study are as follows: $S = 120, M_0 = 130, T = 1.0, r = 0.4, q = 0.02, \sigma = 0.3$. The exact price of this option is 12.4819.
The simulation results reveal the same trend as the one shown in the application for the barrier options: as the monitoring frequency increases, the estimated price of the option increases, converging toward the theoretical value expected under continuous monitoring.

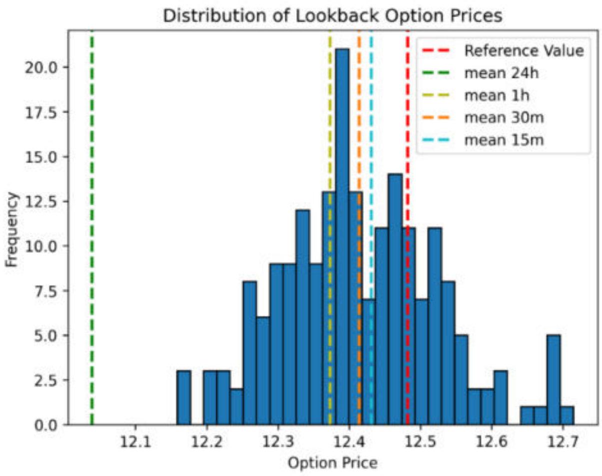


Figure 6: Crude Monte Carlo for Floating-Strike Lookback options at different monitoring frequencies

The settings applied to the function for the application of the Crude Monte Carlo for pricing a Fixed-Strike version of a Lookback Call option, are the following: $S = 100, M_0 = 100, T = 1.0, r = 0.4, q = 0.02, \sigma = 0.3$. The result of the closed formula valuation is 8.6626. Here, too, the same dynamic can be observed: as the monitoring frequency increases, the accuracy of the method becomes significantly better.

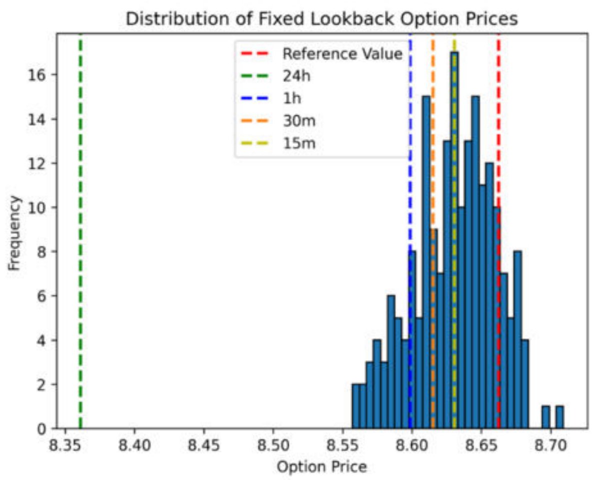
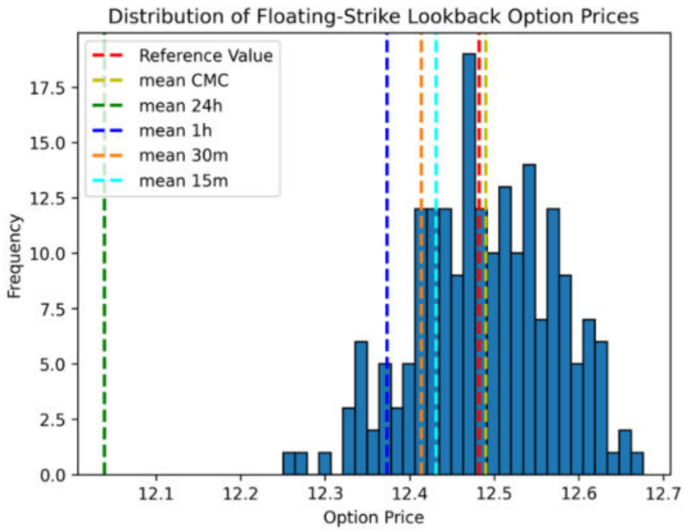


Figure 7: Crude Monte Carlo for Fixed-Strike Lookback options at different monitoring frequencies

The same kind of bias can be observed, as the distribution of the function set at a narrower monitoring frequency shows closer and closer results to the exact one. It can be shown that for lookback options, with the same number of iterations, the approximation is slightly less precise than the one obtained applying the method on the barrier options.

3.2.2) Conditional Monte Carlo application

Although this method accurately replicates the dynamics of the derivative, it is subject to numerical integration errors due to the inability to continuously monitor the underlying asset. Consequently, the Conditional Monte Carlo method, previously introduced in the section on standard barrier options, is often preferred. This method is convenient when the focus is solely on determining the extreme values that the underlying asset might reach within a given time frame, utilizing a numerical technique that adheres to the principles of the Brownian Bridge.
The Conditional method demonstrated the same advantages in both computational speed and precision. The crude Monte Carlo method took increasingly longer with each reduction of the monitoring time window, while the conditional Monte Carlo method only took one minute. By reducing the number of irrelevant paths, the Conditional method required far fewer simulations to reach a given level of accuracy. This reduction in computational effort translates directly into faster runtimes, making the CMC method the suitable choice here as well.



	Mean	Std. Dev
24h	12.0389	0.1226
1h	12.3731	0.1192
30m	12.4139	0.1105
15m	12.4312	0.1147
Cond.	12.4774	0.0413

Figure 8: Conditional Monte Carlo: comparison with the Crude Monte Carlo for Floating-Strike Lookback Option Prices

The graph with the Conditional Monte Carlo results is shown in Figure 9. This, too, has been set to run for two hundred replications, each with a set number of a hundred thousand simulations.

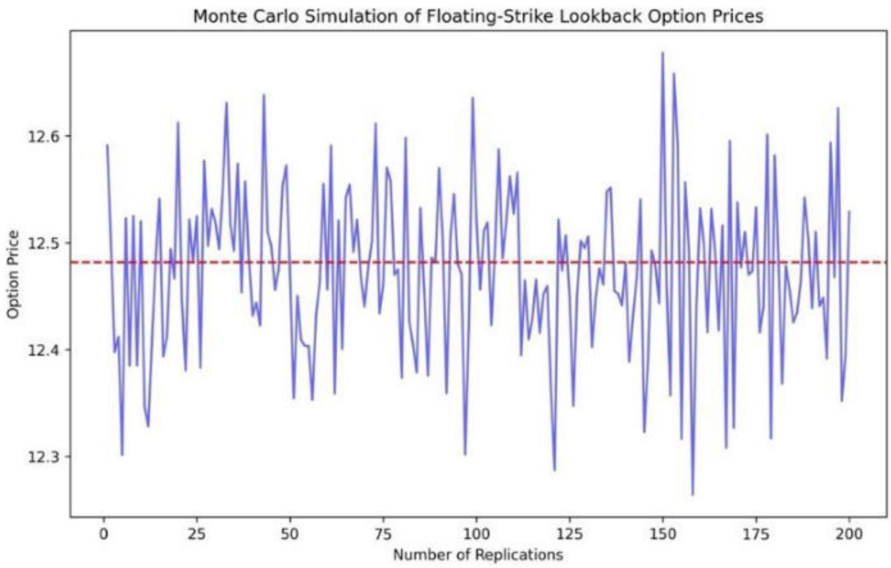
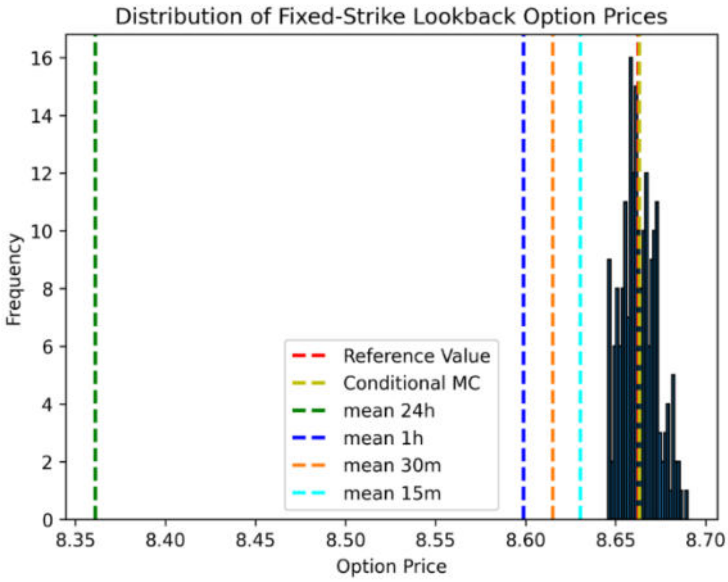


Figure 9: 200 replications of the Conditional Monte Carlo model for Floating-Strike Lookback Option prices

In terms of precision, the Conditional method consistently produced more accurate estimates for the lookback option. The option prices estimated by the Conditional method had a much narrower confidence interval compared to those produced by the Crude Monte Carlo method.



	Mean	Std. Dev
24h	8.3612	0.0387
1h	8.5987	0.0363
30m	8.6152	0.0424
15m	8.6304	0.0297
Cond.	8.6624	0.0209

Figure 10: Conditional Monte Carlo: comparison with the Crude Model for Fixed-Strike Lookback Option prices

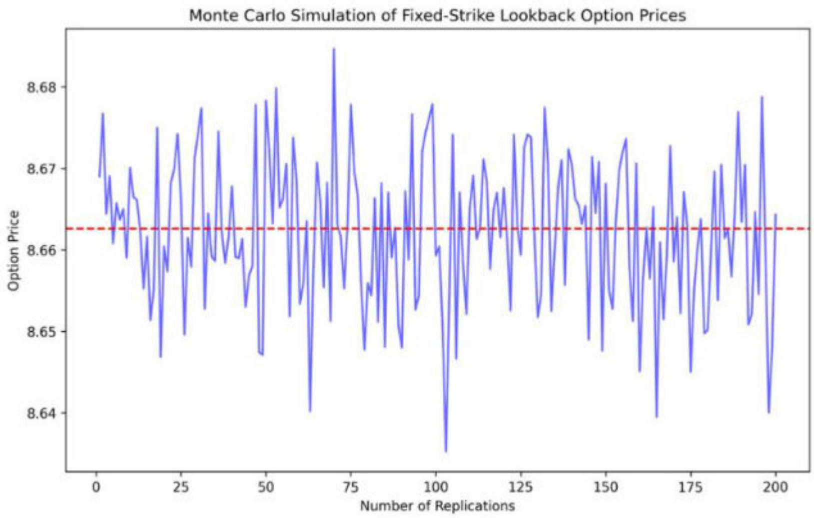


Figure 11: 200 replications of Conditional Monte Carlo on Fixed-Strike Lookback options

3.3) Soft Barrier Options

Another variant of the standard barrier option is the soft-barrier option. It is similar to the latter, but the barrier is no longer defined by a single level. Rather, it is a "soft range" between an upper level, U, and a lower level, L. The main difference between a soft and a standard barrier, is that soft-barrier options are knocked in – or out – proportionally. For instance, consider a soft down-and-out call with a current asset price of 100, with a soft barrier range from $U = 90$ to $L = 80$. If the lowest asset price during the lifetime is 86, then 40% of the call will be knocked out.
Hart and Ross (1994) introduced for the first time the closed formula that can be applied and used to find the fair value of the soft-down-and-in call and soft-up-and-in put options.

$$w = \frac{1}{U-L}\Bigg\{\eta S e^{(b-r)T} S^{-2\mu} \frac{(SK)^{\mu+0.5}}{2(\mu+0.5)}\left[\left(\frac{U^2}{SK}\right)^{\mu+0.5} N(\eta d_1) - \lambda_1 N(\eta d_2) - \left(\frac{L^2}{SK}\right)^{\mu+0.5} N(\eta e_1) + \lambda_1 N(\eta e_2)\right] + \\ -\eta K e^{-rT} S^{-2(\mu-1)} \frac{(SK)^{\mu-0.5}}{2(\mu-0.5)}\left[\left(\frac{U^2}{SK}\right)^{\mu-0.5} N(\eta d_3) - \lambda_2 N(\eta d_4) - \left(\frac{L^2}{SK}\right)^{\mu-0.5} N(\eta e_3) + \lambda_2 N(\eta e_4)\right]\Bigg\} \quad (96)$$

where η is set to 1 for a call and -1 for a put, and

$$d_1 = \frac{\ln\left(\frac{U^2}{SK}\right)}{\sigma\sqrt{T}} + \mu\sigma\sqrt{T} \ (97), \ d_2 = d_1 - (\mu+0.5)\sigma\sqrt{T} \ (98)$$

$$d_3 = \frac{\ln\left(\frac{U^2}{SK}\right)}{\sigma\sqrt{T}} + (\mu-1)\sigma\sqrt{T} \ (99), \ d_4 = d_3 - (\mu-0.5)\sigma\sqrt{T} \ (100)$$

$$e_1 = \frac{\ln\left(\frac{L^2}{SK}\right)}{\sigma\sqrt{T}} + \mu\sigma\sqrt{T} \ (101), \ e_2 = e_1 - (\mu+0.5)\sigma\sqrt{T} \ (102)$$

$$e_3 = \frac{\ln\left(\frac{L^2}{SK}\right)}{\sigma\sqrt{T}} + (\mu - 1)\sigma\sqrt{T} \ (103), \quad e_4 = e_3 - (\mu - 0.5)\sigma\sqrt{T} \ (104)$$

$$\lambda_1 = e^{-0.5\left[\sigma^2 T(\mu+0.5)(\mu-0.5)\right]} \ (105), \qquad \lambda_2 = e^{-0.5\left[\sigma^2 T(\mu-0.5)(\mu-1.5)\right]} \ (106)$$

$$\mu = \frac{b+\frac{\sigma^2}{2}}{\sigma^2} \ (107)$$

For the valuation of the price of a soft down-and-out call, the value of a soft down-and-in call must be subtracted to a standard call. Similarly, the value of a soft up-and-out put is equal to the value of a standard put, minus a soft up-and-in put.
Standard barrier options become increasingly difficult to delta hedge as the asset price nears the barrier.
This occurs due to an increase in gamma risk, which reflects how sensitive the option delta is to changes in the price of the underlying asset.
A higher gamma implies that even small fluctuations in the asset price can cause significant shifts in delta, making it harder to maintain an effective hedge.
Delta hedging refers to the strategy of adjusting the position in the underlying asset to neutralize the option price sensitivity (delta) to asset movements.
In contrast, soft-barrier options generally exhibit lower gamma risk, meaning their delta is less reactive to price changes, thus simplifying the hedging process.

3.3.1) Crude Monte Carlo application

A Monte Carlo method has been applied in the same way it has been applied on the other options seen previously.
The Crude Monte Carlo method directly replicates the working principle of a soft-barrier option. For soft-barrier options, the main feature is that the barrier is not hit instantly; instead, the payoff is determined by whether the asset price crosses a 'soft' boundary within a certain range or time window, rather than a rigid threshold. This is crucial for the simulation, as it allows the model to capture the probabilistic nature of the barrier behavior. The soft-barrier feature is embedded in the Monte Carlo framework by modifying the payoff function to account for the gradual approach to the barrier.
At each time step of the simulation, the algorithm checks whether the underlying asset price has crossed the soft-barrier range, storing this information to determine whether the option is activated or not. The final payoff is computed based on whether the asset price has crossed the barrier, and if so, how far it has moved within the soft-barrier zone. This process is repeated for many simulated paths, each representing a potential future evolution of the asset price.
Once all the paths are simulated, the average of the payoffs across all paths is calculated. This average represents the expected payoff of the option under the assumed stochastic process.
A Down-and-Out Call has been used for the simulation. The settings of the option price used to conduct the Monte Carlo simulations are the followings: $S = 100, K = 100, U = 95, L = 90, T = 0.5, r = 0.1, b = 0.05, \sigma = 0.2$.
The precision of the Crude Monte Carlo approach depends heavily on the number of simulations performed. A higher number of simulations typically leads to more accurate pricing, but it increases the computational cost. In practice, variance reduction techniques may be employed alongside the basic Monte Carlo algorithm to improve efficiency and reduce the error margin, while maintaining the accuracy of the price estimate (Bottasso *et al.*, 2023).
The monitoring frequencies have been set at twenty-four hours and one hour. Differently from before, only twenty-four and one-hour frequencies have been set for two reasons: the first is that approaching from twenty-four to one hour – the widest "jump" – shows only minimal improvements in the results of the simulations; the other is that calculation times of the higher frequencies would have been very long. As always, the number of simulations at each iteration is set to 10.000; the loop went for 200 iterations. Each scenario was run through the Monte Carlo simulation to estimate a Soft Down-and-Out Call option price. The paths have been shown in a different kind of plot to show the bias in a clearer way. The exact price of this option is 5.5616.
The Monte Carlo simulation for the soft barrier at a twenty-four hours monitoring frequency is shown in Figure 12.

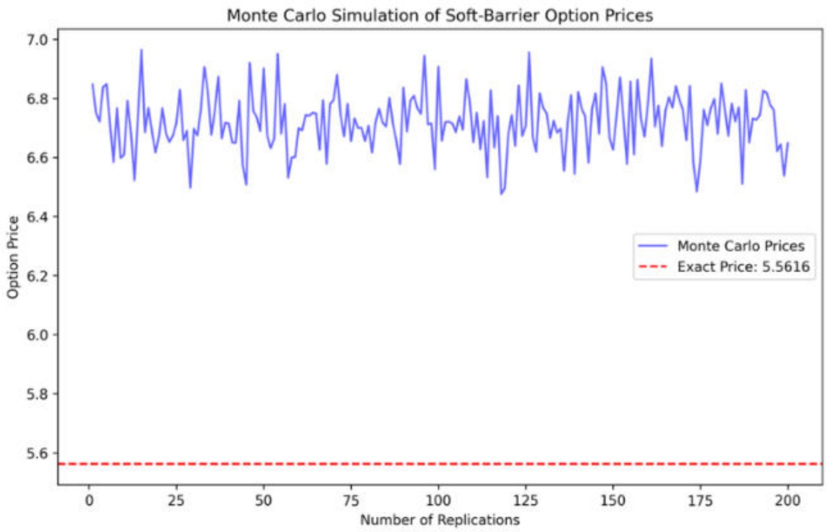


Figure 12: Monte Carlo Simulation for soft-barrier options prices, twenty-four-hour frequency monitoring.

The Monte Carlo simulation for the option at a one-hour monitoring frequency is shown in Figure 13.

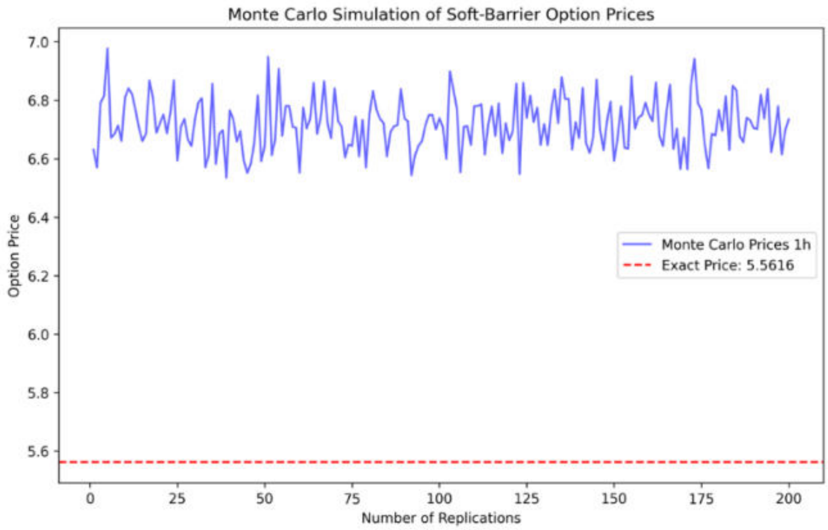


Figure 13: Monte Carlo Simulation for soft-barrier options prices, one-hour frequency monitoring.

The simulation results reveal a huge bias, similarly to the ones shown in the previous applications: as the monitoring frequency increases, though, the estimated price of the option does not increase as much as the other simulations, making it difficult to see the convergence toward the theoretical value that would be expected under a continuous monitoring.

3.3.2) Conditional Monte Carlo application

When we analyze this option, the bias resulting from the Crude Monte Carlo method is more evident, as it is subject to heavier numerical integration errors due to the inability to continuously monitor the underlying asset. As previously noted, the Babsiri-Noel's Conditional Monte Carlo method, is often preferred.
This approach proves particularly beneficial when the primary objective is to estimate the maximum or minimum values that the underlying asset may achieve over a specified time horizon.
By employing a numerical simulation, the method accurately tracks the asset price evolution while leveraging the properties of the Brownian Bridge. This mathematical construct enables the model to interpolate intermediate price points between two known values—typically the start and end points of the simulation period.
In doing so, the simulation can capture the probability distribution of the asset path more effectively, particularly when focusing on extreme movements.
As such, this technique provides a reliable way to assess the likelihood of the asset breaching certain levels within a predefined time frame, without the need for continuous monitoring, which would be computationally demanding. The Brownian Bridge framework ensures that even with discrete time steps, the method retains a high level of precision in determining the potential range of price fluctuations during the life of the option.

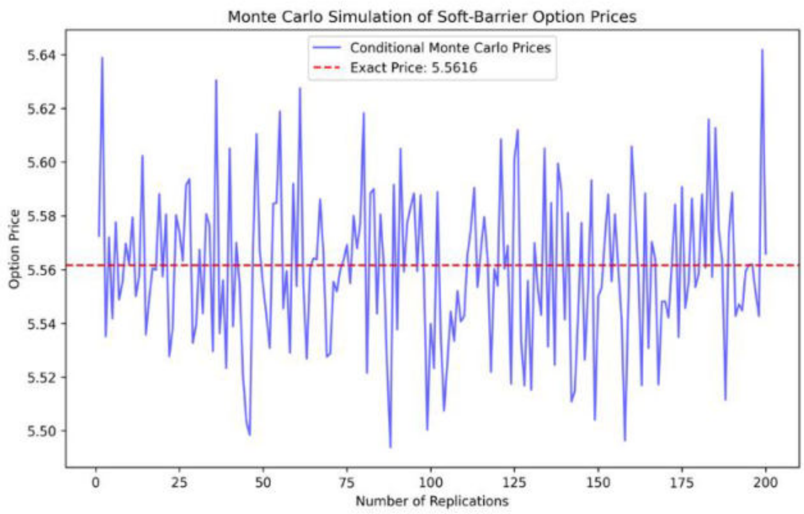
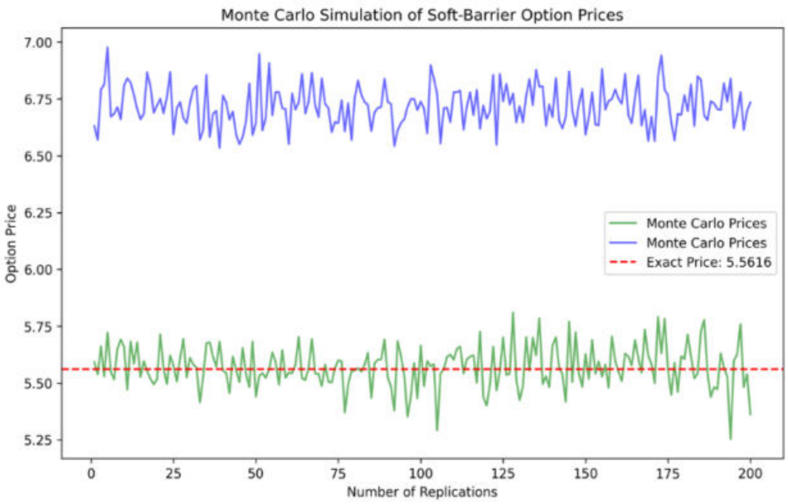


Figure 14: Conditional Monte Carlo simulation for Soft-Barrier Option prices

The Conditional method in Figure 14 shows the same advantages in both computational speed and precision. The Crude Monte Carlo method took increasingly longer with each reduction of the monitoring time window, in the Soft-Barrier case even longer than in the other cases, while the Conditional Monte Carlo method only took one minute, again.
By reducing the number of irrelevant paths, the Conditional method required far fewer simulations to reach a given level of accuracy. This reduction in computational effort translates directly into faster runtimes, making the CMC method the suitable choice here as well.
The plot is shown as follows instead of the histogram illustrated in the previous sections for the other kinds of options for the sake of clarity, being the results of the Crude Monte Carlo further than the results of the other cases.



	Mean	Std. Dev
24h	6.7202	0.0517
1h	6.7197	0.0469
30m	-	-
15m	-	-
Cond.	5.5895	0.0962

Figure 15: Comparison of Crude MC results with Conditional MC results.

3.4) Double Barrier Options

A double Barrier option is characterized by two barriers: one positioned above and one below the current stock price. It is classified as a path-dependent option, like the other barriers, as the holder's payoff is determined by the stock price interactions with these barriers. The contract specifies three distinct payoffs based on whether the stock price breaches the upper barrier, the lower barrier, or neither during the life of the option.
A barrier is a knock-out type if, upon being hit, the resulting payoff is a rebate (which may vary depending on the timing of the breach). Conversely, it is a knock-in type if hitting the barrier triggers a new option for the holder. The barrier feature can apply to the entire lifespan of the option or only to a portion of it.
A wide variety of double barrier options can be constructed to meet different risk management objectives by using different structural design. Similar to single barrier options, investors may use the exotic features of double barrier options to lower premiums, align with their expectations about future stock price movements, or meet specific hedging requirements.
Let τ_{up} and τ_{down} represent the first passage times at which the stock price breaches the upper and lower barriers, respectively, with T denoting the option maturity date. Double barrier options can be categorized based on the payoff structure, which depends on the relationship between τ_{up}, τ_{down} and T:

$\tau_{up} < \min(\tau_{down}, T)$:

This scenario arises when the upper barrier is breached before the lower barrier during the life of the option. For an up-barrier knock-in double barrier option, the holder receives a new option if the upper barrier is breached before the lower one. Otherwise, the option expires worthless.

$\tau_{up} < \tau_{down} < T$:

Here, the upper barrier is breached before the lower barrier within the life of the option. An example is the sequential double barrier option, where the option is knocked out when both the upper and lower barriers are breached in sequence. Essentially, once the upper barrier is hit, the option becomes a down-and-out single barrier option.

$\min(\tau_{up}, \tau_{down}) < T$:

This indicates that one of the barriers is breached before the maturity date. In a one-touch knock-out double barrier option, the option is knocked out if at least one barrier is hit, potentially resulting in a rebate payout.

$\max(\tau_{up}, \tau_{down}) < T$:

In this case, both the upper and lower barriers are breached within the life of the option. A double-touch knock-out option is knocked out only if both barriers are breached before maturity.
It is notable that combining an up-barrier knock-in option with an up-barrier knock-out option results in a standard European option. Similarly, a one-touch knock-in option can be split into an up-barrier knock-in and a down-barrier knock-in option. More complex payoff structures can be created based on the order in which barriers are breached. For instance, a double up-and-in call option expires worthless if neither barrier is breached during its life. If the upper barrier is breached first, the option transforms into a vanilla European call. However, if the lower barrier is hit first, the option becomes an up-and-in call option with a new upper barrier and strike price, effectively converting the option with adjusted parameters.
In the case of an occupation time derivative with double barriers, the payoff depends on the time that the stock price remains within a specified range or corridor. The option defines a corridor $[a, b]$ for the stock price, and if one of the barriers is breached, the option terminates, and the holder receives a payout proportional to the time the stock price spent within the corridor.
A double-barrier option is knocked either in or out if the underlying price touches the lower boundary L or the upper boundary U prior to expiration. The closed formulas shown below are for double knock-out options. The price of a double knock-in call is equal to the portfolio of a long standard call and a short double knock-out call, with identical strikes and time to expiration. In a similar way, a

double knock-in put is equal to a long standard put and a short double knock-out put. Double-barrier options are priced with the Ikeda and Kuintomo closed formula (1992).

Call Up-and-Out-Down-and-Out:
Payoff: $c(S, U, L, T) = \max(S - K; 0)$ if $L < S < U$ before T else 0.

$$c = Se^{(b-r)T} \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{U^n}{L^n}\right)^{\mu_1} \left(\frac{L}{S}\right)^{\mu_2} [N(d_1) - N(d_2)] - \left(\frac{L^{n+1}}{U^n S}\right)^{\mu_3} [N(d_3) - N(d_4)] \right\}$$
$$- Ke^{-rT} \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{U^n}{L^n}\right)^{\mu_1 - 2} \left(\frac{L}{S}\right)^{\mu_2} \cdot \left[N\left(d_1 - \sigma\sqrt{T}\right) - N\left(d_2 - \sigma\sqrt{T}\right)\right] \right.$$
$$\left. - \left(\frac{L^{n+1}}{UnS}\right)^{\mu_3 - 2} \left[N\left(d_3 - \sigma\sqrt{T}\right) - N\left(d_4 - \sigma\sqrt{T}\right)\right] \right\} \ (108),$$

Where:

$$d_1 = \frac{\ln\left(\frac{SU^{2n}}{KL^{2n}}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \ (109)$$
$$d_2 = \frac{\ln\left(\frac{SU^{2n}}{FL^{2n}}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \ (110)$$
$$d_3 = \frac{\ln\left(\frac{L^{2n+2}}{KSU^{2n}}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \ (111)$$
$$d_4 = \frac{\ln\left(\frac{L^{2n+2}}{FSU^{2n}}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \ (112)$$

$$\mu_1 = \frac{2[b - \delta_2 - n(\delta_1 - \delta_2)]}{\sigma^2} + 1 \ (113),$$
$$\mu_2 = 2n\frac{\delta_1 - \delta_2}{\sigma^2} \ (114)$$
$$\mu_3 = \frac{2[b - \delta_2 + n(\delta_1 - \delta_2)]}{\sigma^2} + 1 \ (115),$$

$$F = Ue^{\delta_1 T} \ (116)$$

Where δ_1 and δ_2 determine the curvature L and U. The case of:

1. $\delta_1 = \delta_2 = 0$ corresponds to two flat boundaries.
2. $\delta_1 < 0 < \delta_2$ corresponds to a lower boundary exponentially growing as time elapses, while the upper boundary will be exponentially decreasing.
3. $\delta_1 > 0 > \delta_2$ corresponds to a convex downward lower boundary and a convex upward upper boundary.

Put Up-and-Out-Down-and-Out:

Payoff: $p(S, U, L, T) = \max(K - S; 0)$ if $L < S < U$ before T else 0.

$$p = +Ke^{-rT} \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{U^n}{L^n}\right)^{\mu_1 - 2} \left(\frac{L}{S}\right)^{\mu_2} \left[N\left(y_1 - \sigma\sqrt{T}\right) - N\left(y_2 - \sigma\sqrt{T}\right)\right] - \left(\frac{L^{n+1}}{U^n S}\right)^{\mu_3 - 2} \left[N\left(y_3 - \sigma\sqrt{T}\right) - N\left(y_4 - \sigma\sqrt{T}\right)\right] \right\}$$
$$- Se^{(b-r)T} \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{U^n}{L^n}\right)^{\mu_1} \left(\frac{L}{S}\right)^{\mu_2} [N(y_1) - N(y_2)] - \left(\frac{L^{n+1}}{U^n S}\right)^{\mu_3} [N(y_3) - N(y_4)] \right\} \ (117),$$

Where:

$$y_1 = \frac{\ln\left(\frac{SU^{2n}}{EL^{2n}}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \ (118)$$
$$y_2 = \frac{\ln\left(\frac{SU^{2n}}{KL^{2n}}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \ (119)$$
$$y_3 = \frac{\ln\left(\frac{L^{2n+2}}{ESU^{2n}}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \ (120)$$
$$y_4 = \frac{\ln\left(\frac{L^{2n+2}}{KSU^{2n}}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \ (121)$$
$$E = Le^{\delta_2 T} \ (122)$$

The double-barrier options are expressed as infinite series of weighted normal distribution functions.

3.4.1) Crude Monte Carlo application

To model the presence of the two barriers, the Monte Carlo process involves simulating multiple potential future price paths for the underlying asset and checking, at each time step, whether the asset price has breached either of the two barriers.

For each simulated path, the algorithm tracks if and when the asset hits the upper or the lower barrier. If either barrier is crossed, the option is knocked out, and the payoff is set to zero. Conversely, if neither barrier is breached, the payoff is determined based on the specific option type (e.g., call or put).

The usage of the Crude Monte Carlo method is particularly effective for this application due to its ability to handle the multiple possible paths the asset price might take, especially under stochastic processes.

An Up-and-Out-Down-and-Out Call has been used for the simulation. The settings of the option price used to conduct the Monte Carlo simulations are as follows: $S = 100, K = 100, U = 150, L = 50, T = 0.25, r = 0.1, b = 0.1, \sigma = 0.35$.

Similarly to the simulation of the soft-barrier option, twenty-four hours, one hour and thirty minutes frequencies have been set: approaching from twenty-four hours to thirty minutes has shown very minimal improvements in the results of the simulations.

The number of simulations at each iteration is set to 10.000; the loop went for 200 iterations.

Each scenario was run through the Monte Carlo simulation to estimate the option price. As in the last section, the paths have been shown in the following graphs to show the bias in a clearer way. The exact price of this option is 7.0373.

The Monte Carlo simulation for the double barrier option at a twenty-four hours monitoring frequency is shown in Figure 16.

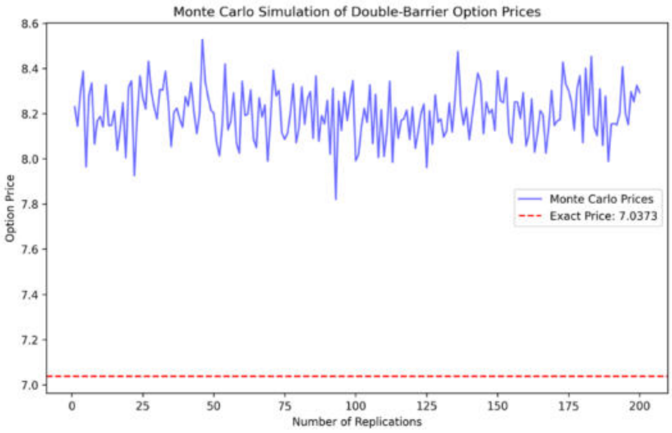


Figure 16: Crude Monte Carlo simulation at 24 hours for Double-Barrier Option prices

And the Monte Carlo simulation for the double barrier option at a one-hour monitoring frequency is shown in Figure 17.

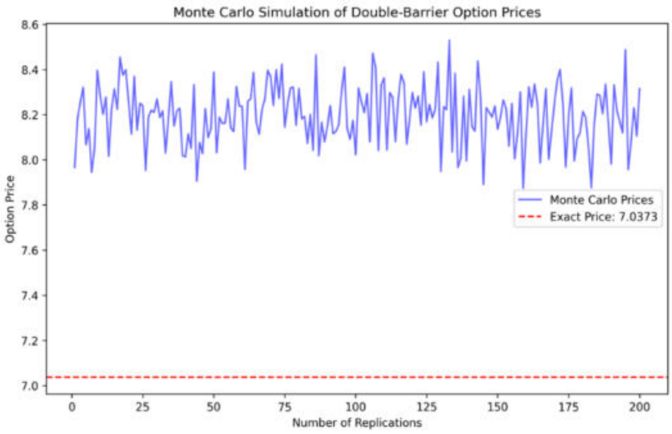


Figure 17: Crude Monte Carlo simulation at one hour monitoring frequency for Double-Barrier Option prices

In Figure 18, the simulation has a monitoring frequency of thirty minutes.

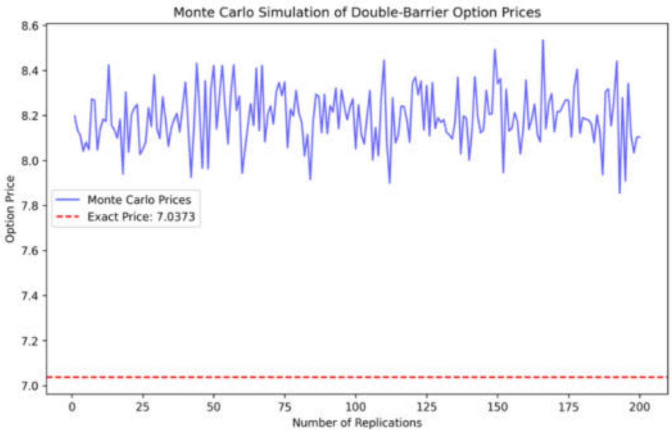


Figure 18: Crude Monte Carlo simulation at 30 minutes monitoring frequency for Double-Barrier Option prices

It can clearly be noted that not only is there a huge bias in all the three simulations, but the convergence to the exact value as the monitoring frequency increases is close to non-existent. The results can be defined as precise as the ones obtained in the soft-barrier option section, thus the different kind of plot rather than the histogram.

3.4.2) Conditional Monte Carlo application

As in the section dedicated to the soft-barrier options, the results obtained through the simulation of Crude Monte Carlo are non-optimal, even at narrower monitoring frequency time frames. The bias is too evident, and numerical integration errors due to the inability to continuously monitor the underlying are too heavy even at high frequencies. The Conditional Monte Carlo method is to be preferred here, too.
A simulation and an analysis are conducted on this kind of options, and the plots are presented below, shown in a similar way as the graphs illustrated in the soft-barrier section (see Figures 19 and 20).
First, a Conditional Monte Carlo simulation on the same option is shown, to illustrate the huge improvement on the calculation of the fair value. Then, comparison graphs are computed to show the difference between the Crude and the Conditional methods, for the sake of clarity and information.

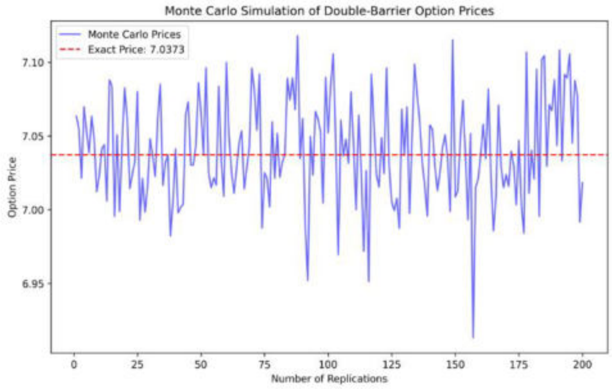
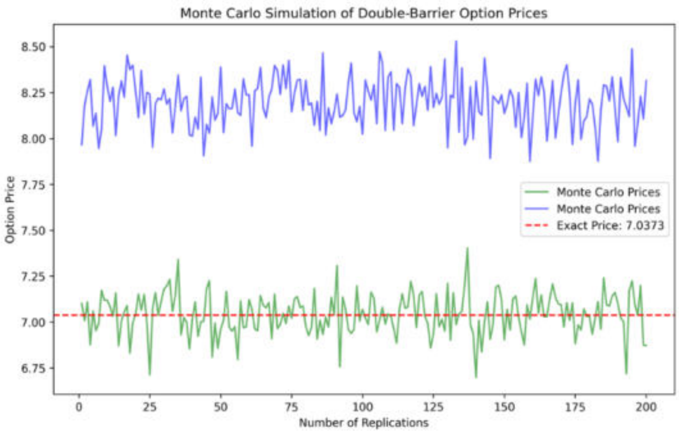


Figure 19: Conditional Monte Carlo simulation for Double-Barrier Option prices

As in the previous sections, the Conditional method in Figure 19 demonstrated the same advantages in both computational speed and precision.
The comparison between the Crude and the Conditional methods is illustrated, at 10.000 iterations each. The comparison is made on the thirty-minute monitoring frequency only, as the other graphs basically show the same divergence.



	Mean	Std. Dev
24h	8.2005	0.1160
1h	8.1981	0.1328
30m	8.1982	0.1328
15m	-	-
Cond.	7.0263	0.0962

Figure 20.a: Comparison between Crude and Conditional Monte Carlo methods for Double-Barrier Option prices

4) Market Case Study

In this section, the pricing of an investment certificate is implemented, highlighting that the Conditional Monte Carlo allows to obtain a fair value of the instrument unbiased and more aligned with market expectations. In order to perform this analysis, the structured product characterized by ISIN NLBNPIT1XYW7 issued by BNP Paribas was considered. All information is available both on the issuer's website and on the website of Borsa Italiana, so only the most important data used for pricing are reported here.

Issue Date: 22/12/2023

Exercise Date: 20/12/2024

Observation Period: from 22/12/2023 to 19/12/2024 (included)

Monitoring Style: American (i.e. Continuous time)

Currency: EUR

Notional Amount: EUR 100

Strike Price: EUR 6.668

Barrier Level: EUR 5.3344

Bonus Level: EUR 7,53484

Bonus Percentage: 113.00%

Cap Level: EUR 7.53484

Cap Percentage: 113.00%

Type of Settlement: Cash

The evaluation of the "Bonus Cap" product having Enel (IT0003128367) as underlying was conducted with the market data of February 16, 2024.

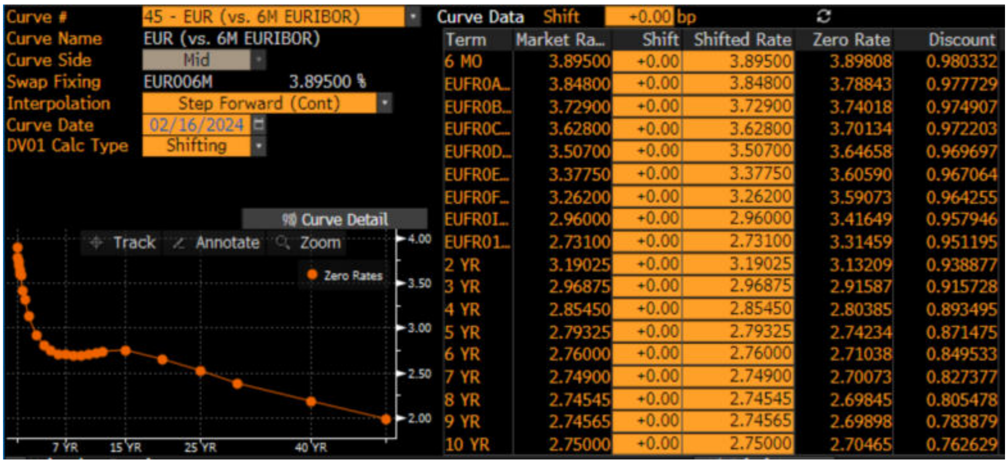
On the Settlement Date, the holder receives the following, for each certificate:

- If the Barrier Event has not occurred, a cash payment equal to the Notional Amount multiplied by the Bonus Percentage Level.

- Otherwise, if the Barrier Event has occurred, a cash payment equal to the lesser of (i) the Notional Amount multiplied by the Underlying Performance and (ii) the Notional Amount multiplied by the Percentage Cap Level.

The Underlying Performance is equal to the Underlying Reference Price divided by the Strike price. In such a case, the holder receives an amount less than the Notional Amount. Finally, the Barrier Event will be deemed to have occurred if the Price of the Underlying is at or below the Barrier Level at least once during the Observation Period.

The market data used for pricing are from info-provider Bloomberg, as of the valuation date. Figure 21 shows the interest rate term structure used for forwarding the underlying projections and discounting the terminal payment. The risk-free rate (r) has been interpolated considering the maturity of the product.

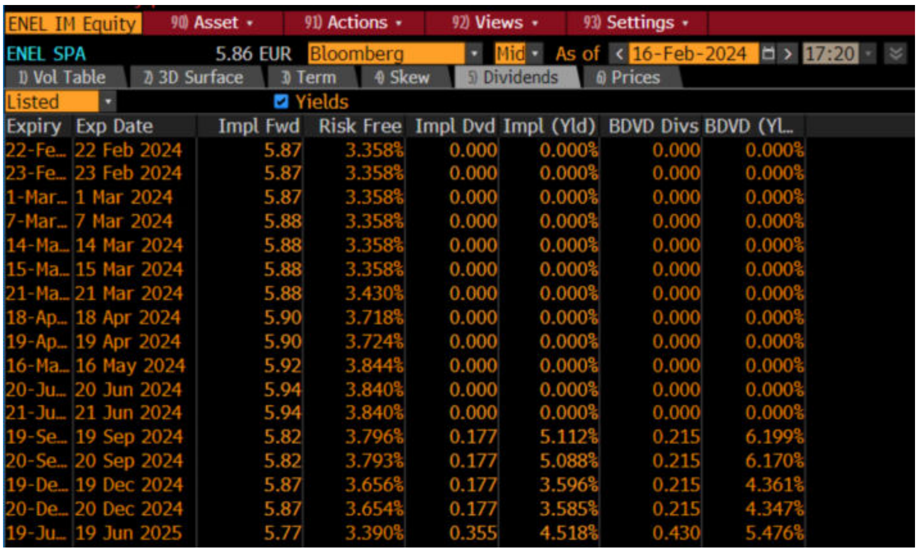


Term	Market Ra...	Shift	Shifted Rate	Zero Rate	Discount
6 MO	3.89500	+0.00	3.89500	3.89808	0.980332
EUFR0A...	3.84800	+0.00	3.84800	3.78843	0.977729
EUFR0B...	3.72900	+0.00	3.72900	3.74018	0.974907
EUFR0C...	3.62800	+0.00	3.62800	3.70134	0.972203
EUFR0D...	3.50700	+0.00	3.50700	3.64658	0.969697
EUFR0E...	3.37750	+0.00	3.37750	3.60590	0.967064
EUFR0F...	3.26200	+0.00	3.26200	3.59073	0.964255
EUFR0I...	2.96000	+0.00	2.96000	3.41649	0.957946
EUFR01...	2.73100	+0.00	2.73100	3.31459	0.951195
2 YR	3.19025	+0.00	3.19025	3.13209	0.938877
3 YR	2.96875	+0.00	2.96875	2.91587	0.915728
4 YR	2.85450	+0.00	2.85450	2.80385	0.893495
5 YR	2.79325	+0.00	2.79325	2.74234	0.871475
6 YR	2.76000	+0.00	2.76000	2.71038	0.849533
7 YR	2.74900	+0.00	2.74900	2.70073	0.827377
8 YR	2.74545	+0.00	2.74545	2.69845	0.805478
9 YR	2.74565	+0.00	2.74565	2.69898	0.783879
10 YR	2.75000	+0.00	2.75000	2.70465	0.762629

Figure 21: Interest rates term structure, tenor: 6 months. Source: Bloomberg®

Figure 22 shows the strip of implied dividend yields summarized from the call-put parity of actively traded options on the ENEL stock.

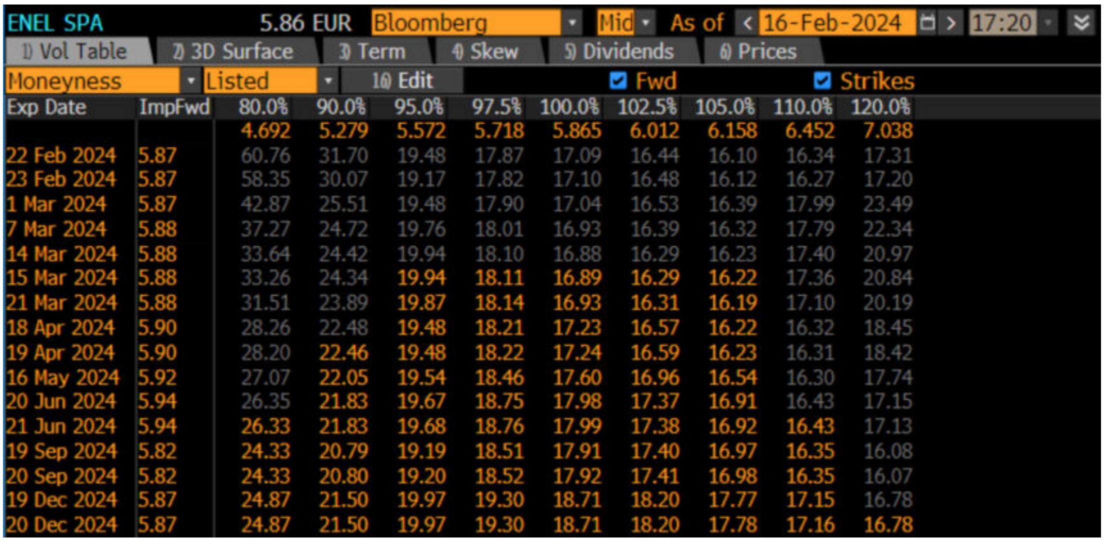
The dividend yield used (q) was interpolated by the time to maturity of the structured product. The reference spot price (S) for the calculations is the closing price on Feb. 16, 2024 (shown at the top left of the Figure).



Expiry	Exp Date	Impl Fwd	Risk Free	Impl Dvd	Impl (Yld)	BDVD Divs	BDVD (YL...
22-Fe...	22 Feb 2024	5.87	3.358%	0.000	0.000%	0.000	0.000%
23-Fe...	23 Feb 2024	5.87	3.358%	0.000	0.000%	0.000	0.000%
1-Mar...	1 Mar 2024	5.87	3.358%	0.000	0.000%	0.000	0.000%
7-Mar...	7 Mar 2024	5.88	3.358%	0.000	0.000%	0.000	0.000%
14-Ma...	14 Mar 2024	5.88	3.358%	0.000	0.000%	0.000	0.000%
15-Ma...	15 Mar 2024	5.88	3.358%	0.000	0.000%	0.000	0.000%
21-Ma...	21 Mar 2024	5.88	3.430%	0.000	0.000%	0.000	0.000%
18-Ap...	18 Apr 2024	5.90	3.718%	0.000	0.000%	0.000	0.000%
19-Ap...	19 Apr 2024	5.90	3.724%	0.000	0.000%	0.000	0.000%
16-Ma...	16 May 2024	5.92	3.844%	0.000	0.000%	0.000	0.000%
20-Ju...	20 Jun 2024	5.94	3.840%	0.000	0.000%	0.000	0.000%
21-Ju...	21 Jun 2024	5.94	3.840%	0.000	0.000%	0.000	0.000%
19-Se...	19 Sep 2024	5.82	3.796%	0.177	5.112%	0.215	6.199%
20-Se...	20 Sep 2024	5.82	3.793%	0.177	5.088%	0.215	6.170%
19-De...	19 Dec 2024	5.87	3.656%	0.177	3.596%	0.215	4.361%
20-De...	20 Dec 2024	5.87	3.654%	0.177	3.585%	0.215	4.347%
19-Ju...	19 Jun 2025	5.77	3.390%	0.355	4.518%	0.430	5.476%

Figure 22: Implied Dividend Yield. Source: Bloomberg®

The implied volatilities surface is shown in Figures 23 and 24. The volatility (σ) has been interpolated considering the strike price and the time to maturity of the investment certificate.



ENEL SPA 5.86 EUR Bloomberg Mid As of 16-Feb-2024 17:20

1) Vol Table 2) 3D Surface 3) Term 4) Skew 5) Dividends 6) Prices

Moneyness Listed 10) Edit Fwd Strikes

Exp Date	ImpFwd	80.0%	90.0%	95.0%	97.5%	100.0%	102.5%	105.0%	110.0%	120.0%
		4.692	5.279	5.572	5.718	5.865	6.012	6.158	6.452	7.038
22 Feb 2024	5.87	60.76	31.70	19.48	17.87	17.09	16.44	16.10	16.34	17.31
23 Feb 2024	5.87	58.35	30.07	19.17	17.82	17.10	16.48	16.12	16.27	17.20
1 Mar 2024	5.87	42.87	25.51	19.48	17.90	17.04	16.53	16.39	17.99	23.49
7 Mar 2024	5.88	37.27	24.72	19.76	18.01	16.93	16.39	16.32	17.79	22.34
14 Mar 2024	5.88	33.64	24.42	19.94	18.10	16.88	16.29	16.23	17.40	20.97
15 Mar 2024	5.88	33.26	24.34	19.94	18.11	16.89	16.29	16.22	17.36	20.84
21 Mar 2024	5.88	31.51	23.89	19.87	18.14	16.93	16.31	16.19	17.10	20.19
18 Apr 2024	5.90	28.26	22.48	19.48	18.21	17.23	16.57	16.22	16.32	18.45
19 Apr 2024	5.90	28.20	22.46	19.48	18.22	17.24	16.59	16.23	16.31	18.42
16 May 2024	5.92	27.07	22.05	19.54	18.46	17.60	16.96	16.54	16.30	17.74
20 Jun 2024	5.94	26.35	21.83	19.67	18.75	17.98	17.37	16.91	16.43	17.15
21 Jun 2024	5.94	26.33	21.83	19.68	18.76	17.99	17.38	16.92	16.43	17.13
19 Sep 2024	5.82	24.33	20.79	19.19	18.51	17.91	17.40	16.97	16.35	16.08
20 Sep 2024	5.82	24.33	20.80	19.20	18.52	17.92	17.41	16.98	16.35	16.07
19 Dec 2024	5.87	24.87	21.50	19.97	19.30	18.71	18.20	17.77	17.15	16.78
20 Dec 2024	5.87	24.87	21.50	19.97	19.30	18.71	18.20	17.78	17.16	16.78

Figure 23: Implied Volatility Table. Source: Bloomberg®

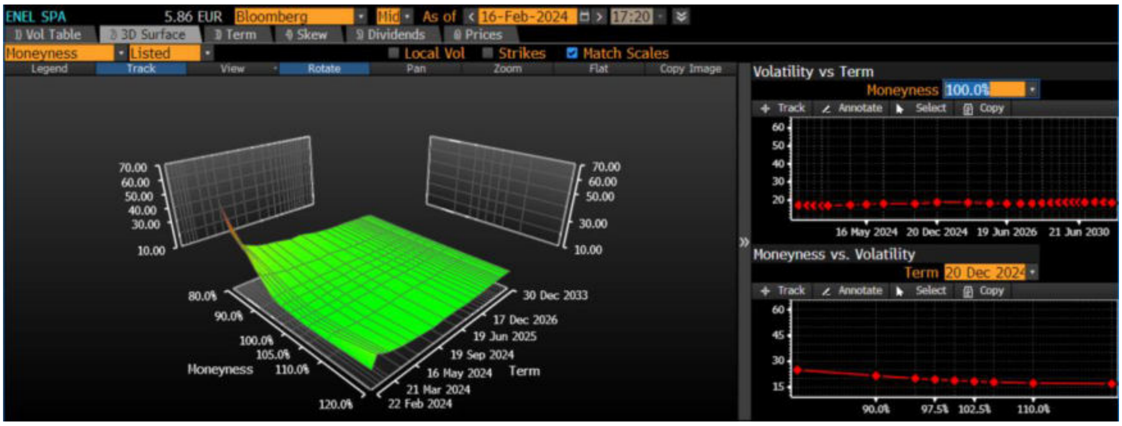


Figure 24: Implied Volatility Surface. Source: Bloomberg®

The spread to be applied to the discount factors so that the creditworthiness of the issuer is properly considered was derived from the one-year Senior CDS curve (Figure 25).

Figure 25: BNP Paribas SA EUR Senior CDS Curve. Source: Bloomberg®

By evaluating the certificate using formulas that allow for the application of the Conditional Monte Carlo, a fair value aligned with market expectations at the analysis date is obtained (Figure 26). Specifically, the expected value achieved with 100 replications of 20,000 paths each is 93.70 ± 0.12. If a Quant were to estimate the price using the Crude Monte Carlo, the impact on pricing would lead to a distortion of up to 1.5.

Figure 26: Market quote for the certificate with ISIN: NLBNPIT1XYW7. Source: Borsa Italiana

5) Conclusions

This study highlights that the implementation of the Conditional Monte Carlo, if the underlying of the option follows a Geometric Brownian Motion and the financial instrument involves continuous monitoring of a threshold, entails two advantages: 1) the certainty of not introducing a numerical error resulting from an incorrect discretization of the motion; 2) the greater celerity in the processing of the pay-off by the calculation algorithm.

It is important to highlight that this methodology remains valid if we work under the assumption of valuating derivatives under the Black-Scholes-Merton pricing framework.

It is deemed interesting for the continuation of this study to verify the proper functioning of the methodology when applied to other second-generation options that involve continuous monitoring of a level and to quantify the evaluative bias accordingly. It would also be interesting to continue and analyze other investment certificates traded in the secondary market and characterized by a continuous monitoring. A proper valuation is also crucial, of course, for estimating sensitivity measures (Greeks), which constitute an essential tool for performing dynamic portfolio hedging.

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