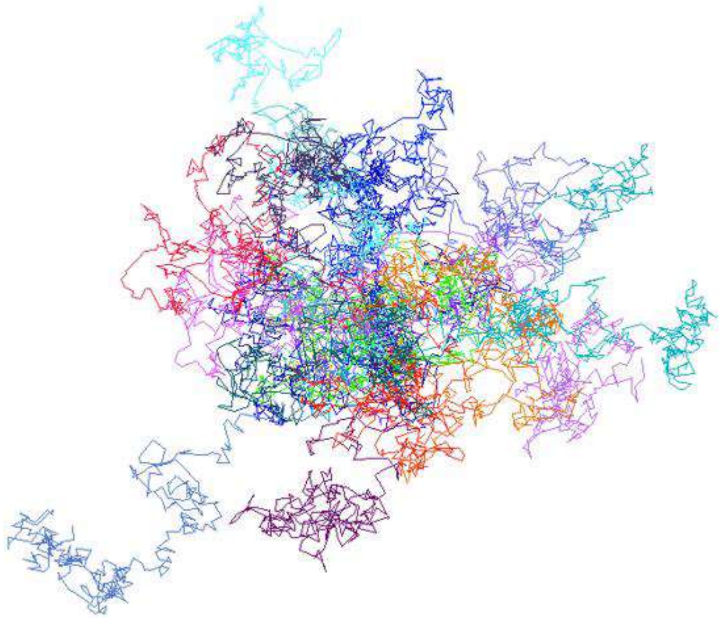
Notes on Quantitative Financial Analysis



Pier Giuseppe Giribone

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Second Edition

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Notes on Quantitative Financial Analysis – Second Edition

AIFIRM Edizioni – Educational Book Series

ISBN Print: 979-12-80245-26-7

ISBN Online: 979-12-80245-25-0

DOI: 10.47473/2016eda0003

Printed on: 10th July 2024

Listening and Observing, Meditating introspectively on the facts.

Heraldic motto of the Perno di Caldera family

Vittorio Amedeo II di Savoia (1688)

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INTRODUCTION

This volume aggregates a set of notions which introduce the fundamentals of quantitative financial analysis in a clear and concise way, providing a very practical approach, as demonstrated by the discussion of numerous case studies. All material can be freely used, quoting the source. Slides can be downloaded from the author's website and the codes, if not protected by copyright, are available upon request. A specific background is not strictly required for the reader, although basic notions of economics and statistics would be recommended. The book is divided into eight sections and each of them has a chapter structure. Below is a brief summary of the covered topics:

Part I: Fixed Income Instruments

The first chapter is a summary of the main concepts of financial mathematics underlying quantitative analysis, up to the modeling of the interest rates term structure.
The second chapter shows the different types of bonds present in the financial markets, together with the assessment of the risks that an analyst must manage.
The third part explores the heart of quantitative analysis, introducing the best practices for estimating the fair value of a bond, together with its risk measures (duration, modified duration and convexity).

Part II: Futures and Forwards

After the description of the basic concepts for understanding this category of derivatives, the second chapter introduces the specific quantitative analysis of these instruments, with a particular focus on pricing and hedging.

Part III: Options

Given the inherent variety of topics connected to options, this section has been thoroughly covered. In addition to the description of the standard pay-off, the first chapter deals with the foundations of this derivative and introduces the mathematical properties, including the put-call parity.
The second chapter concerns the pricing of plain-vanilla options. The well-known Black-Scholes-Merton pricing framework has been introduced, showing how it can be applied to options written on different underlyings (equity, index, rates, futures and currencies). In addition to the fair value, the sensitivities (Greeks) are also estimated.
The third chapter deals with option strategies: combinations of plain-vanilla options with underlying and with other options, in order to create specific hedging and trading strategies. Among the strategies, covered call, protective put, bull/bear spread, butterfly spread, straddle, strip, strap and strangle are covered.
The fourth chapter reviews the main non-standard (i.e. exotic) options, characterized by special pay-offs. The lognormal pricing framework is extended to these types of options; among them: forward start, cliquet, digital, chooser, compound and path-dependent (barrier, Asian and lookback) options.

Not all options can be adequately priced using a closed formula. For those characterized by particularly non-linear pay-offs or by early exercise features, a numerical methodology has to be implemented.
Chapter 5 is therefore dedicated to binomial stochastic trees, particularly useful for dealing with derivatives characterized by the possibility of being exercised in advance, while chapter 6 is dedicated to the Monte Carlo technique, which is considered suitable for representing any type of pay-off, thanks to its flexibility. The working principle, the internal consistency, the pricing estimation, and the computation of the most important risk measures are illustrated for both algorithms. Once the reader has become confident on the correct approach for the quantitative analysis of the derivative, it is time to focus on the inputs of the model.
Finally, Chapter 7 centers on determining the inputs for the previously exposed techniques. A particular focus has been given to the estimation of volatility (both historical and implied) and to the correlation.

Part IV: Swaps

Similarly to the previous scheme, the section dedicated to swaps is divided into two parts: the first chapter describes the fundamentals of the different types of swaps, while the second deals with the quantitative analysis of the instrument. Particular attention is paid to Interest Rate Swaps (IRS) and Currency Swaps. Two distinct valuation approaches are provided, i.e. considering the derivative as a portfolio of forward contracts, or as two positions (one long and one short) in two bonds. The second chapter concludes with the derivation of long-term spot rates from Interest Rate Swaps, a process known as swap curve stripping.

Part V: Credit Derivatives

This section consists of only one chapter in which Credit Default Swaps (CDS) are presented. It describes how premiums can be used to compute risk-adjusted discount factors in a fixed income instrument pricing context. The chapter ends with an introduction of the most popular models among analysts for the pricing of these derivatives.

Part VI: Inflation

This section covers the main inflation-linked derivatives: Zero-Coupon Inflation-Indexed Swap (ZCIIS) and Year-on-Year Inflation-Indexed Swap (YYIIS). The standard market approach is presented to simulate the prospective values of the CPI preparatory to the pricing of these instruments with particular focus on the modeling of seasonality. The chapter concludes with the case study of "BTP Italia", an exotic security linked to Italian inflation characterized by a non-standard pay-off.

Part VII: Aggregate Risk Measures

The risk measures discussed so far have addressed the single instrument and can hardly be extended to a portfolio, characterized by instruments of a different financial nature. Considering this need, the most common approaches to estimating Value-at-Risk have been introduced: parametric, full-evaluation, Monte Carlo backward and forward looking. The Expected Shortfall and the importance of conducting stress tests and back tests are briefly presented as well.

Part VIII: Credit Risk

The first chapter analyzes the determinants of the Demand and Supply of credit and provides a summary for the core elements that constitute a mortgage/loan: interest rate, repayment plans, mode of extinction, amount and Loan-to-Value, guarantees, duration and the Global effective annual rate.
The second chapter focuses on the definition and on the mathematical models for estimating counterparty risk, which can be interpreted by its nature as a hybrid between financial risk and credit risk.
In particular, it has been shown that the probability of default can be inferred from Credit Default Swap (CDS) premiums, listed bond spreads or stock prices using the KMV (Kealhofer, Merton and Vasicek) model.
The last part of the chapter highlights the structural limits of counterparty risk, validating the need to provide a more complete definition of credit risk. Credit risk is based on three pillars: the probability of default (PD), the Loss Given Default (LGD) and the Exposure at Default (EAD). An effectual discussion is dedicated to each of these three important components.
The third chapter presents the statistical approaches that allow the estimation of PD starting from historical data (not necessarily market data), among which, the Altman's Z-Score, the Logit-Probit and the CreditGrades models are covered.
The fourth chapter introduces the regression models suitable for estimating and forecasting the Loss Given Default.
The fifth chapter deals with the estimation and the predictive models for EAD. In this context, a Monte Carlo model is introduced for the determination of the Credit Valuation Adjustment (CVA) with particular attention to the modeling of the Expected Exposure to the various future time buckets. Once the reader has acquired the required knowledge for a correct credit measurement, we move on to the concept of rating systems.
The sixth chapter introduces Rating Agencies and provides the basic notions for creating transition matrices. The Cohort approach and the Hazard approach are adequately discussed with the relative methods of calculating confidence intervals.
The seventh chapter deals with credit risk managed not on a single position, but at portfolio level. In this phase asset correlation has to be presented and, to this end, the Moment matching and the Maximum Likelihood approaches are explained. An example of estimating a Monte Carlo VaR and a C-VaR is also provided in the credit context.
The part dealing with credit concludes with the main methods for validating credit models. Among those, the Cumulative Accuracy Profile (CAP), the Receiver Operating Characteristics (ROC), the binomial test and the Brier Score are covered.

At the end of each chapter, further food for thought is provided through a bibliography of reference papers or books, which allow useful insights into each topic covered.

ACKNOWLEDGEMENTS

I would like to express my heartfelt thanks to the AIFIRM Editorial Board for the opportunity to work on this project.

My gratitude also goes to my colleagues and friends in BPER Banca (Financial Administration) and in UNIGE (Department of Economics) for providing many interesting insights.

My heartfelt thanks and affection goes to my parents: I could not aspire to have better guides on the complex journey of life.

Last but certainly not least, I dedicate this book to the memory of my beloved grandparents: Piero, Celeste, Mario and Giuseppina.

PART I: FIXED INCOME INSTRUMENTS

Yield to Maturity and Expected Return
ESG factors and “Greenium” estimation
Annuity formulas
Perpetual bond pricing
Price and reinvestment risk
Weighted Average Maturity (WAM)
Weighted Average Cash Flow (WACF)
Macaulay Duration
Modified Duration
Fischer and Weil Duration
Convexity
First and second order price approximation
Immunization
Fixed income portfolio sensitivity
Asset and Liability Management

I.1 TIME VALUE OF MONEY

The **time value of money principle** states that any amount of money today is worth more than the same amount tomorrow. This statement is based on a number of reasons, which can be summarized with the consideration that the present is certain and known, while the future is not. For example, in the presence of the inflation phenomenon it means that the same product that costs €100 today may be worth more than €100 after a year.
In addition, leaving out inflation, there is an intrinsic "**opportunity cost**" in deferring a sale: the desired object may not only be more expensive afterwards, but it may even no longer be available.
Philosophically, current consumption is often considered to be preferable to deferred gratification.
It is therefore necessary to add an appropriate incentive to equalize this gap: this incentive is represented by the **interest rate**.
Economically, it can be said that the presence of an interest is justified by the need to compensate for the loss of utility linked to the existence of **risk**. Interest rates are usually expressed on an annual basis, and they express the remuneration to be paid by the borrower to the lender. There are different types of interest rates: the simple and the compound interest rate.
The **simple interest** rate assumes that no interest is received on the interest earned. It can be computed using the formula:

Simple Interest = initial value * interest rate * Time (*Eq. I.1*)

The initial value is constituted by the principal on which interest is paid over a defined period of time.
Following the most widespread convention, the interest rate is expressed per annum, thus the time must be in year fractions. For example, the simple interest calculated on €10,000 invested at 6% p.a. after 9 years is equal to: 10,000 * 0.06 * 9 = € 5,400.
However, the formula is too rough, in the real world the payments received as interest are usually re-invested in order to earn more in the following periods.
The **compound interest** assumes that the interest is being re-invested. It can therefore be seen as the simple interest to which the interest calculated on the previous interest amount is added. This concept can be expressed in mathematical terms using:

Compound interest = Simple interest + interest earned on interest (*Eq. I.2*)

The easiest formula for calculating compound interest on an initial amount over a given time period is given by:

Compound interest = initial value * [(1+interest rate)^T - 1] (*Eq. I.3*)

Following the previous example, the compound interest calculated on €10,000 invested at 6% p.a. after 9 years is equal to: € 10,000 * [(1 + 0.06)^9 - 1] = € 6,894.78959.
The same result can be obtained applying directly the definition of compound interest, and again, the accrued interest at the end of the ninth year is € 6,894.78959.

Year	Amount at the beginning of the period [A]	Simple Interest [6%] [B]=0.06*[A]	Amount at the end of the period [C]=[A]+[B]
1	10000.00000	600.00000	10600.00000
2	10600.00000	636.00000	11236.00000
3	11236.00000	674.16000	11910.16000
4	11910.16000	714.60960	12624.76960
5	12624.76960	757.48618	13382.25578
6	13382.25578	802.93535	14185.19112
7	14185.19112	851.11147	15036.30259
8	15036.30259	902.17816	15938.48075
9	15938.48075	956.30884	16894.78959

Table I.1 The compound interest rate

The significant discrepancy is also evident between the method of calculating simple interest compared to the compound one which in our example amounts to € 1,494.78959.

Time [A]	Simple Interest [B]	Compound Interest [C]	Discrepancy [D]
0.25	150.00	146.74	-3.26
0.5	300.00	295.63	-4.37
0.75	450.00	446.71	-3.29
1	**600.00**	**600.00**	**0.00**
1.25	750.00	755.54	5.54
1.5	900.00	913.37	13.37
1.75	1050.00	1073.51	23.51
2	1200.00	1236.00	36.00
2.25	1350.00	1400.88	50.88
2.5	1500.00	1568.17	68.17
[B]=10,000*0.06*[A]; [C]=10,000*((1+0.06)^[A]-1); [D]=[C]–[B]			

Table I.2 Simple versus Compound interest rate

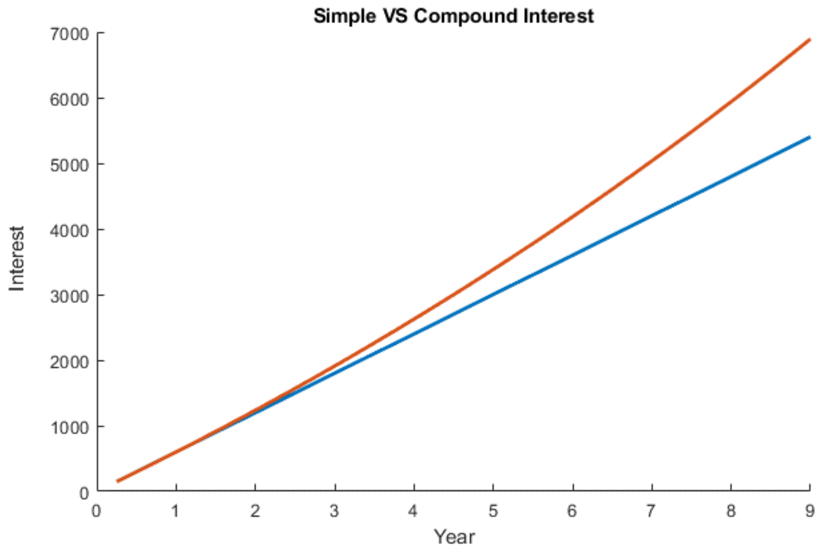


Figure I.1 Simple versus Compound Interest

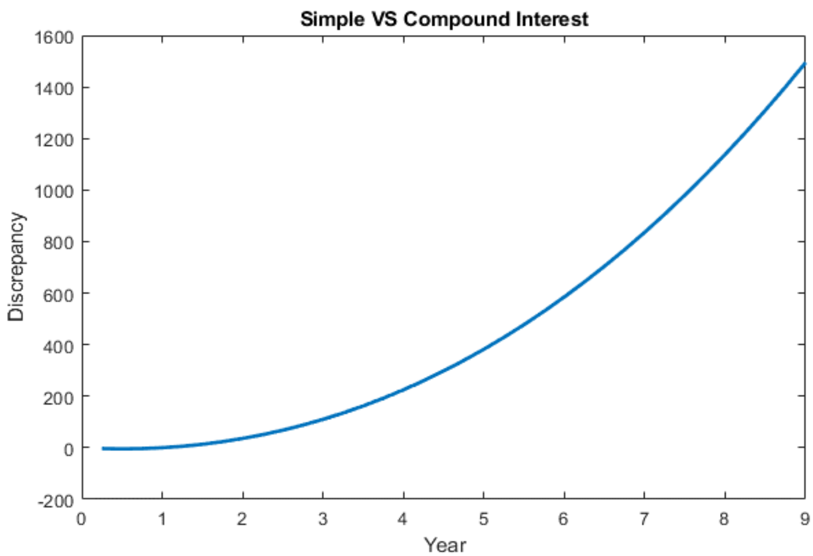


Figure I.2 Simple versus Compound Interest Discrepancy

The future value of a certain amount of money is therefore given by:

Future value = Present value + Interest (*Eq. I.4*)

In the case of a simple regime, the future value is equal to:

Future value = Present value + Present value * interest rate * T
= Present value (1+interest rate * T) (*Eq. I.5*)

In the case of a compound regime, the future value is equal to:

Future value = Present value + Present value * [(1+interest rate)^T - 1]
= Present value * (1+[(1+interest rate)^T - 1])
= Present value * (1+interest rate)^T (*Eq. I.6*)

The process of determining the future value of a payment or a series of cash flows using the concept of compound interest is called **compounding** and such procedure is certainly the most widespread in financial practice. On the other hand, the process of determining the present value of a future payment or a series of cash flows is called **discounting**. The present value is obtained by inverting the previous formulas, therefore, in the case of a simple regime, the present value is equal to:

Present value = Future value / [(1+interest rate*T)] (*Eq. I.7*)

and in the case of a compound regime, it is equal to:

Present value = Future value / [(1+interest rate)^T] (*Eq. I.8*)

The interest rate used to discount future cash flows is called **discount rate,** while the quantity **1/[(1+interest rate*T)]** in simple regime and **1/[(1+interest rate)^T]** in compound regime is defined as the **discount factor**. Clearly, the present value of a future cash flow is inversely proportional to both the rate and the time period. It is good to know both interest regimes, but it should be highlighted that it is standard market practice to use the compound regime.

Let us consider € 100 in a deposit with an interest rate of 6% per annum.

At the end of the first year, we have:

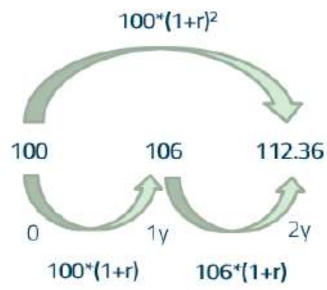
€ 100 *(1+0.06) = € 106

At the end of the second year, we have:

€ 106 *(1+0.06) = € 112.36

which an be equivalently written as:

€ 100*(1+0.06)^2= € 112.36



As the interest rate increases and the time period increases (T>1), a compound interest rate regime allows for greater accumulation of money, since compound interests, compared to simple ones, follow a power law. In the previous case, by parameterizing the time, the following relations hold:
$100 * 1.06 * T$ for the simple interest and $100 * (1.06)^T$ for the compound interest.
The compounding process can take place more frequently than annually in which case the interest accrues m times during the year, for n years.
A **convertible rate** i_c is generally derived from the annual rate, according to the relationship: $i_c = i / m$.
For the calculation of the future value, the same formulas presented in the case of a simple/compound regime can be applied, adjusting the time period at which the convertible rate is defined. Therefore:

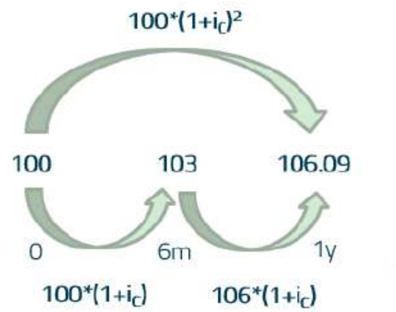
Future Value = Present Value*(1+ i_c * m * n), in simple regime (*Eq. I.9*)
Future Value = Present Value*(1+ i_c)$^{(m * n)}$, in compound regime (*Eq. I.10*)

For example, let us consider € 100 in a deposit with an interest rate of 6% per year that is paid not annually, as in the previous case, but every six months.
Therefore: i = 0.06, m=2, n=1 → i_c =i/m=0.06/2=0.03
Future Value = € 100*(1+0.03*2*1) = € 106, in simple regime
Future Value = € 100*$(1.03)^{2*1}$ = € 106.09, in compound regime.

Compound regime:

Period	Present Value	i_C	m	Future Value
Year 1	100	0.03	2	106.09

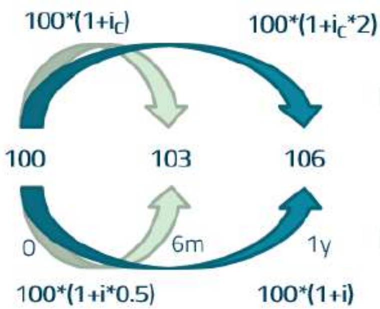
Period	Present Value	i_C	m	Future Value
Semester 1	100	0.03	1	103.0
Semester 2	103	0.03	1	106.9



Simple regime:

Period	Present Value	i_C	m	Future Value
Semester 1	100	0.03	1	103.0
Semester 2	100	0.03	2	106.0

Period	Present Value	i_C	m	Future Value
Semester 1	100	0.06	0.5	103.0
Semester 2	100	0.06	1	106.0



Let us consider an annual rate **i** and suppose to divide the year into **m** periods, and at the end of each period, a fraction of the interest for the entire year equal to i_C = **i / m** is paid, and is then immediately reinvested.

$$M(n)=C(1+i/m)^{(n*m)} \quad (Eq.\ I.11)$$

where M is the future value, C is the initial amount of money (Present value) and n is the number of years. Going to the limit for m which tends to infinity ($m \to \infty$), we have the case in which a continuous flow of payments is **reinvested continuously**.

$$M(n) = \lim_{m\to\infty} C\left(1+\frac{i}{m}\right)^{n\cdot m} \quad (Eq.\ I.12)$$

Setting $\kappa = \frac{m}{i}$ and remembering the Nepero limit: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e = \exp(1) = 2.71828$ we reach:

$$M(n) = \lim_{\kappa\to\infty} C\left(1+\frac{1}{\kappa}\right)^{\kappa\cdot i\cdot n} = \lim_{\kappa\to\infty} C \cdot \left[\left(1+\frac{1}{\kappa}\right)^{\kappa}\right]^{i\cdot n} = Ce^{i\cdot n} \quad (Eq.\ I.13)$$

Future value = Present value * exp(i*n) (*Eq. I.14*)

Present value = Future value * exp(-i*n) (*Eq. I.15*)

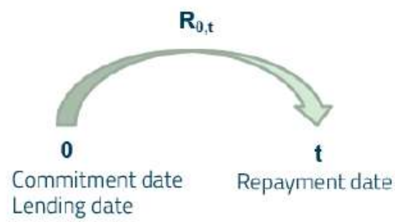
i is generally expressed on an annual basis and **n** in year fraction, while **exp(-i*n)** is the discount factor.

Two rates are defined as **equivalent** if they provide the same future amount when applied to the same capital for the same period of time.
Starting from the convertible capitalization formulas and considering a yearly time horizon (n = 1), the following relationships hold:

In simple regime (*Eq. I.16*)	**In compound regime** (*Eq. I.17*)
$(1+ i_A)=(1+i_m * m)$	$(1+ i_A)=(1+i_m)^m$
$i_A= i_m * m$	$i_A = (1+i_m)^m - 1;$
$i_m= i_A /m$	$i_m = (1+ i_A)^{(1/m)} - 1$

Where: $\mathbf{i_A}$ is the annual rate; $\mathbf{i_m}$ is the periodic rate; and **m** indicates the number of times in which interest is capitalized during the year.

In order to correctly determine an interest rate, it is necessary to specify three different dates:
The **Commitment date** is the date on which the borrower and the lender agree on the interest rate.
The **Lending date** is the effective date on which the amount of money is lent.
The **Repayment date** is the effective date on which the amount of money is returned.



The **spot rate** (or **zero rate**), denoted by $\mathbf{R_{0,t}}$ is defined as the annual interest rate received on a zero-coupon security which expires at time **t**. The spot rate is therefore the interest rate required to lend the money from time **0** to **t** if there is a **single final payment**, including the principal and the interest accrued. As known, the yield is typically expressed per annum.

The two main characteristics of spot rates are the facts that the commitment date coincides with the lending date and that the repayment date is unique.

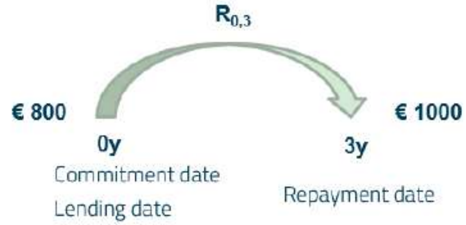
Let us take a bond, as an example, which requires an initial investment of € 800 and returning a capital of € 1,000 after 3 years.

The 3y spot rate, $\mathbf{R_{0,3}}$ is equal to:

$800=1000/(1+R_{0,3})^3$

$R_{0,3} = (1000/800)^{1/3} - 1$

$R_{0,3} = 7.7217\%$

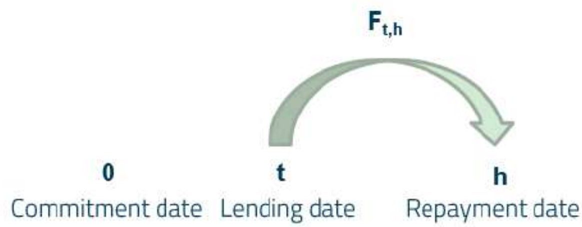


With this available data, no direct inferences can be made on the 1- or 2-year spot rates, thus the only solution is to perform an interpolation starting from the 3y zero rate.

The **forward rate**, denoted by $F_{t,h}$, is the interest rate of a bond whose commitment date does not occur at time 0, but is instead deferred in t.

If a 2-year contract is entered into today starting in one year, the annualized interest rate from year 1 to year 3 (i.e. from year t to h) is a forward rate, $F_{t=1,h=3}$.

Forward rates are characterized by the fact that the commitment date is today, but the lending date is postponed, and here again, the repayment date is unique.



Forward rates are interest rates on loans or bonds that provide for a single payment to the investor after h-t years from the lending date. Let us be more specific through an example: a bond provides for an initial investment of € 850 to be paid in one year and after 3 years, from the date of stipulation, it returns a capital equal to € 1,000.

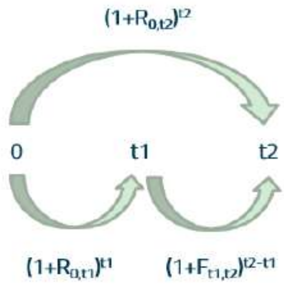
The forward rate between t=1y and h=3y, $\mathbf{F_{1,3}}$ is:

$850=1000/(1+F_{1,3})^2$

$F_{1,3} = (1000/850)^{1/2} - 1$

$F_{1,3} = 8.4652\%$

The relationship between spot rates and forward rates between two time periods t_1 and t_2 is the following:



$$\left(1+R_{0,t1}\right)^{t1}\left(1+F_{t1,t2}\right)^{t2-t1}=\left(1+R_{0,t2}\right)^{t2} \quad (Eq.\ I.18)$$

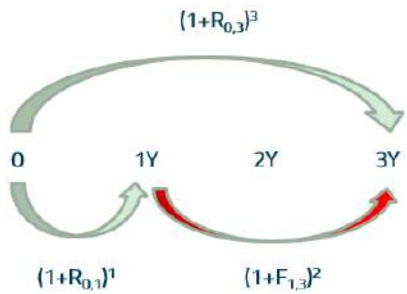
$$F_{t1,t2}=\sqrt[t2-t1]{\frac{\left(1+R_{0,t2}\right)^{t2}}{\left(1+R_{0,t1}\right)^{t1}}}-1=\left[\frac{\left(1+R_{0,t2}\right)^{t2}}{\left(1+R_{0,t1}\right)^{t1}}\right]^{\frac{1}{t_2-t_1}}-1 \quad (Eq.\ I.19)$$

By construction, a spot rate can be seen as the geometric mean of the implicit consecutive forwards:

$$\left(1+R_{0,t}\right)=\left[\left(1+R_{0,1}\right)\left(1+F_{1,2}\right)\left(1+F_{2,3}\right)...\left(1+F_{t-1,t}\right)\right]^{\frac{1}{t}} \quad (Eq.\ I.20)$$

Let us make a few more examples.

We have $R_{0,1Y}=1.5\%$ p.a. , $R_{0,3Y}=2.5\%$ p.a., and we wish to calculate $F_{1Y,3Y}$:

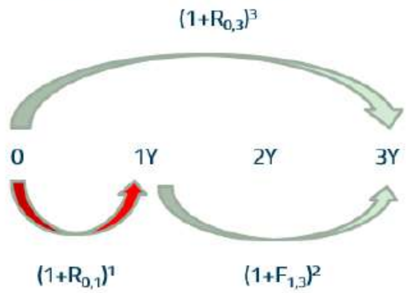


$$\left(1+R_{0,1}\right)^{1}\left(1+F_{1,3}\right)^{3-1}=\left(1+R_{0,3}\right)^{3}$$

$$\left(1+F_{1,3}\right)^{2}=\frac{\left(1+R_{0,3}\right)^{3}}{\left(1+R_{0,1}\right)^{1}} \rightarrow F_{1,3}=\left(\frac{\left(1+R_{0,3}\right)^{3}}{\left(1+R_{0,1}\right)^{1}}\right)^{\frac{1}{2}}-1$$

$$F_{1,3}=\left(\frac{(1+0.025)^{3}}{(1+0.015)^{1}}\right)^{\frac{1}{2}}-1=\sqrt{\frac{1.025^{3}}{1.015}}-1=3.004\%$$

We have $F_{1Y,3Y}=3.004\%$ p.a. , $R_{0,3Y}=2.5\%$ p.a., and we wish to calculate $R_{0,1Y}$:



$$\left(1+R_{0,1}\right)^{1}\left(1+F_{1,3}\right)^{3-1}=\left(1+R_{0,3}\right)^{3}$$

$$1+R_{0,1}=\frac{\left(1+R_{0,3}\right)^{3}}{\left(1+F_{1,3}\right)^{2}} \rightarrow R_{0,1}=\frac{\left(1+R_{0,3}\right)^{3}}{\left(1+F_{1,3}\right)^{2}}-1$$

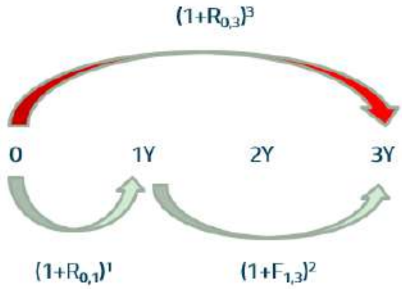
$$R_{0,1}=\frac{(1+0.025)^{3}}{(1+0.03004)^{2}}-1=\frac{(1.025)^{3}}{(1.03004)^{2}}-1=1.5\%$$

We have $R_{0,1Y} = 1.5\%$ p.a. , $F_{1Y,3Y} = 3.004\%$ p.a., and we wish to calculate $R_{0,3Y}$:

$$\left(1 + R_{0,1}\right)^{1}\left(1 + F_{1,3}\right)^{3-1} = \left(1 + R_{0,3}\right)^{3}$$

$$\left(1 + R_{0,3}\right) = \left[\left(1 + R_{0,1}\right)^{1}\left(1 + F_{1,3}\right)^{2}\right]^{\frac{1}{3}}$$

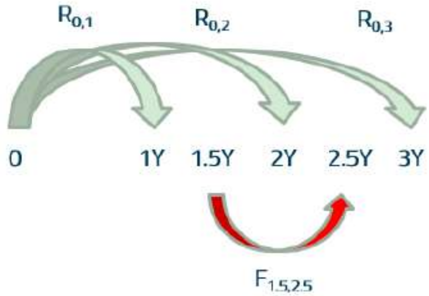
$$R_{0,3} = [(1.015)^{1}(1.03004)^{2}]^{\frac{1}{3}} - 1 = \sqrt[3]{1.076897} - 1 = 2.5\%$$

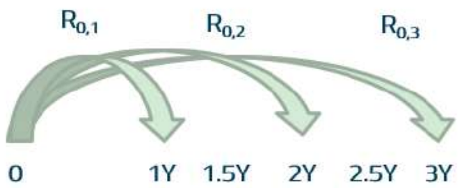


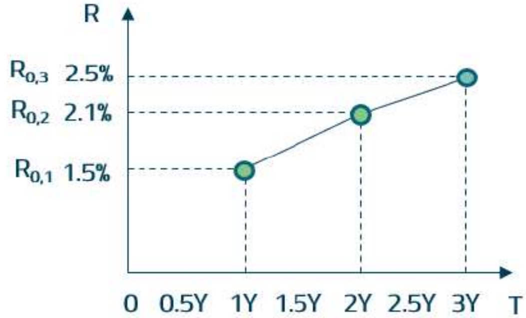
On the other hand, knowing $R_{0,1Y} = 1.5\%$, $R_{0,2Y} = 2.1\%$ and $R_{0,3Y} = 2.5\%$, is it possible to exactly determine $F_{1.5Y,2.5Y}$?

The answer in this case is negative, as there is no mathematical formula that allows to calculate it, but we can use an approximation to estimate $F_{1.5Y,2.5Y}$.

The key idea is thus to perform an interpolation on the known zero rates.







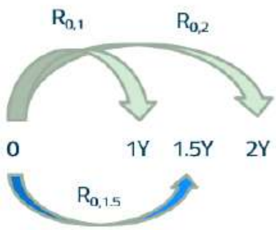
In order to apply the known formula that links spot rates and the implied forward rates, $R_{0,1.5Y}$ and $R_{0,2.5Y}$ should be estimated through interpolation.

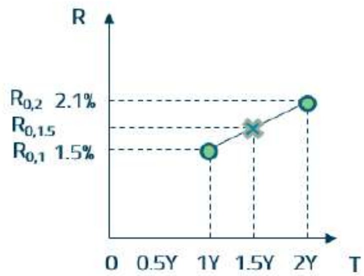
Linear Interpolation – $R_{0,1.5Y}$

$$R_{0,1.5} = R_{0,1} + \frac{R_{0,2} - R_{0,1}}{t_2 - t_1}\left(t_{1.5} - t_1\right)$$

$$R_{0,1.5} = 0.015 + \frac{0.021 - 0.015}{2 - 1}(1.5 - 1)$$

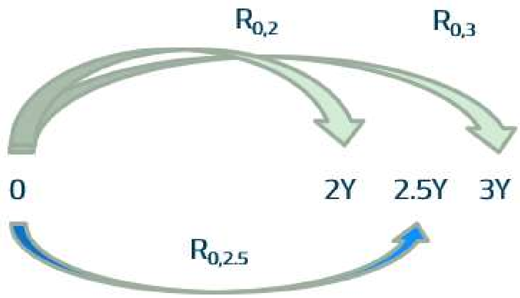
$$R_{0,1.5} = 0.018 = 1.8\%$$

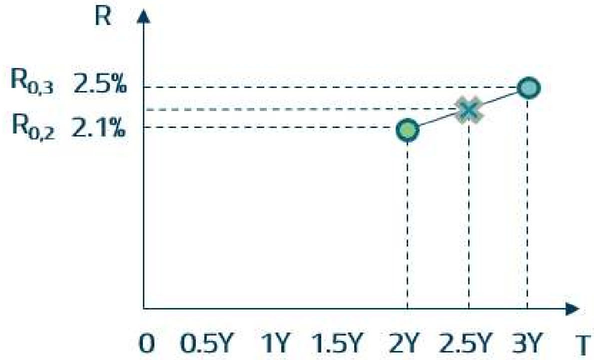




Linear Interpolation – $R_{0,2.5Y}$

$$R_{0,2.5} = R_{0,2} + \frac{R_{0,3} - R_{0,2}}{t_3 - t_2}(t_{2.5} - t_2)$$
$$R_{0,2.5} = 0.021 + \frac{0.025 - 0.021}{3 - 2}(2.5 - 2)$$
$$R_{0,2.5} = 0.023 = 2.3\%$$

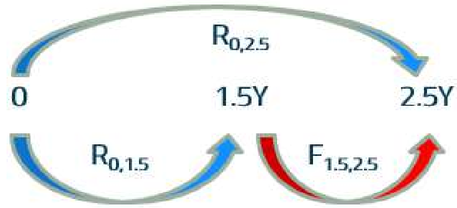




Forward Computation – $F_{1.5Y,2.5Y}$

$$(1 + R_{0,1.5})^{1.5}(1 + F_{1.5,2.5})^{2.5-1.5} = (1 + R_{0,2.5})^{2.5}$$

$$F_{1.5,2.5} = \left[\frac{(1+R_{0,2.5})^{2.5}}{(1+R_{0,1.5})^{1.5}}\right]^{\frac{1}{2.5-1.5}} - 1 = \frac{1.023^{2.5}}{1.018^{1.5}} - 1 = 3.0546\%$$



The aim of the computation of a forward rate is to find the future interest rate $F_{1,2}$ for the time period (t_1, t_2), (where t_1 and t_2 are expressed in years), given the spot rate $R_{0,1}$ for the time period $(0, t_1)$ and the other spot rate $R_{0,2}$ for time period $(0, t_2)$. For doing this we have used the property that the proceeds from investing at rate $R_{0,1}$ for time period $(0, t_1)$ and then reinvesting those proceeds at rate $F_{1,2}$ for time period (t_1, t_2) is equal to the proceeds from investing at rate $R_{0,2}$ for time period $(0, t_2)$. It is worth to note that the mathematical relationship used in the previous examples supposes that we are dealing with yearly compounded rates. Under this assumption, the formulas are:

$$(1 + R_{0,1})^{t_1}(1 + F_{1,2})^{t_2-t_1} = (1 + R_{0,2})^{t_2} \rightarrow F_{1,2} = \left(\frac{(1+R_{0,2})^{t_2}}{(1+R_{0,1})^{t_1}}\right)^{\frac{1}{t_2-t_1}} - 1 \quad (\textit{Eq. I.21})$$

The discount factor formula for a generic period $(0, t)$, Δ_t, expressed in years, and rate $R_{0,t}$ for this period is:

$DF(0,t) = \frac{1}{(1+R_{0,t})^{\Delta_t}}$.

The forward rate can also be expressed in terms of discount factors: $F_{1,2} = \left(\frac{DF(0,t_1)}{DF(0,t_2)}\right)^{\frac{1}{t_2-t_1}} - 1$.

For completeness' sake, we also report the other formulas for simple rates and continuously compounded rates. Under the simple rate calculation mode, the relation for the forward rate is:

$$(1 + R_{0,1}t_1)\left(1 + F_{1,2}(t_2 - t_1)\right) = 1 + R_{0,2}t_2 \rightarrow F_{1,2} = \frac{1}{t_2-t_1}\left(\frac{1+R_{0,2}t_2}{1+R_{0,1}t_1} - 1\right) \text{ (Eq. I.22)}$$

The discount factor formula for a generic period $(0, t)$, Δ_t, expressed in years, and rate $R_{0,t}$ for this period is: $DF(0,t) = \frac{1}{(1+R_{0,t}\Delta_t)}$.

Here, again, the forward rate can be expressed in terms of discount factors: $F_{1,2} = \frac{1}{t_2-t_1}\left(\frac{DF(0,t_1)}{DF(0,t_2)} - 1\right)$.

On the other hand, under the continuously compounded rate calculation mode, the relation for the forward rate is:

$$e^{r_{0,2}t_2} = e^{r_{0,1}t_1} \cdot e^{f_{1,2}(t_2-t_1)} \rightarrow e^{r_{0,2}t_2} = e^{r_{0,1}t_1+f_{1,2}(t_2-t_1)} \rightarrow \ln(e^{r_{0,2}t_2}) = \ln\left(e^{r_{0,1}t_1+f_{1,2}(t_2-t_1)}\right) \rightarrow$$

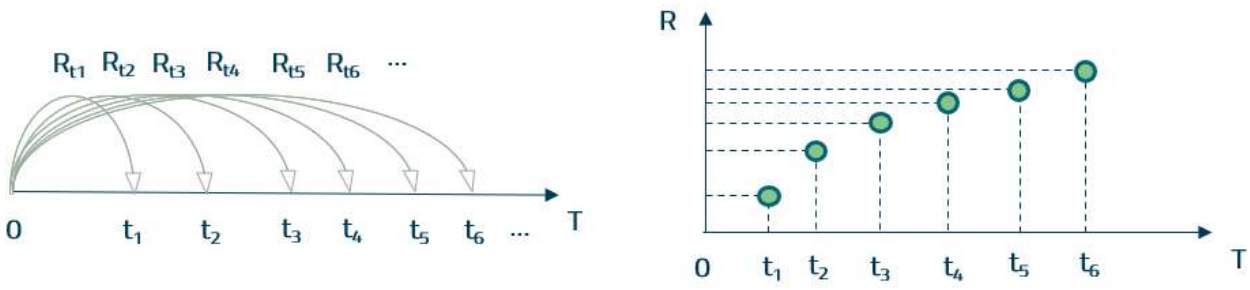
$$r_{0,2}t_2 = r_{0,1}t_1 + f_{1,2}(t_2 - t_1) \rightarrow f_{1,2}(t_2 - t_1) = r_{0,2}t_2 - r_{0,1}t_1 \rightarrow f_{1,2} = \frac{r_{0,2}t_2 - r_{0,1}t_1}{t_2-t_1} \text{ (Eq. I.23)}$$

The discount factor formula for a generic period $(0, t)$, Δ_t, expressed in years, and rate $r_{0,t}$ for this period is: $DF(0,t) = e^{-r_t\Delta_t}$.

In this case, using discount factors, the forward rate can be expressed as follows:

$$f_{1,2} = \frac{\ln(DF(0,t_1)) - \ln(DF(0,t_2))}{t_2-t_1} = \frac{-\ln\left(\frac{DF(0,t_2)}{DF(0,t_1)}\right)}{t_2-t_1}.$$

We can define the **Interest rates term structure** as a **curve model** consisting of **spot rates** rearranged by increasing time to maturity, and this is graphically represented in the below Cartesian plane [t, R_t].



The dots in the plane represent the known zero rates.

Given that markets in the real world are not complete, a relationship between them has to be specified in order to be able to compute forward rates and discount factors for every possible future date, $\mathbf{t_i}$.
Depending on the purposes, we can choose an interpolation methodology (linear, cubic spline, piecewise polynomial), a non-linear parametric model (such as Nelson-Siegel or Svensson model) or aim for a Machine Learning model (such as a shallow Artificial Neural Network). Let us examine the three different alternatives.

Interpolation methodology

In this case, we start from the spot rates, and we implement an interpolation with the aim to calculate the zero rates for the desired period.
Based on the available points, we can choose a more precise methodology than the linear one.
It is anyway worth to note that it is common practice to use the linear interpolation between spot rates for pricing a financial instrument (see the previous example).

Non-linear parametric models

The analysis of term structures is essential for a wide variety of implementations, including understanding the dynamics of the markets, making forecasts on interest rates, managing portfolios, and building hedging strategies.
Therefore, many researchers propose analytical functions for the description of the relationship of the zero rates starting from the historical shapes assumed by these curves.
The most wide-spread representations are the Nelson-Siegel (1987), y_{NS} and the Svensson model (1994), y_{SV}.

$$y_{NS}(t,\vec{\beta},\tau) = \beta_0 + \beta_1\left\{\frac{\tau_1}{t}\left[1-\exp\left(-\frac{t}{\tau_1}\right)\right]\right\} + \beta_2\left\{\frac{\tau_1}{t}\left[1-\exp\left(-\frac{t}{\tau_1}\right)\right] - \exp\left(-\frac{t}{\tau_1}\right)\right\} \quad \text{(Eq. I.24)}$$

$$y_{SV}(t,\vec{\beta},\vec{\tau}) = \beta_0 + \beta_1\left\{\frac{\tau_1}{t}\left[1-\exp\left(-\frac{t}{\tau_1}\right)\right]\right\} + \beta_2\left\{\frac{\tau_1}{t}\left[1-\exp\left(-\frac{t}{\tau_1}\right)\right] - \exp\left(-\frac{t}{\tau_1}\right)\right\} +$$
$$+\beta_3\left\{\frac{\tau_2}{t}\left[1-\exp\left(-\frac{t}{\tau_2}\right)\right] - \exp\left(-\frac{t}{\tau_2}\right)\right\} \quad \text{(Eq. I.25)}$$

Where: y is the interest rates term structure model, t is the time to maturity (independent variable of the fitting problem), $\vec{\beta}$ is the array of the linear parameters and $\vec{\tau}$ is the array of the non-linear parameters.

Machine Learning models

The third alternative is constituted by Machine Learning techniques, which are very flexible and suitable for even the most difficult regression problems, thanks to their “bottom-up” approach.
Unluckily the “black-box” effect is one of the major cons of this approach, therefore it is considered a good practice to adopt ML only when the canonical standard statistical models do not reach the target performance or when they fail.

Term	6 MO	7 MO	8 MO	9 MO	10 MO	11 MO	12 MO	15 MO	18 MO
Zero Rate	0.7751	0.8506	0.9326	1.0049	1.049	1.1013	1.1446	1.2528	1.3362
Term	2 Y	3 Y	4 Y	5 Y	6 Y	7 Y	8 Y	9 Y	10 Y
Zero Rate	1.3954	1.4657	1.5148	1.5712	1.6231	1.669	1.7182	1.7657	1.812
Term	11 Y	12 Y	15 Y	20 Y	25 Y	Interest rates term structure			
Zero Rate	1.8581	1.8993	1.9627	1.8684	1.7343				

Table I.3 Interest rates term structure data

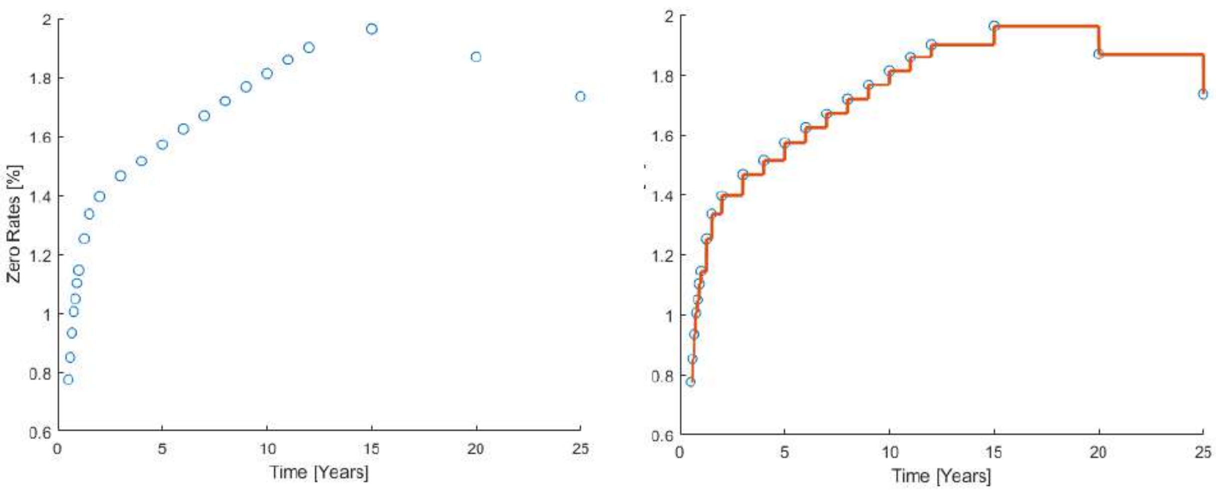


Figure I.3 Zero rates (on the left) and Interest rates term structure using a piece-wise model (on the right)

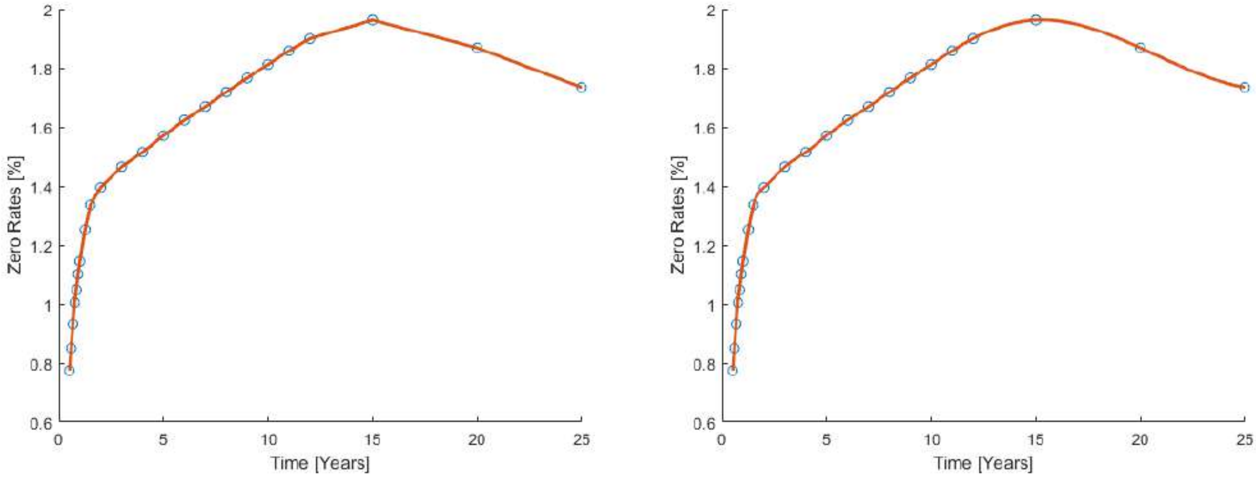


Figure I.4 Interest rates term structure using a linear model (on the left) and a cubic model (on the right)

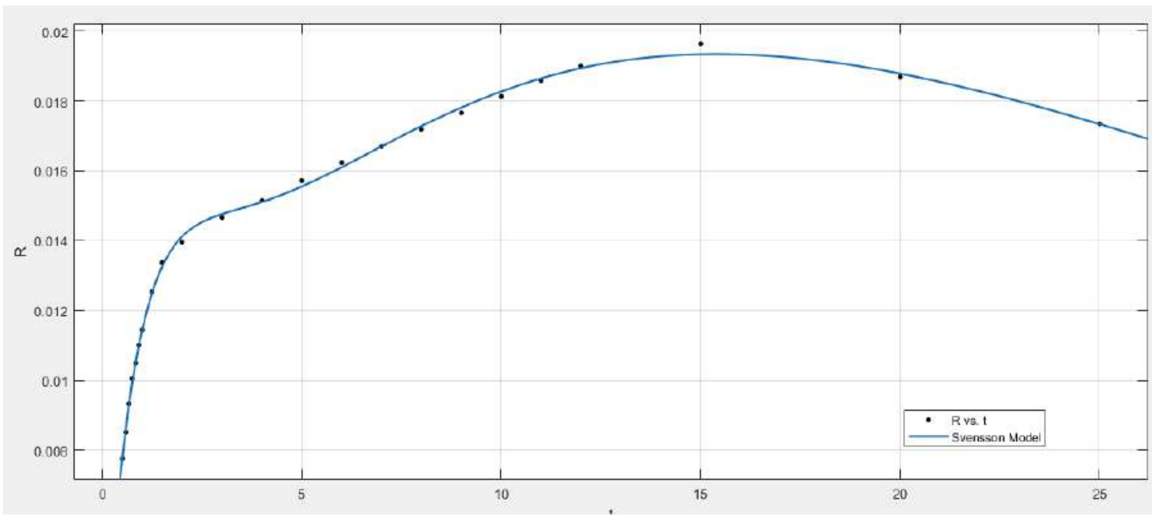


Figure I.5 Interest rates term structure using the non linear parametric model of Svensson.
Parameters: $\beta_0 = -0.0087,\ \beta_1 = 0.0094, \beta_2 = 0.0389, \beta_3 = 0.0848,\ \tau_1 = 0.968, \tau_2 = 10.07$

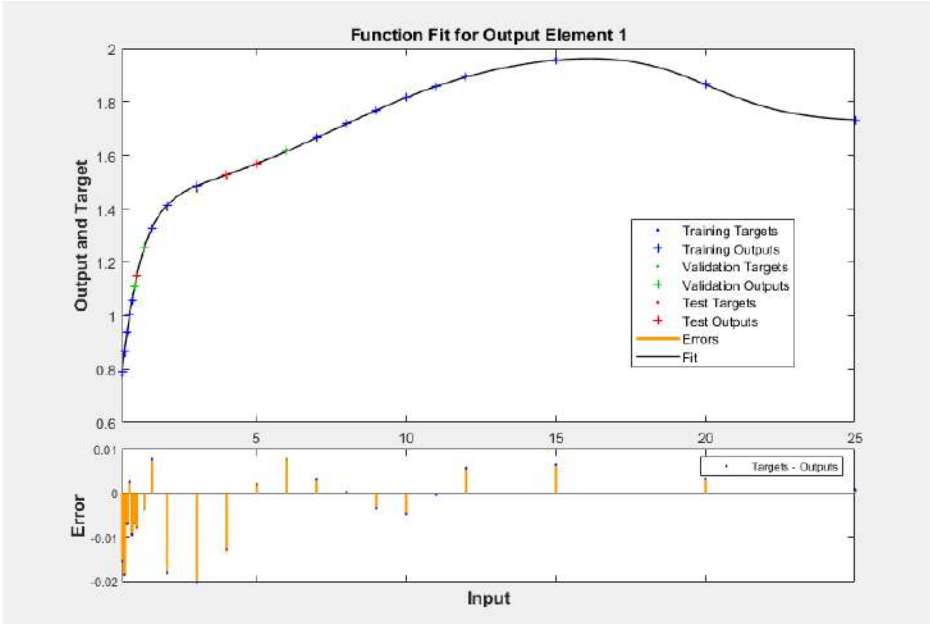


Figure I.6 A One-layered Static ANN with three neurons gives an overall fitting error below 0.02%

The Interest rates term structure is essential for the derivation of zero-rates and, consequently the discount factors, but, as known, markets are not complete, and it is rare to have zero coupons directly quoted for every maturity, especially in the mid-long term. Thus, the process used to convert market rates (par rates) into zero rates is the **bootstrap** or **curve stripping**.

Market rates can be quoted starting from different financial instruments and depending on the maturities we have:

- **Short-term**: deposit, cash rates (up to 18 months) or bonds.
- **Mid-term**: Forward Rate Agreements – FRA contracts and Futures (up to 2-3 years) or bonds.
- **Long-term**: Interest Rates Swaps – IRS (up to 50 years) or bonds.

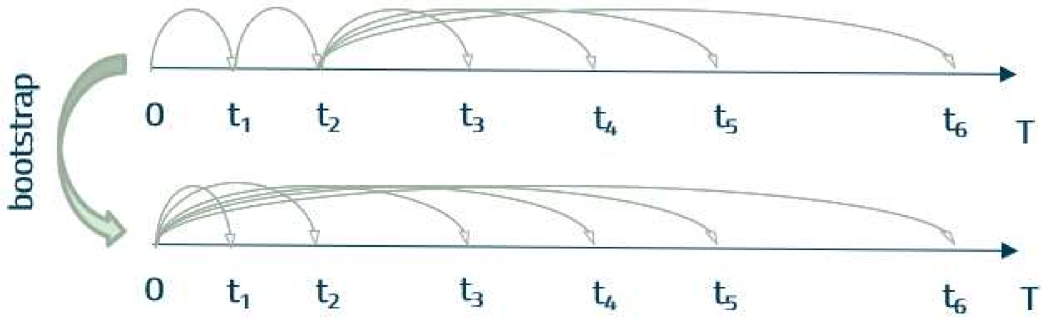
The instruments that are used to imply the zero rates and, consequently, the interest rates term structure, must be chosen among the most representative, i.e. the most **liquid** and they must be characterized by the **same level of creditworthiness**.

The chosen financial instruments, the granulometry of the yield curve and the model that represents the interest rates term structure is subjected to audit and must be declared to Regulators.

The curve stripping process can be summarized as shown:

Figure I.7 Curve stripping procedure

Curve Stripping – Deposit and Cash rates – Money Market (spot lag: 2 days)

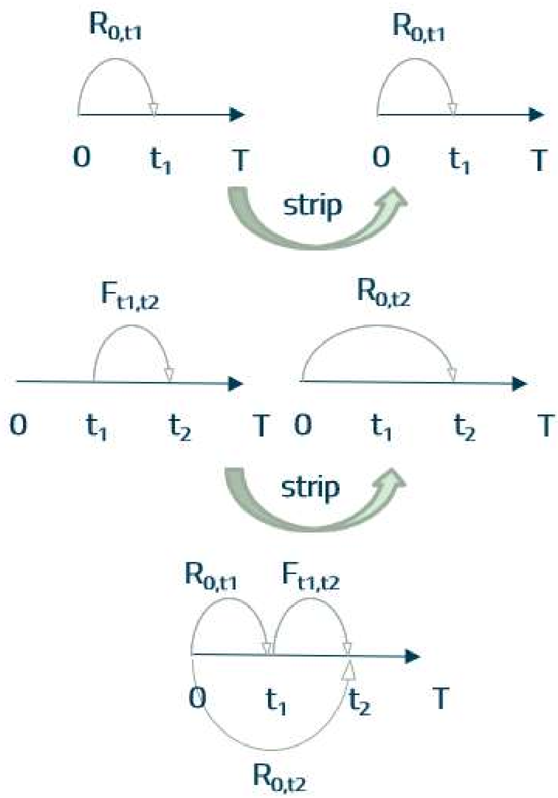


Case	Start Date	End Date	Par Rate
A	0	1 day	2.31
B	1 day	2 days	2.345
C	2 days	1 week	2.375
D	2 days	2 weeks	2.381
E	2 days	3 weeks	2.391
F	2 days	1 month	2.4351

Table I.4 Market Data for the Curve Stripping. Financial Instruments: Deposits and Cash rates

Let us examine various cases.

Case A – t₁



The Par rate 2.31% p.a. is a Zero Rate ($R_{0,t1}$).

Consequently, we can directly use it for deriving the Discount Factor, $DF_{0,t1}$

$$DF_{0,t1} = \frac{1}{(1+R_{0,t1})^{\frac{1}{365}}} = \frac{1}{(1+0.0231)^{\frac{1}{365}}} = 0.999937$$

Case B – t₂

$$\left(1+R_{0,t1}\right)^{t1}\left(1+F_{t1,t2}\right)^{t2-t1} = \left(1+R_{0,t2}\right)^{t2}$$

$$R_{0,t2} = \left[\left(1+R_{0,t1}\right)^{t1}\left(1+F_{t1,t2}\right)^{t2-t1}\right]^{\frac{1}{t_2}} - 1$$

$$R_{0,t2} = \left[(1+0.0231)^{\frac{1}{365}}(1+0.02345)^{\frac{1}{365}}\right]^{\frac{365}{2}} - 1$$

$$R_{0,t2} = [(1.0231)^{0.002739} \cdot (1.02345)^{0.002739}]^{182.5} - 1$$

$$R_{0,t2} = 2.3269\%,$$

$$DF_{0,t2} = \frac{1}{(1+R_{0,t2})^{t_2}} = \frac{1}{(1+0.023269)^{\frac{2}{365}}} = 0.999874$$

Case C – t₃

$$\left(1+R_{0,t2}\right)^{t2}\left(1+F_{t2,t3}\right)^{t3-t2} = \left(1+R_{0,t3}\right)^{t3}$$

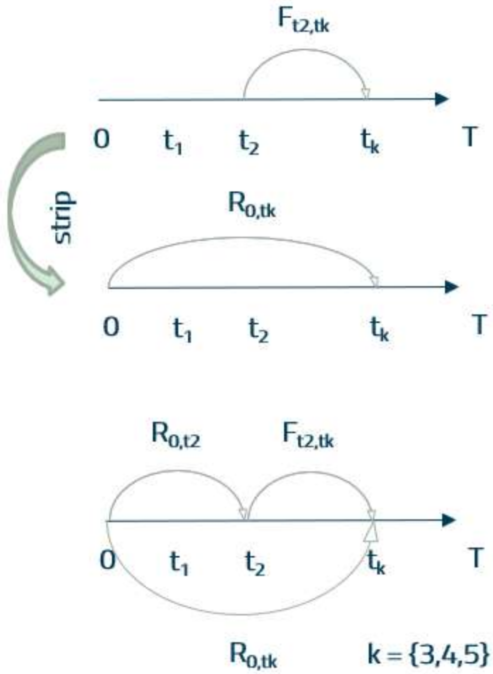
$$R_{0,t3} = \left[\left(1+R_{0,t2}\right)^{t2}\left(1+F_{t2,t3}\right)^{t3-t2}\right]^{\frac{1}{t3}} - 1$$

$$R_{0,t3} = \left[(1+0.023269)^{\frac{2}{365}}(1+0.02375)^{\frac{7-2}{365}}\right]^{\frac{365}{7}} - 1$$

$$R_{0,t3} = [(1.023269)^{0.005479} \cdot (1.02375)^{0.013699}]^{52.14286} - 1$$

$R_{0,t3} = 2.36119\%,$

$DF_{0,t3} = \frac{1}{(1+R_{0,t3})^{t3}} = \frac{1}{(1+0.0236119)^{\frac{7}{365}}} = 0.999553$



Case D – t$_4$

$R_{0,t4} = \left[\left(1 + R_{0,t2}\right)^{t2}\left(1 + F_{t2,t4}\right)^{t4-t2}\right]^{\frac{1}{t4}} - 1$

$R_{0,t4} = \left[(1 + 0.023269)^{\frac{2}{365}}(1 + 0.02381)^{\frac{14-2}{365}}\right]^{\frac{365}{14}} - 1 =$
2.37324%

$DF_{0,t4} = \frac{1}{(1+R_{0,t4})^{t4}} = \frac{1}{(1+0.0237324)^{\frac{14}{365}}} = 0.999101$

Case E – t$_5$

$R_{0,t5} = \left[\left(1 + R_{0,t2}\right)^{t2}\left(1 + F_{t2,t5}\right)^{t5-t2}\right]^{\frac{1}{t5}} - 1$

$R_{0,t5} = \left[(1 + 0.023269)^{\frac{2}{365}}(1 + 0.02391)^{\frac{21-2}{365}}\right]^{\frac{365}{21}} - 1 =$
2.38487%

$DF_{0,t5} = \frac{1}{(1+R_{0,t5})^{t5}} = \frac{1}{(1+0.0238487)^{\frac{21}{365}}} = 0.998645$

Case F – t$_6$

$R_{0,t6} = \left[\left(1 + R_{0,t2}\right)^{t2}\left(1 + F_{t2,t6}\right)^{t6-t2}\right]^{\frac{1}{t6}} - 1$

$R_{0,t5} = \left[(1 + 0.023269)^{\frac{2}{365}}(1 + 0.024351)^{\frac{30-2}{365}}\right]^{\frac{365}{30}} - 1 = 2.42787\%$

$DF_{0,t5} = \frac{1}{(1+R_{0,t5})^{t5}} = \frac{1}{(1+0.0242787)^{\frac{30}{365}}} = 0.998030$

Case	Start Date	End Date	Time to Maturity [Years]	Par Rates [%]	Zero Rates [%]	Discount Factors
A	0	1 day	0.002740	2.31	2.31	0.999937
B	1 day	2 days	0.005479	2.345	2.3269	0.999874
C	2 days	1 week	0.019178	2.375	2.3612	0.999553
D	2 days	2 weeks	0.038356	2.381	2.3732	0.999101
E	2 days	3 weeks	0.057534	2.391	2.3849	0.998645
F	2 days	1 month	0.082192	2.4351	2.4279	0.998030

Table I.5 The stripped curve – Par rates, Zero rates and Discount Factors

In the example we have used an ACT/365-day basis convention for estimating the year fraction t. Typically the convention to be adopted depends on the financial characteristics of the instrument.

Curve Stripping – Zero Coupon bonds

In this case, the yield of the zero coupon bond can be directly used for building the interest rates term structure.

Case	Tenor	Market Price	Yield
ZC_A	3 Months	99.972	0.1121%
ZC_B	6 Months	99.749	0.5039%
ZC_C	1 Year	99.052	0.9571%

Table I.6 Zero Coupon bonds stripping

$$Price = \frac{Face\ Amount}{(1+Yield)^T} \rightarrow Yield = \left(\frac{Face\ Amount}{Price}\right)^{\frac{1}{T}} - 1 \quad (Eq.\ I.26)$$

$Yield_A = \left(\frac{100}{99.972}\right)^{\frac{1}{0.25}} - 1 = 0.1121\%$; $Yield_B = \left(\frac{100}{99.749}\right)^{\frac{1}{0.5}} - 1 = 0.5039\%$ and $Yield_C = 0.9571\%$

It is interesting to note that the Market Price of a zero coupon bond, expressed on a 100% basis (Face Amount = 100%), is the discount factor.

Present value = Future value / [(1+interest rate)^T] = Future value * Discount Factor (*Eq. I.27*)

The Present value is the quoted market price, and the future value is equal to 1. For this reason, a popular notation for representing the discount factor between two dates is $P(t_0, t_1)$, where typically $t_0 = 0$.

Generally speaking, economists view the spot rate curve as the **pure time value of money**.

Given that a nominal interest rate can be broken down into three components, it follows that there are more than one interest rates term structures for a single fixed reference currency.

Nominal Interest Rate = Real I.R. + Inflation premium + risk premium (*Eq. I.28*)

The **real interest rate** is the compensation given to the investor for the future procrastination of the consumption of the loaned money (time value of money).

The **inflation premium,** as the name suggests, is intended as the additional quantity required by the investor to protect his future purchasing power.

The third component, i. e., the **risk premium** protects the investor against all the potential risks that may be incurred over the period of time considered (default risk, market risk, interest rate risk ....)

The **"risk-free" curve** is thus the term structure that best expresses the pure time value of money and it is theoretically unique in terms of currency and tenor of the reference rate.
For example, the ESTR (Euro Short-term rate) curve is normally taken as the risk-free curve for the Euro area.

On the other hand, we should consider that risk is present in every yield calculated from the return of a zero coupon. Therefore, the following cases are observed on the market:

- Higher yield curves for issuers that have lower credit quality and lower yield curves for issuers that have a higher rating.

- Higher yield curves for callable bonds compared to equivalents without optionality.

- Lower yield curves for puttable bonds compared to equivalents without optionality.

The **Interest rates term structure shape** can assume four classified canonical forms:

- Positively sloped term structure

- Negatively sloped term structure

- Flat

- Humped term structure

The part of the curve formed by short maturities is mainly influenced by monetary policy, while the part with longer maturities is more sensitive to inflationary factors.
An upward sloping curve indicates strong economic times, while flat or downwards shows the opposite.
To try and explain the shape of the interest rates term structure three main theories have been developed:

- Expectation hypothesis

- Liquidity preference

- Market segmentation theory

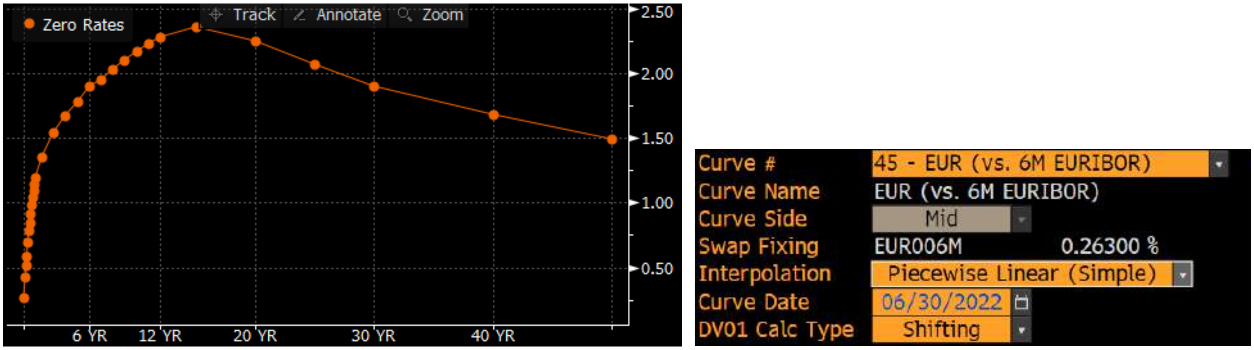


Figure I.8 A humped interest rates term structure. Source: Bloomberg®

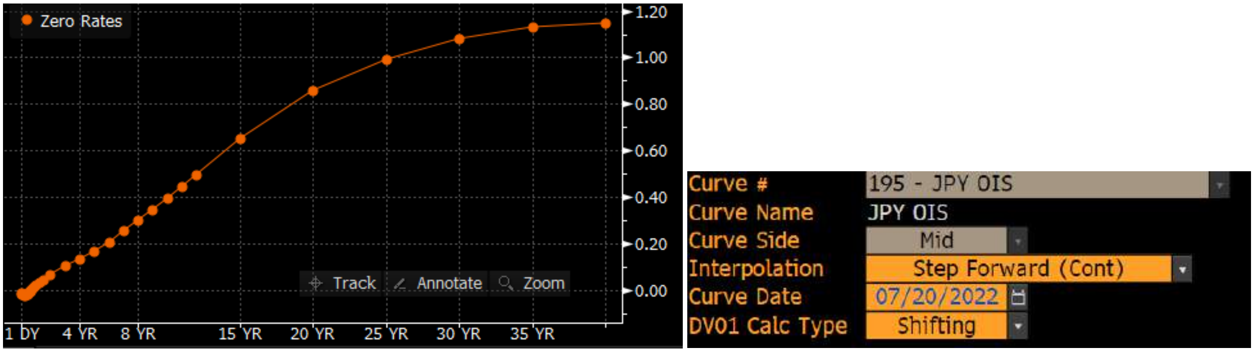


Figure I.9 A positively sloped interest rates term structure. Source: Bloomberg®

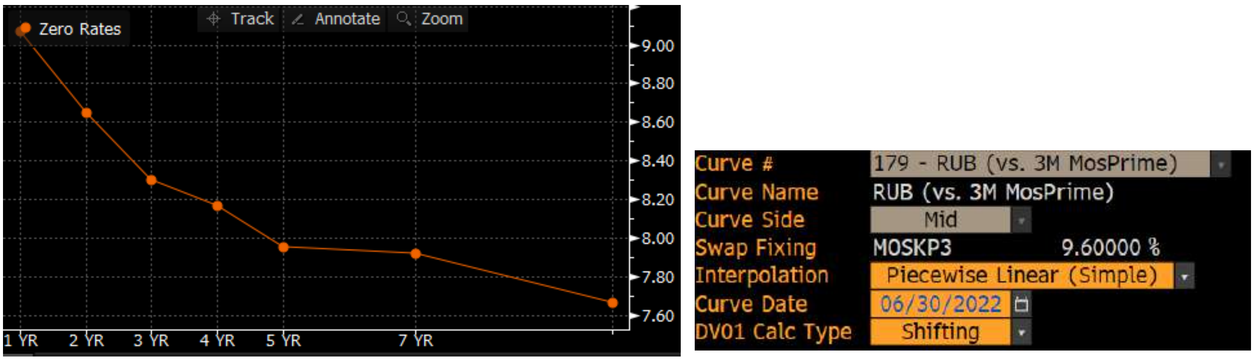


Figure I.10 A negatively sloped interest rates term structure. Source: Bloomberg®

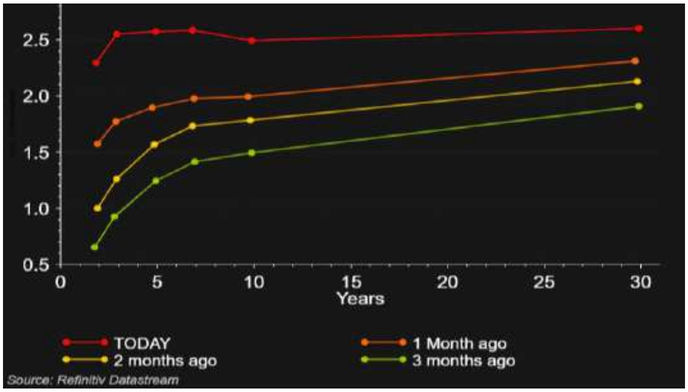


Figure I.11 A Yield curve flattening of the US zero curves. Source: Refinitiv - Datastream®. Reference Date: 29th March 2022

Let us examine those three theories. According to the **Expectation hypothesis**, the shape of the curve reflects the market's opinion on future levels of interest rates. Therefore, the implied forward rate on the curve that can be observed today is an unbiased estimation of the future spot rate: $\mathbf{F_{t,h} = E(R_{t,h})}$.
This formulation presents some criticalities though. In fact, if it were accepted that future spot rates are equal to the forward rates that can be calculated today, it would imply that each bond can be perfectly replaced by any other bond, regardless of the latter's maturity. Cox, Ingersoll and Ross had formally demonstrated the internal inconsistency of this statement by highlighting that it can be consistent with the financial equilibrium only for short time intervals. This scaled-down version is known as the **Local Expectations hypothesis**.
The son theory is the **Liquidity preference theory,** which states that investors prefer to hold more liquid securities and, consequently, characterized by a shorter duration. In order to induce an investor to buy longer-term securities, it becomes thus necessary to offer an additional risk premium: a liquidity or term premium.
In this case, two factors come into play:

- The future expected short-term spot rate, in accordance with the local expectations hypothesis; and
- A positive premium for liquidity.

We can express this in mathematical terms:

$$\mathbf{F_{t,t+1} = E(R_{t,t+1}) + L_{t,t+1} \text{ for } t > 0 \text{ and } L_{t,t+1} > 0} \quad (\textit{Eq. I.29})$$

According to this theory, the longer the maturities, the greater the premium required for liquidity. Consequently, interest rates term structures should mostly be with a positive slope.

Finally, the third theory, the **market segmentation** or **preferred habitat theory** considers as if there were as many parallel and distinct markets as there are maturities making up the term structure. In such context, each investor, according to their preference and risk appetite, decides on which maturity to operate; money is considered as a commodity and the meeting point between supply and demand for a given expiration constitutes the interest rate. In accordance with this theory, each point of the spot curve is given by a distinct market that reflects the investor's preferences of the moment. In this case, then, the sign of the risk premium, $\boldsymbol{\Pi_{t,t+1}}$, for an investor who wants to invest in a desired maturity **t** has no a priori definable sign:

$$\mathbf{F_{t,t+1} = E\ (R_{t,t+1}) + \Pi_{t,t+1}} \quad (\textit{Eq. I.30})$$

This theory allows to explain all the forms that a term structure can take:

- If the curve has a positive slope, investors prefer to invest in the short-term segment.
- In the presence of a negative slope, investors prefer to invest over the long term.
- If the curve is flat, it is indifferent.
- The term structure can admit humps depending on the preferences in the different market segments.

The interest rates term structure is obviously an **essential tool** for many sectors related to the banking activity. We can cite the departments that use it daily for their analyzes:

- Risk Management (stress test, sensitivity, future projections)
- Asset and Liability Management, ALM (what-if analysis on financial statements)
- Financial Engineering (pricing, fair value estimation, hedging strategies)
- Wealth Management (customized structured products design)
- Regulatory (Supervisory bodies – ECB, Bank of Italy, Internal Audit – requests to conduct scenarios under particular or extreme conditions of market inputs)
- Trading (comparison between theoretical and market prices, market expectations on rates)

Given its importance, it is essential to define the relevant parameters to be set before use, which are firstly, the market instruments for implying the spot rates, and the number of points to use (i.e., the granulometry) for defining the term structure. Then, the kind of model/interpolation to adopt if a value is not directly provided by the market is also to be defined, and finally, the curve to use for discounting the future cash-flows also has to be determined, since it must reflect the counterparty risk.

In accordance with the literature, the movements of an interest rate curve have been classified and labeled as follows:

Shift: when they refer to parallel movements of the yield curve.
Twist: if they refer to changes in the slope of the curve.
Butterfly: when the rates in the short and long term move in the same direction, while those in the medium term move in the opposite direction.

The yield curve strategies use the distribution of the fixed-income portfolio maturities in order to take advantage of expected future movements of the zero rates. The yield curve strategies that a portfolio manager can pursue are typically categorized into three groups:

Bullet strategy: the bond portfolio is concentrated in a single maturity (i.e. in a single point of the interest rates term structure).
Barbell strategy: the portfolio is concentrated in the two extremes of the yield curve.
Ladder strategy: the portfolio has maturities equally spaced on the curve.

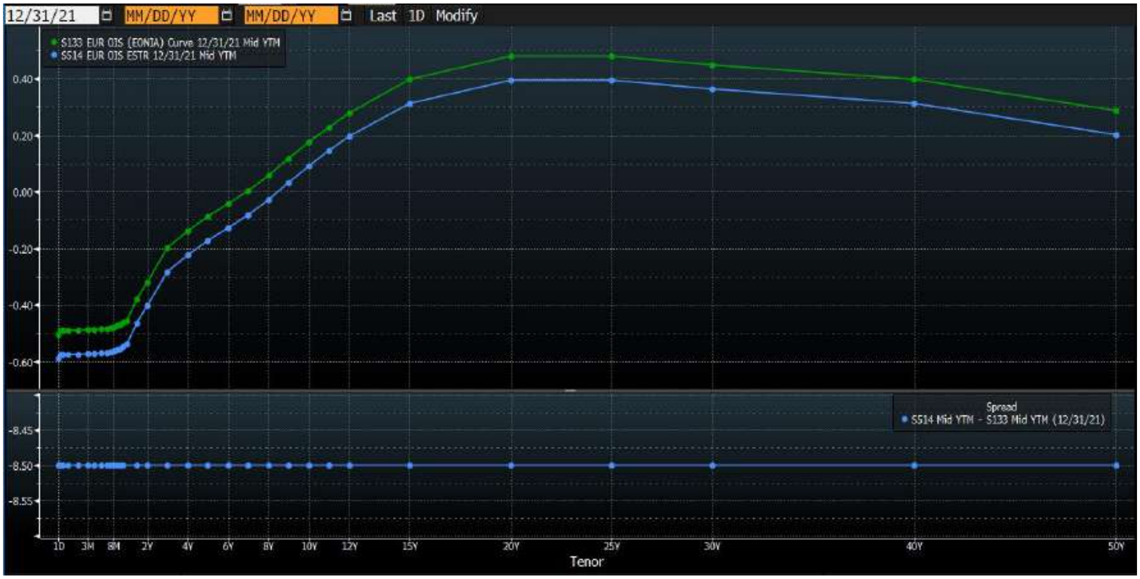


Figure I.12 Curve movements: shift. Source: Bloomberg®

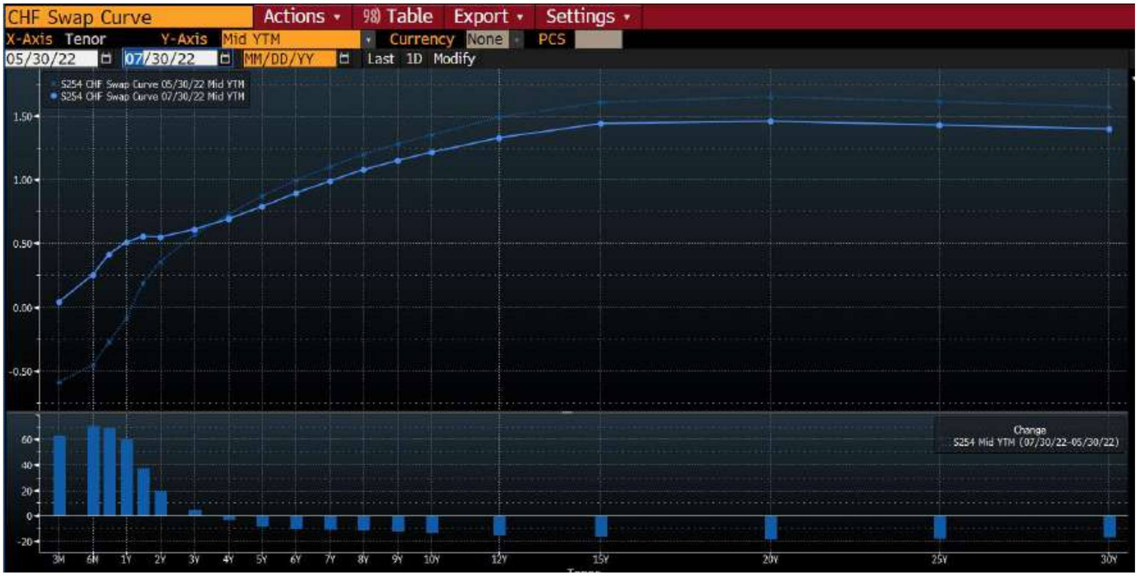


Figure I.13 Curve movements: twist. Source: Bloomberg®

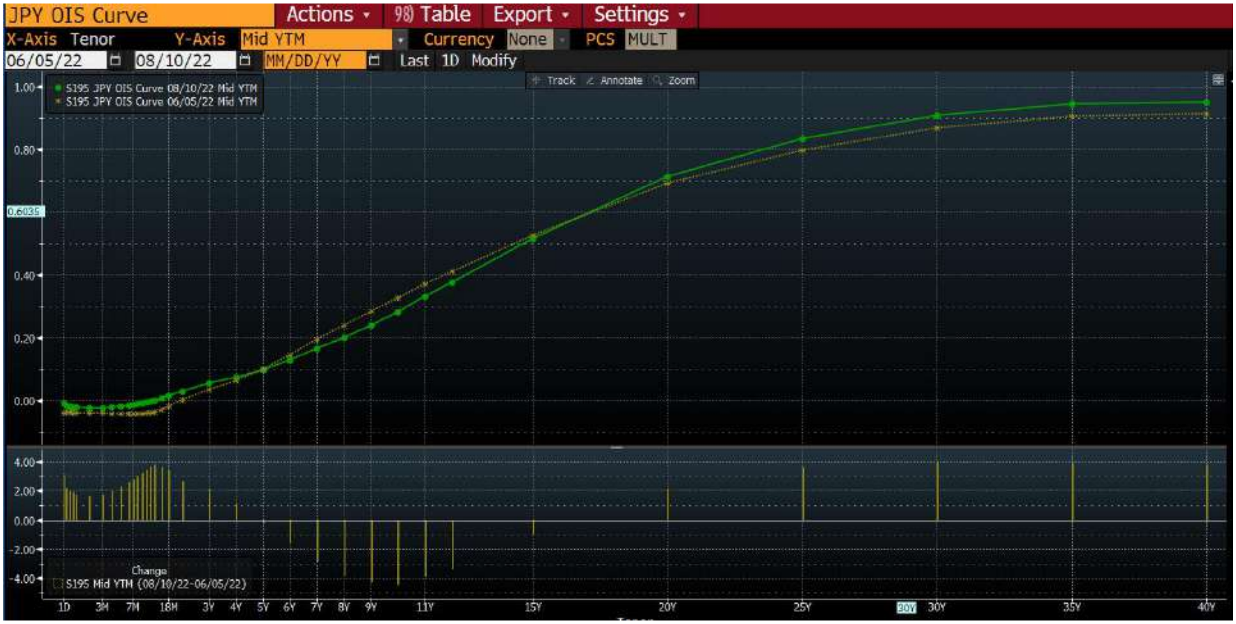


Figure I.14 Curve movements: butterfly. Source: Bloomberg®

FURTHER READINGS

Interest rates term structure model under extreme conditions

Cafferata A., Giribone P. G., Neffelli M., Resta M. – "Yield curve estimation under extreme conditions: do RBF networks perform better?" – Chapter 22 in book: "Neural Advances in Processing Nonlinear Dynamic Signals" – Springer (2019).

Cafferata A., Giribone P. G., Resta M. – "Interest rates term structure models and their impact on actuarial forecasting" – QFW18: Quantitative Finance Workshop 2018 (UniRoma3 - Rome).

Parametric models for Interest rates term structure

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Nelson C. R. and Siegel A. F. – "Parsimonious modeling of yield curves" – The Journal of Business, Vol. 60, No. 4 (1987).

Svensson L. E. O. – "Estimating and interpreting forward interest rates: Sweden 1992-4", NBER Working Paper Series, no 4871. (1994).

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Machine Learning models for Interest rates term structure

Delucchi A., Giribone P. G. – "Modeling the interest rates term structure using Machine Learning: a Gaussian process regression approach" – Risk Management Magazine Vol. 18, N. 3.
Gaggero G., Giribone P. G., Gualandi S., Martelli D. – "Improving the performance of traditional interest rates term structure models using Artificial Intelligence" – CGRM 2024: Corporate Governance and Risk Management in Financial Institutions (Università degli Studi di Bari)
Giribone P. G. – "Mathematical modeling in Quantitative Finance and Computational Economics", PhD Thesis in Economics, Part I: Artificial Intelligence and Machine Learning Techniques, Chapter 2 (2021).

I.2 BONDS

Bonds are debt securities issued by companies or public bodies that give the holder the right to the reimbursement of a sum of money lent to the issuer plus accrued interest. The normal practice for a standard bond (**straight bullet**) provides that upon maturity, the issuer pays the creditor the entire principal in a single solution, while periodic payments are paid on a regular basis (hence the term **fixed income**) typically quarterly, semi-annual or yearly for the interest.
These earnings received by the investor are called **coupons** and a failure to pay a coupon to the bondholder is considered an event of **default**.
It is substantially the **legal structure** that provides for the differentiation between the nominal value of a bond (or the principal that must be paid at maturity) and the coupons (which are to be considered as the payment of interest), and such distinction has fiscal repercussions in those countries that apply a different tax scheme based on the origin of financial income: capital gains tax or income tax.
As known, from a purely financial point of view, a bond is a series of cash flows.
In a straight bullet bond (or a standard bond whose principal is repaid in full at maturity) the cash-flows are **equally spaced,** and they are characterized by the **same amount**, with a final followed by and higher one paid at maturity.
As shown in the figure, the coupons are represented by have regular flows, while the final disbursement represents the payment of the last coupon together with the repayment of the principal.

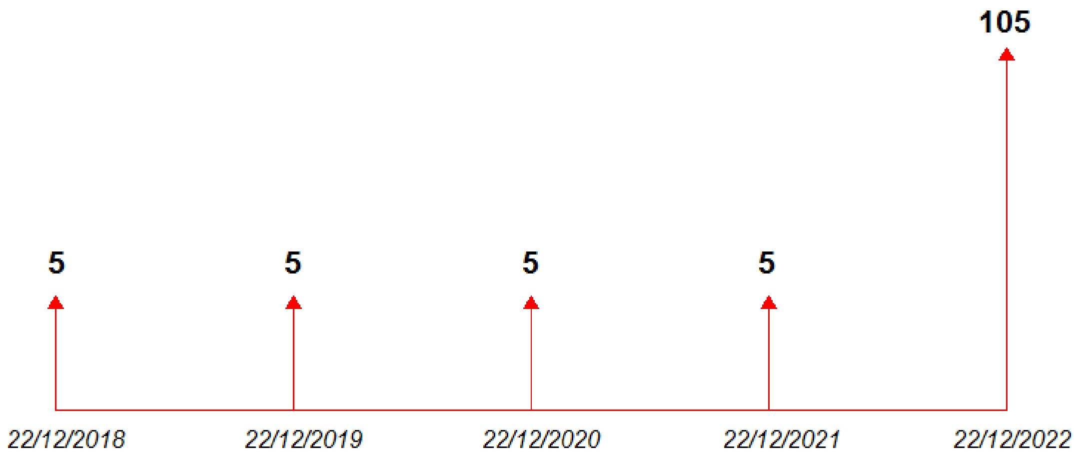


Figure I.15 Cash Flow example for a Straight Bullet Bond

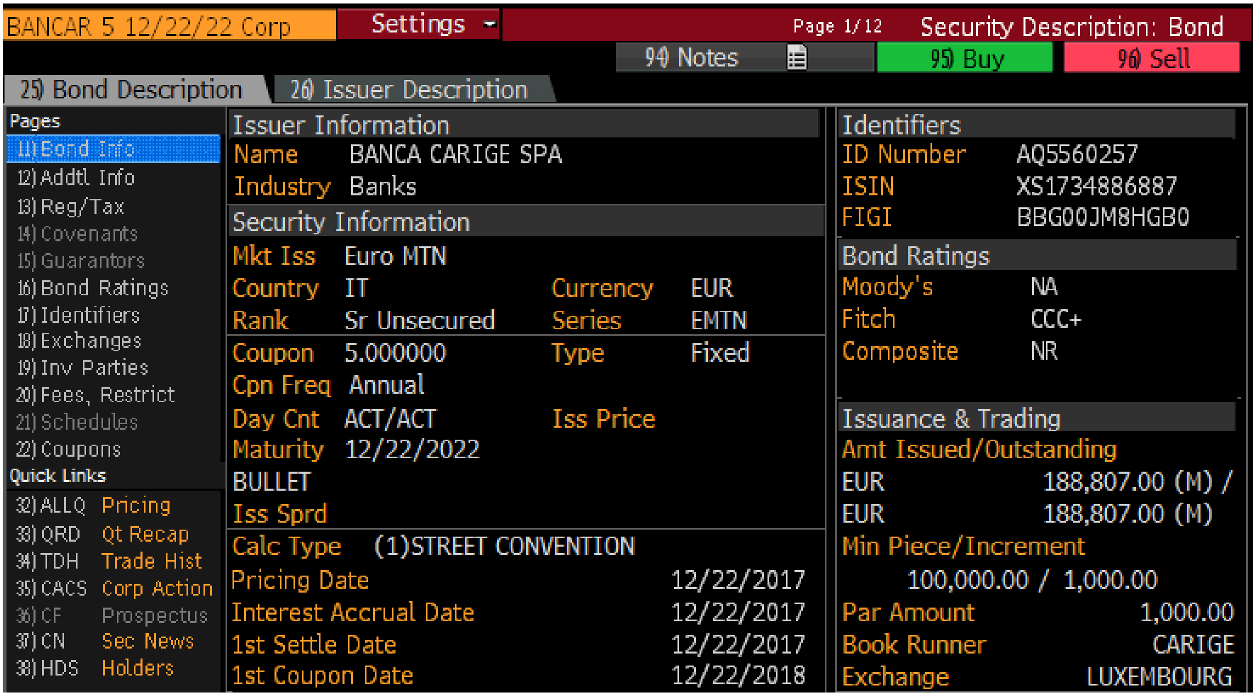


Figure I.16 Straight bullet bond. Source: Bloomberg®

The price of a bond constitutes the **market value** at which it is **currently traded**, and it is generally quoted as a percentage of the nominal amount (**face** or **par value**).
Therefore, in order to convert the market price of a bond into a monetary amount, it is sufficient to multiply the face value by the percentage and divide the result by 100. For example, if a bond with a par value of EUR 5,000 traded at 98.71, its value is:

$$(98.71 * 5000)\ /100 = \text{EUR } 4{,}935.5$$

Certain bonds, with particular reference to those issued by the US Treasury (American Treasury bonds or **American T-bonds**), are quoted as a percentage of their face value and in 32ths of a percentage. Thus, a T-bond quoted 102-14 means that its market price is equal to:

$$102 + 14/32 = 102 + 0.4375 = 102.4375.$$

This pricing method derives from the trading convention of adopting the 1/32nd of a point (in decimal 0.03125) as the minimum allowed price variation for this type of financial instrument.

BTPS 2 $\frac{1}{2}$ 12/01/32

Range	04/27/2022	-	08/19/2022	Period	Daily
Market	Bid Px		Ask Px	Currency	EUR
View	Price Table			Source	BGN

Date	Bid Px	Ask Px
Fr 08/19/22	91.672	91.806
Th 08/18/22	93.052	93.228
We 08/17/22	93.157	93.258

PCS	Firm Name	Bid Px / Ask Px
MSG1	MSG Quotes	/ 96.891
BVAL	BVAL (Score: 10)	91.750 / 91.765
	Last Trade	91.831
AKRO	BANCA AKROS SPA	91.670 / 91.790
MTS	MTS S.P.A.	91.670 / 91.790
GETX	GETTEX	91.670 / 91.810
MILA	MILAN EXCH	91.7600 / 91.8300
TWMA	Tradeweb MTF EU Com	92.183 /
BMFQ	BBG MTF Pre-Trade	91.843 /
PRBV	MTS Bondvision Pre	91.815 /
BPEG	BNP PARIBAS EURO G...	91.794 / 91.959
BTFQ	BBG MTF EU PreTrade	91.767 /
BBGX	BBVA	91.765 / 91.795
IXEP	IBOXX-EURO CORP CO...	91.76150 / 91.81900
BRLN	BERLIN EXCHANGE	91.760 / 91.860
GERM	GERMAN EXCHANGE	91.760 / 91.860
IMIG	INTESA SANPAOLO IM...	91.758 / 91.818
DAB	DEUTSCHE BK-DAB	91.757 / 91.797
DUSS	DUSSELDORF EXCH	91.740 / 91.810
FRNK	FRANKFURT EXCH	91.730 / 91.820
HMTF	HI-MTF Quote Driven	91.7300 / 91.8500
DZBK	DZ BANK	91.704 / 91.774
DEKA	DEKABANK	91.689 / 91.789
BGN	BLOOMBERG GENERIC ...	91.672 / 91.806
MNCH	MUNICH EXCHANGE	91.670 / 91.810

T 3 02/15/48

Range	08/20/2021	-	08/19/2022	Period	Daily
Market	Bid Px		Ask Px	Currency	USD
View	Price Table			Source	BGN

Date	Bid Px	Ask Px
Fr 08/19/22	92-17+	92-21
Th 08/18/22	93-30+	94-02
We 08/17/22	93-24	93-27
Tu 08/16/22	94-31	95-02+
Mo 08/15/22	94-23	94-26

PCS	Firm Name	Bid Px / Ask Px
BVAL	BVAL (Score: 10)	92-16 / 92-18+
	Last Trade	
SCBX	STANDARD CHARTERE...	92-15+ / 92-23
NONY	Nomura UST & Agenc...	92-15$\frac{1}{4}$ / 92-17$\frac{1}{4}$
CNTF	CANTOR FITZGERALD ...	92-17 / 92-17
MCAP	MCAP	92-15 / 92-16$\frac{3}{8}$
BBNY	BBVA	92-17+ / 92-21
EDFA	ED&F MAN	92-17+ / 92-21
BULL	BULLTICK LLC	92-17+ / 92-21
LCM	LOOP CAPITAL MARKE...	92-17+ / 92-21
ACAM	Academy Securities	92-17+ / 92-21
FCMT	FIDELITY CAPITAL MKT	92-14+ / 92-24
CBRA	Cabrera Capital Mkts	92-17+ / 92-21
BREA	BREAN CAPITAL	92-17 / 92-21
RILY	B RILEY FBR INC	92-17$\frac{1}{8}$ / 92-21$\frac{3}{8}$
ODEO	ODEON CAPITAL GROU...	92-17+ / 92-21
MFGI	MISCHLER FINANCIAL	92-17+ / 92-21
CG	CITIGROUP	92-16+ / 92-21$\frac{7}{8}$
GETX	GETTEX	92-16$\frac{15}{16}$ / 92-20$\frac{13}{16}$
TRDG	TradeGate	92-00$\frac{1}{16}$ / 93-03+
DAB	DEUTSCHE BK-DAB	92-18 / 92-20+
BGN	BLOOMBERG GENERIC ...	92-17+ / 92-21
MNCH	MUNICH EXCHANGE	92-16$\frac{15}{16}$ / 92-20$\frac{13}{16}$
STGF	Stuttgart Ex FXplus	92-16 / 92-19+

Figure I.17 Listing Conventions. Source: Bloomberg®

Comparing the prices of two bonds may not be straightforward because the coupon payment dates vary from bond to bond. Consequently, prices are generally **quoted net of accrued interest**. Practically, however, when a bond is bought between two payment dates, the investor also pays, in addition to the market price, the **accrued interest,** pro rata temporis. When a security is traded, the coupon quota already accrued in the time interval between the last coupon payment date and the date of purchase of the security is taken into account, in accordance with the relevant day count convention. This calculation is fair because the bond holder will receive the full coupon on the following payment date.

The price at which the bond is traded is called Tel Quel (or full purchase price or gross price or dirty price) and it is given by the sum of the Quoted price (or flat price or clean price) plus the accrued interest.

Full Price of bond = Quoted price + Accrued Interest (*Eq. I.31*)

For example, we want to buy the previously described straight bullet bond (5%) on 19th August 2022 for an amount equal to EUR 1,000,000. This security is traded at 101.1202, the previous coupon payment was on 22nd December 2021. The coupon is paid yearly.

Clean Price = 101.1202%
Accrued Interest = 5% * 240/365 = 3.287671%
Dirty Price = 101.1202% + 3.287671% = 104.407871%
Total Amount to be paid = Dirty Price * Face Amount = EUR 1,000,000 * 104.407871% = EUR 1,044,078.71.

Figure I.18 Clean and Dirty Prices

The **day-count convention** is the system used to consider days when calculating the interest accrued between two consecutive payment dates. Each financial instrument has its own methodology which varies, for example, depending on the type of bond, the interest rate (fixed or floater) or the country in which it has been issued. The numerous day calculation rules are standardized and codified and the most common are:

ACT/ACT or Actual/Actual: effective days on effective days. The numerator is given by the number of effective days between two dates, while the denominator is 365 for the non-leap year, otherwise 366.

ACT/360 or Actual/360: effective days on 360 days. The numerator is given by the number of effective days between two dates, while the denominator is 360.

ACT/365 or Actual/365: effective days on 365 days. The numerator is given by the number of effective days between two dates, while the denominator is 365.

30/360 or 30 on 360 days. Each month has 30 days and each year has 360 days.

For Swiss bonds and Eurobonds, it is usually assumed that a year consists of 360 days, or 12 months of 30 days each (Day Basis: **30/360**).

The formula for calculating the accrued interest is then:

Accrued Interest = C*(30*m+d)/360 (*Eq. I.32*)

Where **C** is the coupon rate, **m** the number of months and **d** days since the payment of the last coupon. Here is an example related to a Swiss bond:

Trading Date: 23rd August 2022
C = 0.5%
m = 2 (from 30th May 2022 to 30th July 2022)
d = 23 (from 30th July 2022 to 23rd August 2022)
Accrued Interest = 0.5%*(30*2+23)/360 = 0.115278%
Face Amount = CHF 1,000,000
Accrued = CHF 1,152.78

Figure I.19 Day-count convention and Accrued Interest – Swiss Bond. Source: Bloomberg®

For bonds issued by the US Treasury, the ACT/ACT day basis is usually adopted. The formula for calculating the accrued interest, in this case, can be rewritten as:

Accrued Interest = C / 2 * (Exact number of days since the last coupon / Exact number of days between coupons) (*Eq. I.33*)

When the coupon is paid every six months, as in this case, the coupon rate expressed on an annual basis must be halved. Here comes an example of a US Treasury bond:

Trading Date: 23rd August 2022
C = 2.875%
num = 100 (from 15th May 2022 to 23rd August 2022)
den = 184 (from 15th May 2022 to 15th November 2022)
Accrued Interest = 2.875% / 2 * 100/184 = 0.78125%
Face Amount = USD 1,000,000
Accrued = USD 7,812.5

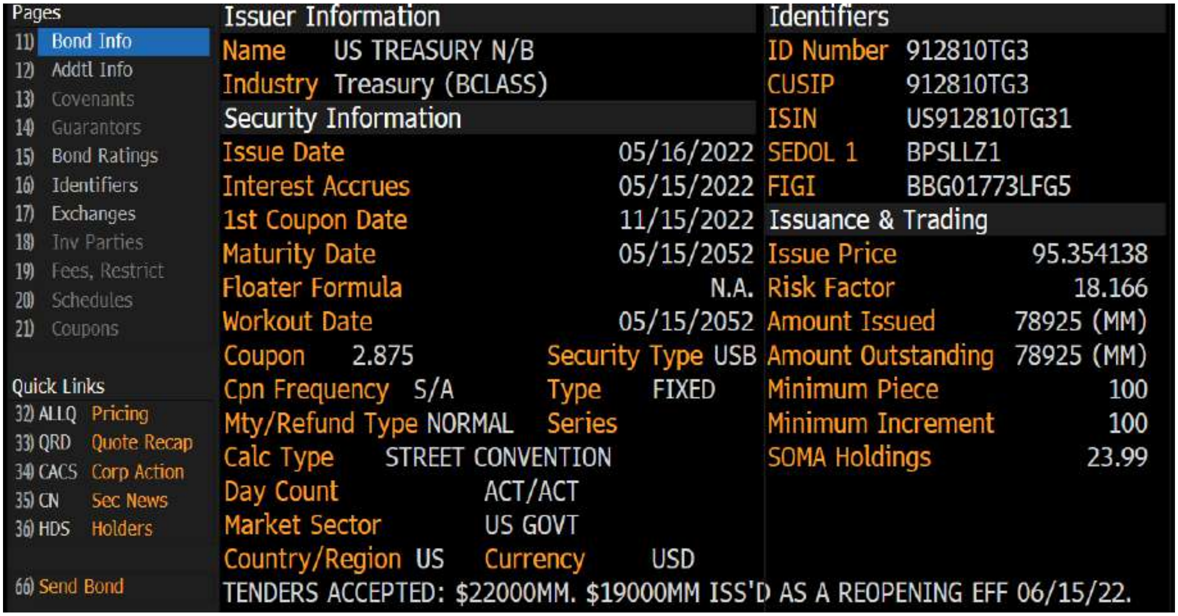


Figure I.20 Day-count convention and Accrued Interest – US Treasury Bond. Source: Bloomberg®

Certain bullet straight bonds may have an irregular first coupon (Short or Long). This does not change the calculation rules for the accrued interest. As shown in the picture, the other coupon payment dates are regular till the maturity of the bond. Here is an example of such a case:

Trading Date: 23rd August 2022
C = 2.15%
num = 175 (from 1st Mar 2022 to 23rd August 2022)
den = 184 (from1st Mar 2022 to 1st September 2022)
Accrued Interest = 2.15% / 2 * 175/184 = 1.02242%
Face Amount = EUR 1,000,000
Accrued = EUR 1,022.42

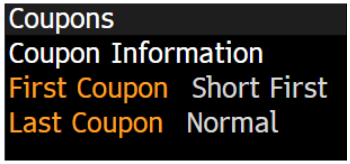


Figure I.21 Day-count convention and Accrued Interest – Italian bond

For bonds issued by US companies (corporate bonds), the basis is 30/360 and the payment frequency is usually half-yearly. Thus the formula for calculating the accrued interest is:

Accrued Interest = (C/2)*(30*m+d)/180 *(Eq. I.34)*

Where **C** is the coupon rate, **m** the number of months and **d** days since the payment of the last coupon. Here is an example regarding a US corporate bond:

Trading Date: 23rd August 2022
C = 4.125%
m = 4 (from 9th April 2022 to 9th August 2022)
d = 14 (from 9th August 2022 to 23rd August 2022)
Accrued Int. = 4.125% /2*(30*4+14)/180 = 1.535417%
Face Amount = USD 1,000,000
Accrued = USD 15,354.17

Figure I.22 Day-count convention and Accrued Interest – US Company

Figure I.23 Bloomberg® module check.

If the coupon payment date falls on a public holiday, the type of adjustment to be made, in accordance with the target calendar, must be contractually specified. As with all other terms, the business day conventions are also codified and the most important are indicated here:

Following: the date is adjusted to the following business day.
Preceding: the date is adjusted to the previous business day.
Modified Following: the adjusted date is the following business day unless it would then fall in the following month in which case, it is adjusted to the previous business day.

BPER: Banca

BPER BANCA S.P.A.

(a bank incorporated as a joint-stock company (società per azioni) in the Republic of Italy)

Business Day Convention, in relation to any particular date, has the meaning ascribed to it in the relevant Final Terms and, if so specified in the relevant Final Terms, may have different meanings in relation to different dates and, in this context, the following expressions shall have the following meanings:

(i) **Following Business Day Convention** means that the relevant date shall be postponed to the first following day that is a Business Day;

(ii) **Modified Following Business Day Convention** or **Modified Business Day Convention** means that the relevant date shall be postponed to the first following day that is a Business Day unless that day falls in the next calendar month in which case that date will be the first preceding day that is a Business Day;

(iii) **Preceding Business Day Convention** means that the relevant date shall be brought back to the first preceding day that is a Business Day;

Figure I.24 Covered Bond Prospectus. Source: Bloomberg®

There is a large variety of fixed income securities on financial markets, and these can be classified in different ways, either based on their structure, their numerous types of maturities, repayments or their coupons. In this section, we describe the most important classifications.

A bond that periodically pays fixed interest and repays the notional at maturity is called a **straight bond** (or as previously mentioned **bullet bond**).
A **callable bond** gives the issuer the right to repurchase the security on a certain date (**call date**) at a predetermined price (**call price**).
The call price is often equal to the par value plus a premium (**call premium**): the lower the premium, the greater the reliability of the issuer.
There are particular features typical of these bonds, for example, callable bonds from issuers with an excellent credit profile - high-quality issuers - can be called "at par", i.e. at 100.
Obviously, the call dates and the respective call prices must be specified on the prospectus drafted at the issue of the security.

R	Name	Ticker	Issue Date	Coupon	Maturity	BB R...	Mty Type	Curr	Bid Pri...	Ask Px	Source	Collateral	ISIN
		BPEIM											
1)	BPER Banca	BPEIM	07/25/2019	8.750	PERPETUAL	NA	CONV/CA...	EUR	101.831	105.002	BGN	JR SUBORDINATED	IT0005380263
2)	Cassa di Risparmio di Bra S...	BPEIM	03/30/2012	FLOAT	PERPETUAL	NA	PERP/CA...	EUR			BGN	JR SUBORDINATED	IT0004809411
3)	BPER Banca	BPEIM	06/20/2022	0.000	12/20/2032	NA	CALLABLE	EUR	99.799	101.080	BVAL	BONDS	IT0005499063
4)	BPER Banca	BPEIM	01/25/2022	3.875	07/25/2032	BB-	CALLABLE	EUR	83.321	84.313	BGN	SUBORDINATED	XS2433828071
5)	BPER Banca	BPEIM	11/30/2020	3.625	11/30/2030	BB-	CALLABLE	EUR	90.408	91.412	BGN	SUBORDINATED	XS2264034260
6)	BPER Banca	BPEIM	08/12/2022	ZERO	08/12/2027	NA	BULLET	EUR	950.000		EXCH	UNITS	IT0005500985
7)	BPER Banca	BPEIM	05/09/2022	ZERO	05/10/2027	NA	BULLET	EUR	920.000		EXCH	UNITS	IT0005493579
8)	BPER Banca	BPEIM	03/31/2021	1.375	03/31/2027	BB+	CALLABLE	EUR	87.523	88.202	BGN	SR UNSECURED	XS2325743990
9)	BPER Banca	BPEIM	03/31/2022	ZERO	03/31/2027	NA	BULLET	EUR	895.000		EXCH	UNITS	IT0005490062
10)	BPER Banca	BPEIM	03/31/2022	ZERO	03/31/2027	NA	BULLET	EUR	885.000		EXCH	UNITS	IT0005490070
11)	BPER Banca	BPEIM	03/01/2022	ZERO	03/01/2027	NA	BULLET	EUR	875.000		EXCH	UNITS	IT0005482283
12)	BPER Banca	BPEIM	08/26/2008	ZERO	08/26/2026	NA	BULLET	EUR	83.399	84.773	BVAL	UNSECURED	IT0004405319
13)	BPER Banca	BPEIM	06/29/2022	0.500	07/22/2026	NR	CALLABLE	EUR	100.000	100.000	EXCH	COMPANY GUARNT	IT0005498404
14)	BPER Banca	BPEIM	06/27/2022	ZERO	06/29/2026	NA	BULLET	EUR	940.000		EXCH	UNITS	IT0005496192
15)	BPER Banca	BPEIM	03/19/2019	1.125	04/22/2026	NR	BULLET	EUR	97.104	97.363	BGN	COVERED	IT0005365710
16)	BPER Banca	BPEIM	11/11/2021	FLOAT	10/28/2025	NR	CALLABLE	EUR	100.064	100.442	BVAL	COVERED	IT0005467201
17)	BPER Banca	BPEIM	11/11/2021	0.500	10/28/2025	NR	CALLABLE	EUR	94.688	95.066	BVAL	COVERED	IT0005467185

Schedules
Call Schedule
Once only Call
Last Call Date 11/30/2025

Callable only on date(s) shown

Date	Price
11/30/2025	100.000

Figure I.25 European Callable bond. Source: Bloomberg®

The option that makes a bond "repurchasable" by the issuer only on certain specific dates (discrete dates) is called the **European Call**. If the issuer has the right to buy back the security at any time starting from a certain date at a predetermined price, it is called **American Call**. In cases when interest rates fall sharply on the market, the callable bond issuer can exercise the call option allowing for refinancing by issuing new bonds, at a later date, at a lower fixed coupon rate. Therefore, a callable bond protects the issuer from being forced to pay high coupons when interest rates fall. On the other hand, **Puttable bonds** are the opposite of callable bonds: in this case the holder of the bond has the right to sell the bond to the issuer at a predetermined price (**put price**). The presence of the put increases the value of the bond in favor of the investor as it ensures that the security cannot be traded below the put price. Bullet bonds are bonds that are not callable.

Corp | Govt | Loan | Pfd | CDS | CDS Idx | Muni | Futr | Opt | IRS | IRS Vol | Gen Govt | Muni Issu...

60) Exclusions: (71) Matured/Called, Non-Verified Bonds 62) Advanced Search | SRCH » 61) Column Settings

R	Name	Ticker	Coupon	Maturity	BB R...	Mty Type	Curr	Ask Px	Source	Collateral
		ENELIM								
1)	Enel SpA	ENELIM	6.500	01/10/2074	BB+	CALLABLE	EUR	106.375	BGN	JR SUBORDINATED
2)	Enel SpA	ENELIM	2.500	11/24/2078	BB+	CALLABLE	EUR	93.279	BGN	JR SUBORDINATED
3)	Enel SpA	ENELIM	8.750	09/24/2073	BB+	CALLABLE	USD	104.838	BMRK	JR SUBORDINATED
4)	Enel Finance International NV	ENELIM	4.250	09/14/2023	BBB	BULLET	USD	97.945	BMRK	COMPANY GUARNT
5)	Enel Finance International NV	ENELIM	2.875	05/25/2022	BBB	BULLET	USD	94.635	BMRK	COMPANY GUARNT

Figure I.26 Bermudan Callable bond. Source: Bloomberg®

Figure I.27 Puttable bond. Source: Bloomberg®

Certain bonds may include an amortization plan (**sinking-fund provision**) which requires the issuer to pay the principal not in a single payment at maturity, but at each coupon payment. The **Sinker percentage** is defined as the percentage of the bond which is amortized before maturity. Let us consider as an example a ten-year sinkable bond, with a coupon equal to 2% paid annually, 1,000 Euro of nominal value and a sinker percentage of 90%. The cash-flows are summarized in Table I.7.

Time	Total amount due	Interest payment 2% [A]	Sinking payment [B]	Principal Repayment [C]	Total annual payment [A+B+C]
1	1000	20	100	0	120
2	900	18	100	0	118
3	800	16	100	0	116
4	700	14	100	0	114
5	600	12	100	0	112
6	500	10	100	0	110
7	400	8	100	0	108
8	300	6	100	0	106
9	200	4	100	0	104
10	100	2	0	100	102

Table I.7 Total annual payment for a Sinkable bond

Schedules

52) Sink | 56) Factor Hist

Type	Level	Amount Issued	1,000,000.00 (M)
Frequency	Annual	Schd Amt Outstanding	1,000,000.00 (M)
Voluntary	None	Next Mandatory Sink	07/31/2031
Avg Life	15.81 (03/24/38)	Shortest	16.17 (07/31/38)
		Longest	16.17 (07/31/38)

Table View | Chart View

Date	Amount Cash (M)	%	Total Sunk Cash (M)	%	Remaining Balance Cash (M)	
07/31/2031	66666.67	6.6667	66666.67	6.6667	933333.33	93.333
07/31/2032	66666.67	6.6667	133333.33	13.3333	866666.67	86.666
07/31/2033	66666.67	6.6667	200000.00	20.0000	800000.00	80.000
07/31/2034	66666.67	6.6667	266666.67	26.6667	733333.33	73.333
07/31/2035	66666.67	6.6667	333333.33	33.3333	666666.67	66.666
07/31/2036	66666.67	6.6667	400000.00	40.0000	600000.00	60.000
07/31/2037	66666.67	6.6667	466666.67	46.6667	533333.33	53.333
07/31/2038	66666.67	6.6667	533333.33	53.3333	466666.67	46.666
07/31/2039	66666.67	6.6667	600000.00	60.0000	400000.00	40.000
07/31/2040	66666.67	6.6667	666666.67	66.6667	333333.33	33.333

52) Sink | 56) Factor Hist

Principal Factor History

Date	Factor
04/25/2005	1.000000000
07/31/2031	0.933333333
07/31/2032	0.866666666
07/31/2033	0.800000000
07/31/2034	0.733333333
07/31/2035	0.666666666
07/31/2036	0.600000000
07/31/2037	0.533333333
07/31/2038	0.466666666
07/31/2039	0.400000000
07/31/2040	0.333333333
07/31/2041	0.266666666
07/31/2042	0.200000000
07/31/2043	0.133333333
07/31/2044	0.066666666

Figure I.28 Sinkable bond. Source: Bloomberg®

The **Zero-coupon bond** is a type of bond which pays no interest during its whole lifetime. These bonds generally repay the face value (100%), but are typically issued below par, therefore the financial return of this type of security is derived from the differential between these two prices according to the well-known relationship:

$$Yield = \sqrt[T]{\frac{face\ value}{issue\ price}} - 1 = \left(\frac{face\ value}{issue\ price}\right)^{\frac{1}{T}} - 1 \quad \textit{(Eq. I.35)}$$

Where T is expressed in year fractions. As an example, a zero-coupon issued at 45.73% and which redeems at par (i.e. at 100%), with a duration of 25 years, yields the following interest at maturity: (100/45.73)^(1/25)-1 = 3.179%.

The concept of zero-coupon bonds is central to all financial analysis and risk management. It constitutes a building block for financial engineering, since any bond made up of cash flows can be broken down into a series of zero coupons. Besides, this type of financial instrument is also essential to derive the spot (or zero-curve) interest rate curve from which the discount factors necessary to discount future cash flows are calculated.

Let us make an example and use the above formula:

Bid/Ask Price = 98.931/98.961
Reference Date = 24th August 2022
Maturity Date = 14th August 2023
Day Basis = ACT/360
T = 355/360 = 0.98611

$$Yield = \left(\frac{Face\ Value}{Price}\right)^{\frac{1}{T}} - 1; Yield_{BID} = \left(\frac{100}{98.931}\right)^{\frac{1}{0.98611}} - 1 = 1.096\%; Yield_{ASK} = 1.065\%$$

Figure I.29 Zero Coupon bond. Source: Bloomberg®

A concept derived from zero-coupons is that of **Stripped Bonds,** which are zero-coupon bonds artificially created from risk-free government bonds. Such concept dates back from the eighties, when the first strips were synthesized by Wall Street companies, initially from American government bonds (US Treasury bonds), which soon became widespread enough to be created by the same government issuers. Nowadays, there are strips on the market made from French bonds (**French OAT**s – Obligations Assimilables du Trésor), United Kingdom (**UK GILT**s – Government Issued Long Term Stocks) and American bonds (**US Treasuries**), all sponsored and managed by government issuers themselves.

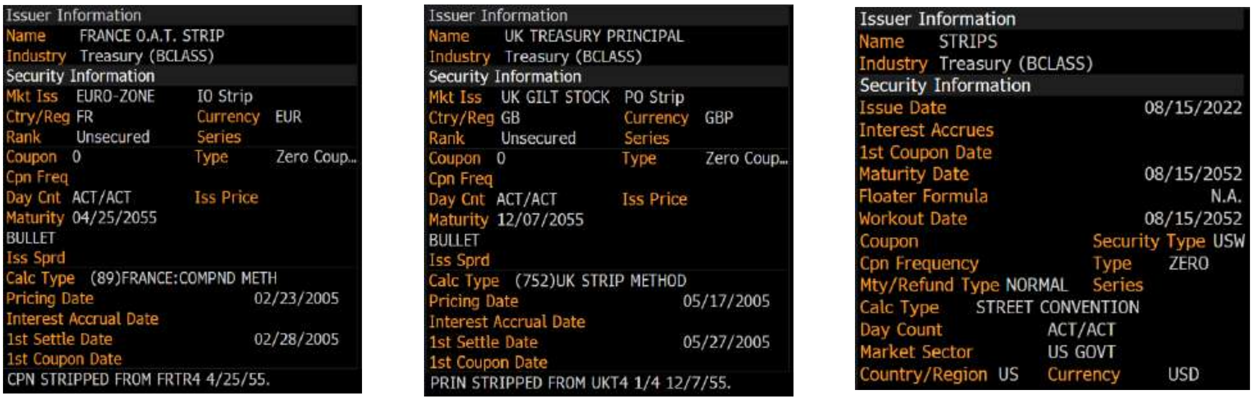


Figure I.30 Stripped Bonds. Source: Bloomberg®

Perpetual bonds (or undated bonds) are securities whose issuer pays interest at a fixed rate "forever", without the principal being ever repaid. These bonds include issues that are quite dated, even centuries old, such as those issued by the Canadian and British governments as war loans.
Although more recently, other issuers, especially in the banking sector, have also issued undated bonds with embedded call options.

Income bonds are securities that pay interests only if the company's profits are sufficiently high. In this particular case, failure to pay a coupon is not considered as an event of default.
Another peculiarity of these bonds is that the holders have the priority to be remunerated through the payment of the coupon before the distribution of dividends to the ordinary shareholders of the company.

Another typology is constituted by **Convertible bonds,** securities that are placed in an intermediate position between a stock and a bond. In fact, these financial instruments offer their subscriber the right to remain a creditor of the issuing company (therefore to maintain the status of bondholder), or to convert the bonds, within certain time frames (**conversion periods**) and on the basis of pre-established **conversion ratios**, into shares of the issuing company or of another company, thus assuming the status of shareholder.
The **conversion option** is at the discretion of the holder of the bond, however, certain convertible bonds provide for the possibility to force the conversion.

Figure I.31 A Perpetual Bond (on the left) and a Convertible Bond (on the right).

Bonds cum warrant constitute a particular category of bonds, which have the main security accompanied by another financial instrument that gives the subscriber the right to obtain a certain quantity of other securities (shares and/or bonds) of the issuing company, or another one, on a due date and within a predetermined period of time, against payment of a certain amount (**warrant premium**). In legal terms, the warrant is configured as a call option to purchase other financial instruments. If the right to purchase can be exercised at any time prior to the expiry date, the exercise type is called "**American**", otherwise if this right can only be exercised on a specific future date the exercise type is called "**European**". Bonds with warrants have a great similarity with convertible bonds, but they have their own specific characteristics:

- The warrant can generally be detached from the bond and negotiated separately from it, while in the convertible bond the conversion right is inseparably linked to the security.

- The option incorporated in the warrant can have a longer life than the bond.

Therefore, three prices can be determined on the market: the full bond (i.e. **cum warrant bond**), the bond without the warrant (**ex warrant bond**) and the **warrant** alone.


Issuer Information
Name BASF SE
Industry Chemicals (BCLASS)
Convertible Information
Mkt of Issue EURO MTN Convertible
Ctry/Reg DE Currency USD
Rank Sr Unsecured Series BAS
Conv Ratio 2158.4130 Conv Price 115.8258
Stock Tkr BAS GR Stock Price 41.459999
Parity 35.5999 Premium 177.5854
Coupon 0.925000 Init Prem 25.000
Type Fixed Freq S/A
Calc Type (49)CONVERTIBLE
Pricing Date 03/02/2017
1st Coupon Date 09/09/2017
Convertible Until 12/12/2022
Maturity 03/09/2023
Identifiers
FIGI BBG00G4XNDL0
ISIN DE000A2BPEU0
ID Number AM7310650
Bond Ratings
Moody's A3
S&P A
Composite A-
Scope A
Issuance & Trading
Amt Issued/Outstanding
USD 850,000.00 (M) /
USD 850,000.00 (M)
Min Piece/Increment
250,000.00/ 250,000.00
Par Amount 250,000.00
Book Runner DB-sole
Reporting TRACE
PRX/SHR EUR 110.8408 (1 USD = 0.9488 EUR). ISS'D W WARRANTS, BOND W/O WRNT: ISIN DE000A2BPEV8. WARRANT: DE000A2BPEW6. CASH SETTLE.

Figure I.32 Cum Warrant Bond. Source: Bloomberg®

Issuer Information				Identifiers	
Name	BASF SE			FIGI	BBG00G4VBCY5
Industry	Chemicals (BCLASS)			ISIN	DE000A2BPEV8
Security Information				ID Number	AM7163299
Mkt Iss	EURO-DOLLAR			Bond Ratings	
Ctry/Reg	DE	Currency	USD	Moody's	A3
Rank	Sr Unsecured	Series		S&P	A
Coupon	0.925000	Type	Fixed	Fitch	A
Cpn Freq	S/A			Composite	A-
Day Cnt	ISMA-30/360	Iss Price	91.20000	Issuance & Trading	
Maturity	03/09/2023			Amt Issued/Outstanding	
BULLET				USD	850,000.00 (M) /
Iss Yield	.925			USD	850,000.00 (M)
Calc Type	(1)STREET CONVENTION			Min Piece/Increment	
Pricing Date			03/02/2017		250,000.00/ 250,000.00
Interest Accrual Date			03/09/2017	Par Amount	250,000.00
1st Settle Date			03/09/2017	Book Runner	DB-sole
1st Coupon Date			09/09/2018	Exchange	FRANKFURT

UNDLY = BAS GR. ISS'D W/O WARRANTS BOND WITH WARRANT: ISIN DE000A2BPEU0. WARRANT: ISIN DE000A2BPEW6.

Figure I.33 Ex Warrant Bond. Source: Bloomberg®

Type	Equity Uncovered American Call			ISIN	DE000A2BPEW6
Issuer	BASF SE				
Pricing		Conversion		Underlying	BAS GR
Price		Strike EUR	109.9056	3) DES	BASF SE
Turnover		Underlying/Wrnt	2,158.143	Price EUR	41.46
Volume	0			High 02/10	69.15
Analytics				Low 07/05	39.325
1) WRNT	% Premium	Settle	Cash	Market Cap	38,080.13M
Gearing		Timeline		Shares Out	918.48M
Eff. Gearing		Expiration Date	03/09/23	Ind Gross Yield	8.2%
Parity	-147,715.400	Days To Exp	197	Hist Volatility	35.511
Underlying/Strike	.377	1st Exercise Date	03/17/17	Liquidity 4) WMON	
2) OVME				Total Warrants Avail	4,332
				Total Warrants Out	56,154.54M
Delta		Issue Date	03/09/17	Wrnts Out 1 Day Chg	-1,189.50M
Normalized Delta		Issue Price EUR	237,192.00	Wrnts Out % 1 Day Chg	-2.07%
		Issue Amount	2K	Total Turnover	.02M
		Outstanding	2,400	Turnover 5 Day Avg	.05M
Moneyness	-165.09%			Total Volume	.05M
		Out 1 Day Chg 0%	0	Volume 5 Day Avg	.21M

USD 250000 DENOM TRANSLATED @ US$ 1 = fr¾¾ 0.9488 - DETACHABLE WARRANTS - EXERCIS PERIOD END 12/12/2022 - MANDATORY EXERCISE THEREAFTER

Figure I.34 Warrant. Source: Bloomberg®

The three prices, even if related, have their own peculiar characteristics, in particular, the price of the warrant will essentially depend on the relationship between the share price and the price set to exercise the subscription right (based on the number of warrants needed to obtain a share). The price of the ex-warrant bond will be close to that of a normal bond with the same financial characteristics.
On the other hand, the price of the cum warrant bond is theoretically equal to the sum of the two previous values, but it is in fact also influenced by other factors such as the duration of the option, the expected performance of the stock on the equity market, the performance of the stock market itself and interest rates on the fixed income market. This formula theoretically might hold:

Th. Cum-Warrant Bond = Ex-Warrant Bond + Warrant Price *(Eq. I.36)*

Covered bonds are securities insured by assets intended to primarily satisfy the rights of the bondholders in the event of the issuer's insolvency. Ceteris paribus, typically a Covered Bond Rating is greater than that of senior or subordinated bonds.

Figure I.35 Covered bond. Source: Bloomberg®

Securities that have prefixed increasing (/ decreasing) coupons are called **step-up** (/ **step-down**) **bonds**.

Another important category is that of **Indexed bonds,** a category of securities with considerably different characteristics, but anyhow attributable to the need to supply the market with securities whose value is protected in whole or in part from the loss of purchasing power of the currency. The indexing feature consists in linking the yield and/or the redemption value of the security to the performance of an index chosen during the issue phase.

Issuer Information			
Name	INTESA SANPAOLO SPA		
Industry	Banking (BCLASS)		
Security Information			
Mkt Iss	EURO MTN		
Ctry/Reg	IT	Currency	EUR
Rank	Sr Unsecured	Series	EMTN
Coupon	1.000000	Type	Step Cpn
Cpn Freq	Annual		
Day Cnt	ACT/ACT	Iss Price	100.0000
Maturity	10/20/2031		
BULLET			
Iss Sprd			
Calc Type	(1311)MULTI-STEP CPN BND		
Pricing Date			09/14/2021
Interest Accrual Date			10/20/2021
1st Settle Date			10/20/2021
1st Coupon Date			10/20/2022

Coupon	End Date
1.000	10/20/2022
1.000	10/20/2023
1.000	10/20/2024
1.000	10/20/2025
.500	10/20/2026
.500	10/20/2027
.500	10/20/2028
.500	10/20/2029
.500	10/20/2030
.500	10/20/2031

Figure I.36 Step Down bond. Source: Bloomberg®

The most widespread parameters used for indexation are the following:

- **monetary:** when indexation is linked to the cost of money on the money market.
- **real**: when indexation is linked to the price of a basket of goods or services.
- **financial:** when it is linked to the cost of capital on the financial market.
- **foreign exchange:** as the name suggests, when it is linked to exchange rate.

It is worth to note that the financial and monetary indexation clauses generally concern coupon interest, while a real and currency indexation tends to refer to the principal. We will analyze the characteristics of each type of indexation mentioned above and provide examples.

Monetary indexing

The variable coupon can include a multiplier (**leverage**) and an additional margin (**spread**), generally expressed in basis points. A typical coupon of a bond indexed to a reference rate of the money market (floating-rate notes) can be expressed as:

leverage * money market reference rate + spread *(Eq. I.37)*

Here is an example:
Reference Date: 25^{th} August 2022
Last Coupon payment date: 15^{th} April 2022
Reset Days Prior = 2 days
Fixing Date = 13^{th} April 2022
Euribor 6 months (EUR006M Index) = -0.328%
Coupon Formula = Leverage * index + spread
Current Coupon = 1 * (-0.328%) + 75 bps = 0.422%

Figure I.37 Floating-rate bond. Source: Bloomberg®

In bonds having a floating coupon, the issuer can define a maximum rate (**cap**) and/or a minimum rate (**floor**), and the interest to be paid cannot go beyond this threshold, while the simultaneous presence of an upper-bound and a lower-bound defines a range of admissible values for the defined coupon. Such corridor is called **collar** (cap + floor = collar).

The value of a floating bond with a cap has a lower intrinsic value than one with the same financial characteristics but without the embedded option. Dually, the value of a floating bond with a floor has a higher intrinsic value than one with the same financial characteristics but without the embedded option. It is in fact

intuitive to understand that the investor is protected if the coupon assumes a value lower than the floor **strike price** (i.e. the coupon lower bound). In the case of a collar, on the other hand, it cannot be concluded a priori: it depends on the agreed strikes. Here is an example with both a cap and a floor:
First Coupon Date: 28th April 2020
Fixing Date = 26th April 2020
EUR003M Index = -0.538%
Coupon Formula = Max(Floor,Min(Cap,index))
First Coupon = Max(0.64,Min(1.05,-0.538)) = 0.64

Taking into consideration that the parameter has been negative till 14th July 2022, the coupons of the bond are equal to the floor strike price, that is 0.64%.

Figure I.38 Capped and Floored Bonds. Source: Bloomberg®

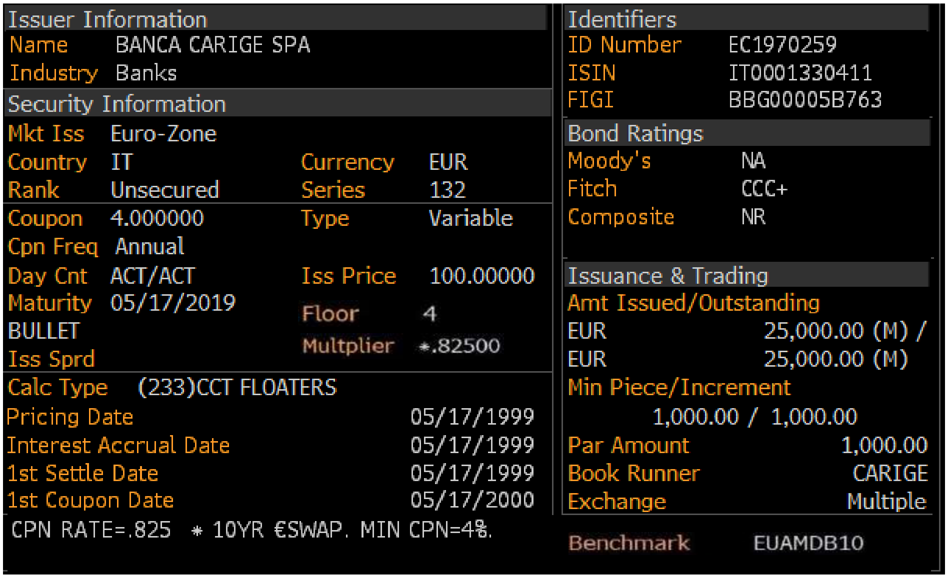


Figure I.39 Deleveraged bonds. Source: Bloomberg®

Real indexing

That is the case of an indexation on goods or services, which we will explain through an example.

Figure I.40 Inflation-indexed bonds. Source: Bloomberg®

2.1 THE INDEXATION COEFFICIENT

The Indexation Coefficient (IC) is calculated on the basis of the rate of inflation measured by the Harmonised Index of Consumer Prices for the Euro area, excluding tobacco, calculated and published each month by Eurostat (Eurostat Index).

The coefficient allows the calculation of adjusted values of the nominal principal amount, for a day d of a month m, on the basis of price inflation. The IC is calculated using the following formula:

$$IC_{d,m} = \frac{\text{Reference Inflation}_{d,m}}{\text{Base Inflation}}$$

where *Reference Inflation* is the rate of inflation described in the following paragraph; and *Base Inflation* is the value of Reference Inflation as at the interest commencement date.

Only six decimal places of the value of the Coefficient calculated in this manner are considered, the value's fifth decimal place is rounded.

On a monthly basis, the Ministry of Economy and Finance publishes daily values for the Indexation Coefficient in the Statistics section of the Public Debt website.

Department of the Treasury

2.2 REFERENCE INFLATION

Reference Inflation for a particular date (a day *d* of a month *m*) is calculated on the basis of the Eurostat Indices for the second and third month prior to the month for which the calculation is being made, as per the following formula:

$$RI_{d,m} = EI_{m\text{-}3} + \frac{d-1}{gg} * (EI_{m\text{-}2} - EI_{m\text{-}3})$$

dove:

- $EI_{m\text{-}3}$ is the Reference Inflation of day *d* of month *m*;
- $EI_{m\text{-}3}$ is the Eurostat Index value for the month three months prior to that being calculated;
- $EI_{m\text{-}2}$ is the Eurostat Index value for the month two months prior to that being calculated;
- *d* is the day of the month being calculated;
- *gg* is the actual number of days in month *m*.

Figure I.41 Inflation Bond Accrued. Source: Bloomberg® and Department of Treasury

Last Coupon Payment Date	03/15/2022
Trading Date	08/25/2022
Next Coupon Payment Date	09/15/2022
CPI Base (09/15/2009)	92.21738
CPI Index (05/01/2022)	115.74
CPI Index (06/01/2022)	116.7
CPI Reference (08/25/2022)	116.48323
Index Ratio	1.26314
Notional (real term) [€]	1,263,140
MKT Price [%]	118.24
MKT Value [%]	1,493,536.74
Coupon – C [%]	2.550%
C/2 [%]	1.275%
Days Numerator and Denominator	163 and 184
Accrued [%]	1.12948%
Accrued [€]	14,266.91
Full Price [€]	1,507,803.65

Table I.8 Accrued interest for an inflation-indexed bond

Let us calculate the CPI Reference as follows:

CPI Reference= CPI(m-3) + (d-1)/(gg)* [CPI(m-2) - CPI(m-3)]=115.7 + (25-1)/31 *[116.7-115.7] = 116.48323
Index Ratio = CPI Reference/CPI Base = 116.48323/92.21738 = 1.26314;
Notional in Real terms = Notional * Index Ratio =1,000,000 * 1.26314 = 1,263,140

The regulation of the markets organized and managed by Borsa Italiana defines **structured bonds**, as the bonds whose repayment and/or interest is indexed to the price level of one of the following financial assets:

- Shares or baskets of shares (**basket linked**).
- Equity indices or baskets of equity indices (**equity index linked**).
- Currencies (**forex linked**).
- Mutual **funds.**
- **Commodities.**

In the category of structured bonds, in addition to the most common embedded options (callable/puttable bonds and floored/capped floaters), non-standard options (called **exotic options**) can also be embedded.
This process studied mainly by financial engineers/structurers allows to provide a higher yield from the fixed income instrument, obviously associated to a greater level of risk.
There is a large number of non-standard options that can be embedded in a security and their classification mainly depends on the characteristics of their **pay-off**. Let us review several of those options:

The **Asian option** is a path-dependent derivative whose pay-off depends on the average price of the underlying asset observed in an agreed time interval during the option life.

The **Barrier option** is a type of derivative which payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price.

Digital options are derivatives whose pay-off is defined in a binary way, all or nothing, depending on the occurrence of an event during the option life.

Lookback options are derivatives whose final value depends on the maximum or minimum level recorded by the underlying asset during a certain time period.

The **Basket option** is a type of financial derivative whose underlying asset is a group, or basket, of commodities, securities, or currencies. This exotic option has all the characteristics of a standard option, but with a strike price based on the weighted values of the basket components.

Lastly, the **Worst of option** is a derivative composed of a bundle of call (put) options all with the same expiration dates but each for a different asset. There must be at least two defined assets, and on the expiry date, only the option of the worst performing asset will be exercised, and then only if it is in-the-money (ITM).

It is also worth mentioning **Drop lock bonds,** securities which are convertible into fixed rate bonds. This particular conversion clause protects the holder from an excessive drop in interest rates. The drop lock bond was born as a standard floating-rate bond, but in which a minimum threshold of interest (called trigger rate) is established. The automatic conversion clause of the fixed rate bond is performed below this threshold. **Mixed-rate bonds** are securities that have floating (/ fixed) coupons in a first phase and fixed (/ floating) coupons in a second phase, which normally lasts until maturity. Unlike drop locks, there is no embedded optionality.

Foreign-exchange indexing

It is the case of an indexation linked to the rate of exchange between different currencies.

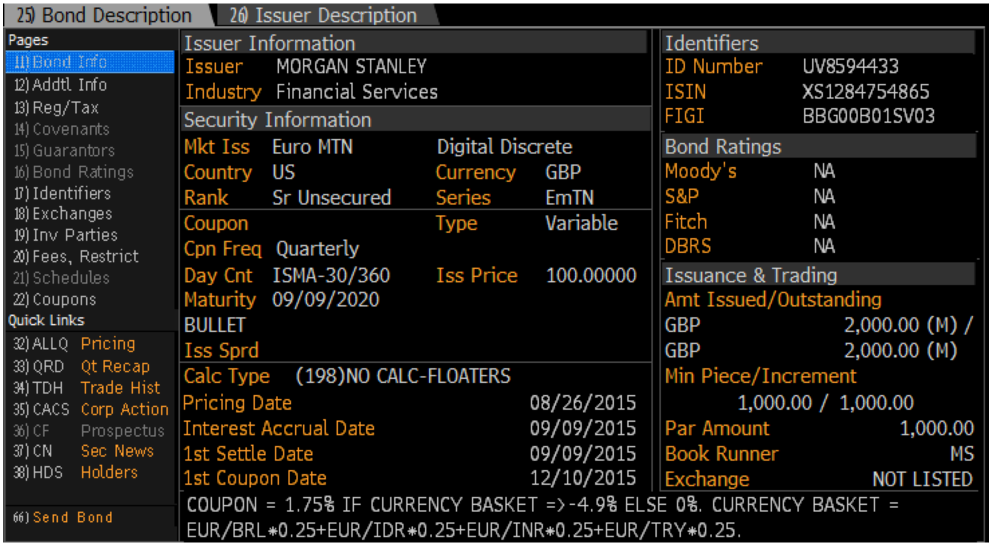


Figure I.42 Basket Option in a forex-linked Bond. Source: Bloomberg®

Financial indexing

It is the case of an indexation linked to the cost of capital on the markets, and it typically relates to the coupons.

Figure I.43 Digital Option in an equity-linked Bond. Source: Bloomberg®

Issuer Information		Identifiers	
Name	UNICREDIT SPA	FIGI	BBG00D0Q0ZP8
Industry	Banking (BCLASS)	ISIN	IT0005185381

Coupon Formula Schedule

Effective Dt	Reset Idx	Spread	Day Cnt	Pay Freq	Fix Freq	Cap	Floor
05/31/16		3.000%	ACT/ACT	Quarterly			
05/31/19	US0003M	1.030%	ACT/ACT	Quarterly	Quarterly		

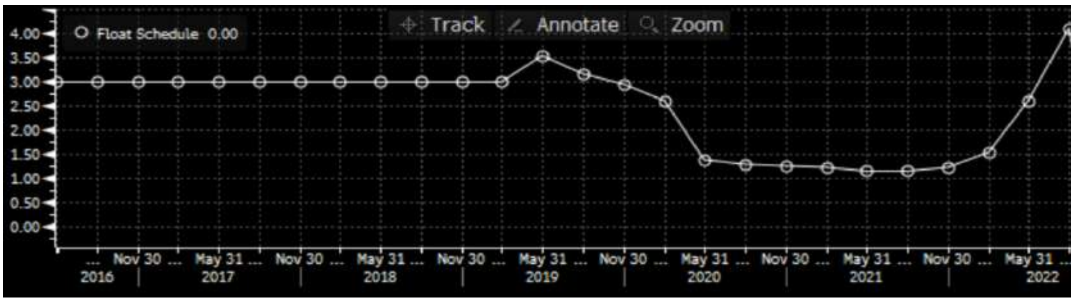


Figure I.44 Mixed rate and Drop lock Bond. Source: Bloomberg®

The constant rate and variable duration bonds are characterized by having a non-prefixed maturity, but variable from a minimum to a maximum period. Such **Extendable bonds** can be modeled as a standard security with a maturity up to the maximum expiration date allowed by the contract, associated with a collection of call options on par.

Issuer Information				Identifiers	
Issuer	CANADIAN IMPERIAL BANK			FIGI	BBG0145F85B9
Industry	Banking (BCLASS)			CUSIP	13607HA28
Security Information				ISIN	CA13607HA285
Mkt Iss	DOMESTIC MTN			Bond Ratings	
Ctry/Reg	CA	Currency	CAD		
Rank	Sr Unsecured	Series	STEP		
Coupon	1.450000	Type	Step-Up F...		
Cpn Freq	S/A				
Day Cnt	30/360	Iss Price	100.0000	Issuance & Trading	
Exp Mat	12/23/2023			Amt Issued/Outstanding	
EXTENDIBLE				CAD	30,000.00 (M) /
Iss Sprd				CAD	30,000.00 (M)
Calc Type	(1311)MULTI-STEP CPN BND			Min Piece/Increment	
Extension Decision Date					5,000.00/ 1,000.00
1st Settle Date			12/23/2021	Par Amount	1,000.00
1st Coupon Date			06/23/2022	Book Runner	CIBCWM-sole
Final Maturity Date			12/23/2023	Exchange	NOT LISTED

Figure I.45 Extendable Bond. Source: Bloomberg®

Bull & bear bonds are instruments that embed a speculative component in relation to the principal amount. The redemption amount is linked to an index, which generally represents the stock market trend, but it can also be referred to a commodity or a specific security or even the exchange ratio between two currencies. As for the interest rate, they ensure a fixed return based on a rate set by the issuer. These bonds can be considered interesting for the subscriber as an innovative form of investment, which mixes the characteristics both of a bond and of a share:

- it guarantees a flow of interest and a repayment of the principal (in a variable measure) as a fixed income instrument, but
- it also gives the possibility of betting on the future trend of the stock market with the risk of capital losses and, specularly, with the possibility of capital gains, as an equity investment.

The **Dual currency bonds**, as the name suggests, are securities whose interest is paid in a currency other than the principal repayment currency.

A broad category is constituted by **Floating rate bonds**, which are proportionally indexed to the reference index, i.e., as the reference rate increases (/ decreases), the paid coupon will have a higher (/ lower) amount. On the other hand, **Reverse floaters** work in the opposite way: as the benchmark increases, the indexed coupon decreases, therefore these instruments are suitable for investors who have an expectation of a decline in the reference index.
These are long-term bonds, whose issue regulation provides for the payment of initial fixed coupons higher than the market rates, and in a second phase, the payment of coupons calculated by applying a variable rate given by the difference between a maximum ceiling determined at issue (i.e. a cap rate) and a floating rate (Euribor). To avoid the risk that the coupon may become zero in the presence of a sharp rise in rates, the bond can provide for a minimum threshold (i.e. a floor rate). The coupon can thus be expressed by the formula:

Coupon Rate = max (floor rate; cap rate – Reference Rate) *(Eq. I.38)*

Let us now consider the type of issuer, in that respect, bonds can be classified into 4 classes:

Domestic bonds, that are issued on the local market by a domestic borrower in the local currency.

Foreign bonds, issued on the local market by a foreign borrower in the local currency. Here are a few examples of them:

- **Yankee bonds,** USD securities issued on the US market by a non-US issuer.
- **Samurai bonds,** that are JPY bonds issued in Japan by non-Japanese companies.
- **Bulldog bonds** which are GBP bonds issued by non-British entities on the UK market.

The third type is the **Euro bond,** a term used for the hypothetical creation of a public debt bond issued by one of the European countries with a single currency, but subscribed by all the states of the Eurozone. In this way, their solvency is jointly guaranteed, reducing the associated risk.

Lastly, the **Global bond** is a type of bond traded both in a domestic market and in a foreign market, whose currency is that of the nation to which the issuer belongs.

Coupon (2019) = 9% - (-0.191%) = 9.191%

Figure I.46 Reverse-Floater. Source: Bloomberg®

Another kind of classification is based on the type of collateral present on the payments, thereby, bonds can be classified into:

Government bonds are securities issued and directly guaranteed by sovereign states.

Government guaranteed bonds are securities issued by entities other than the state but guaranteed by it.

Government Agency bonds are securities issued by affiliated or state-sponsored companies, but which do not benefit from a direct state guarantee.

Supranational Agency bonds are securities issued by a few entities, typically banks, which are controlled by sovereign states. For example: World Bank, European Investment Bank, Asian Development Bank.

Provincial or State bonds are securities issued by sub-national/territorial government bodies (Swiss cantons, municipalities, …)

Last but not least, the category of **Corporate bonds,** securities issued by public or private companies, is quite broad and diversified. In this case, it is assumed that, unlike bonds issued by the state, whose ability to repay the debt is mainly guaranteed by the possibility of raising taxes, for corporate bonds, the company solely relies on its own business. In fact, there are different ways for a company to increase the confidence level in repayments (**bond seniority**). Corporate bonds issued by one single issuer can assume different levels of seniority:

Secured debt: when the debt/bond is guaranteed by specific assets of the issuer. A set of assets are used as a kind of collateral.

Senior debt: it is defined senior when it must be repaid first to the investors, in case of default.

Subordinated: in this case, in the event of default of the issuer, subordinated debt can be repaid only after all the senior notes have been repaid.

In addition, a security can be **backed** in a variety of ways, and depending on the underlying collateral, there are **mortgage-backed** securities, **asset-backed** securities and **covered bonds.**

When an analyst decides to invest in a corporate bond, he must pay close attention to the creditworthiness of the issuer, since, unlike a government bond, it is very variable. Useful information can be found directly from the offer document in which the issuer is required to detail the seniority, the presence of any protection in case of default and the assigned rating. Over time, these assessments made when the bond was issued can become obsolete, so the investor must rely on the score assigned by the rating agencies. By rating we mean a synthetic level of the credit quality of an issuer assigned by an agency that is based on the financial conditions, the reliability of the management and the analysis of the company's balance sheet.
Such opinion can be expressed in a short/medium/long term perspective on the issuer (issuer outlook) and/or on the financial instrument itself. The conventions adopted by the different rating agencies are not exactly the same, but there are equivalence tables between them for uniformity of judgment.

Ticker	Coupon	Maturity	BB R...	Mty Type	Curr	Ask Px	Source	Collateral
BANCAR								
BANCAR	5.000	12/22/2022	NR	BULLET	EUR	91.723	BGN	SR UNSECURED
BANCAR	16.000	11/30/2028	NA	CALLABLE	EUR	19.292	BVAL	SUBORDINATED
BANCAR	1.250	01/28/2021	BBB	BULLET	EUR	100.047	BGN	COVERED
BANCAR	0.000	05/17/2019	NR	BULLET	EUR	101.404	BVAL	UNSECURED
BANCAR	0.000	12/15/2020	NA	BULLET	EUR	94.890	HMTO	UNSECURED
BANCAR	4.000	04/15/2019	NA	BULLET	EUR	99.930	HMTO	SR UNSECURED
BANCAR	4.000	12/03/2019	NA	BULLET	EUR	100.000	HMTO	UNSECURED
BANCAR	13.050	11/30/2028	NA	CALLABLE	EUR	109.574	BVAL	BONDS
BANCAR	FLOAT	06/29/2020	NA	CALLABLE	EUR	22.565	BVAL	SUBORDINATED
BANCAR	0.000	08/20/2019	NA	BULLET	EUR	96.870	HMTO	SR UNSECURED
BANCAR	FLOAT	10/25/2021	BBB	BULLET	EUR	99.585	BVAL	COVERED
BANCAR	FLOAT	02/16/2020	NA	BULLET	EUR	95.020	HMTO	SR UNSECURED
BANCAR	3.500	01/24/2020	NA	BULLET	EUR	98.500	HMTO	SR UNSECURED
BANCAR	5.000	02/15/2020	NA	BULLET	EUR	99.580	HMTO	UNSECURED
BANCAR	FLOAT	04/05/2019	NA	BULLET	EUR	98.000	HMTO	UNSECURED
BANCAR	FLOAT	03/03/2020	NA	BULLET	EUR	94.870	HMTO	SR UNSECURED
BANCAR	5.000	05/03/2020	NA	BULLET	EUR	100.270	HMTO	UNSECURED
BANCAR	0.000	08/10/2019	NA	BULLET	EUR	98.160	HMTO	SR UNSECURED
BANCAR	0.000	08/20/2019	NA	BULLET	EUR	98.040	HMTO	SR UNSECURED

Figure I.47 Classification based on Collateral. Source: Bloomberg®

The main rating agencies, most widely used by companies and trusted by investors are Moody's and Standard & Poor's, while the two main categories of bonds, based on their ratings are the investment grade and the speculative grade bonds, each of them in turn containing several levels of creditworthiness scores.

Investment Grade

Within the Investment Grade category, the different levels of rating considering the Standard & Poor's scale are the following:

AAA: An obligor rated 'AAA' has an extremely strong capacity to meet its financial commitments, it is the highest issuer credit rating assigned by Standard & Poor's.

AA: An obligor rated 'AA' has a very strong capacity to meet its financial commitments, it differs from the previous level only to a small degree, and it includes:

AA+: high quality debt, with a very low credit risk, but susceptibility to long-term risks appears somewhat greater. This level is equivalent to Aa1 in the Moody's rating scale.

AA: equivalent to Moody's Aa2.

AA−: equivalent to Moody's Aa3.

A: An obligor rated 'A' has a strong capacity to meet its financial commitments but is somewhat more

susceptible to the adverse effects of changes in circumstances and economic conditions than obligors in higher-rated categories. This level includes:
A+: equivalent to A1, and
A: equivalent to A2 in the Moody's rating scale.

BBB: An obligor rated 'BBB' has an adequate capacity to meet its financial commitments. However, adverse economic conditions or changing circumstances are more likely to lead to a weakened capacity to repay its debt.

Speculative Grade (or Non-investment)
This category is also composed of several rating levels, as indicated below:

BB: An obligor rated 'BB' is less vulnerable in the near term than other lower-rated obligors. However, it faces uncertainties and is exposed to adverse business, financial, or economic conditions, which could lead to the obligor's inadequate capacity to meet its financial commitments.

B: An obligor rated 'B' is more vulnerable than the previous level, but the obligor currently has the capacity to meet its financial commitments. Adverse business, financial, or economic conditions will likely impair the obligor's capacity or willingness to meet its financial commitments in the future.

CCC: An obligor rated 'CCC' is currently vulnerable, and is dependent upon favorable business, financial, and economic conditions to meet its financial commitments.

CC: An obligor rated 'CC' is currently highly vulnerable.

C: highly vulnerable, perhaps already in bankruptcy or delaying its payments, but still willing to continue paying.

R: An obligor rated 'R' is under regulatory supervision owing to its financial condition. During the pendency of the regulatory supervision, the regulators may have the power to favor one class of obligations over others or to pay some obligations and not others.

SD: has selectively defaulted on some obligations.

D: has defaulted on obligations and S&P believes that it will generally default on most or all obligations.

Generally speaking, the outcome of a financial investment depends on the occurrence of a series of future events that can determine its variability, and the concept of risk is closely linked to this consideration. Risk is in fact inherent to any form of economic activity. There are no financial investments that do not expose to risks. Based on the effects produced by future and uncertain events, a distinction can be made between pure and speculative risk. The **pure risk** is due to events that can only generate damage and losses. The eventuality of a fire, a theft, an accident or an illness are all events that can cause only damage to the exposed person and, therefore, no benefit.

Speculative risk, on the other hand, is linked to future and uncertain events that can have not only unfavorable but also favorable effects. The exposure to risk, typical of financial transactions, is presented under the dual aspect of producing both losses and gains, depending on the type of event, therefore belonging to speculative risk. The overall risk of a financial investment is determined by events and situations of a specific and general nature that highlight various types of risk.

Rating	Moody's	Fitch	S&P	Classificazione
AAA	Aaa	AAA	AAA, AAA-	Investment grade
AA	Aa, Aa1, Aa2, Aa3	AA, AA-, AA+	AA, AA-, AA+	Investment grade
A	A, A1, A2, A3	A, A-, A+	A, A-, A+	Investment grade
BBB	Baa, Baa1, Baa2, Baa3	BBB, BBB-, BBB+	BBB, BBB-, BBB+	Investment grade
BB	Ba, Ba1, Ba2, Ba3	BB, BB- BB+	BB, BB- BB+	Non investment grade
B	B, B1, B2, B3	B, B-, B+	B, B-, B+	Non investment grade
CCC	Caa, Caa1, Caa2, Caa3	CCC, CCC-, CCC+	CCC, CCC-, CCC+	Non investment grade
CC	Ca	CC, CC-, CC+	CC, CC-, CC+	Non investment grade
C	C	C, C-, C+	C	Non investment grade
DDD	DDD	DDD	DDD	Non investment grade
DD	DD	DD	DD	Non investment grade
D	D	D	D	Non investment grade

Table I.9 Rating equivalence table. Source: Borsa Italiana

In fact, the contractors of a financial transaction are exposed to risks attributable both to the behavior of the counterpart (**endogenous factors**), and to situations of a general nature connected to unexpected changes in market variables (**exogenous factors**).

The main types of risk can be summarized as follows:

Insolvency risk: this type of risk occurs first of all when the debtor is unable to honor his debts in full, paying interest and repaying principal.

Migration risk: during the life of the financial transaction, the debtor's creditworthiness may differ from that assigned by the creditor ex-ante, before entering into the contract. A lowering of creditworthiness is determined by an increase in credit risk and vice versa. Even when there are no defaults, the risk of the transaction varies because the creditworthiness changes over time.

Settlement risk: this type of risk is associated with contractual breaches in the settlement of a financial transaction. The counterparty, buyer or seller, of a securities negotiation does not perform the due service, nullifying the exchange intentions of the other operator who must bear the consequences of the cancellation of the operation.

Price/interest risk: changes in interest rates may impact the price at which the financial instrument can be traded before maturity. This risk therefore originates from fluctuations in the sale prices of the security due to changes in market rates.

Reinvestment risk occurs only in the event that the financial instrument provides intermediate income between the time of its purchase and the investor's time horizon. At the time the interest is collected, the operator is unable to reinvest the sum collected at the same return conditions.

Liquidity/negotiability risk, as the name suggests, occurs when unexpected events actually reduce the degree of liquidity of the instrument for the investor. In particular, if the investor needs an early disinvestment, it might prove difficult to implement at a reasonable price and in a reasonable time.

Exchange/currency risk happens when a financial instrument is denominated in a currency different from the reference one, and the investor thus also has to consider the volatility of the exchange rate.

Lastly, **Inflation/monetary risk** occurs because the return on the financial investment is typically expressed in nominal terms, but the holder is often interested in the "real" level of return, net of the inflation rate. If the monetary depreciation was greater than the rate of return on the investment, the final capital shows a depreciation, in real terms.

FURTHER READINGS

Fabozzi F. J. – "The handbook of Fixed Income Securities" – McGraw Hill (2012).
Gabbrielli M., De Bruno S. – "Capire la finanza. Guida pratica agli strumenti finanziari" – Il Sole 24 Ore, Finanza e Mercati (2010).
Giribone P. G. – "L'Educazione Finanziaria ed il Risparmio Gestito: Fixed Income" – CARIGE Academy (2020).
Giribone P. G. – "Dalle obbligazioni standard alle strutturate: definizioni, caratteristiche e tipologie" – CARIGE Academy (2019).

The value of a bond (**fair value**) can be determined using the following criteria written in order of preference:

A) it is equal to the market price observed on **official listed stock exchanges**.

B) it is equal to the price provided by a **contributor** who actively trades the security on the secondary market.

C) if a market price is not available, or if it exists, but is considered unreliable (for example due to lack of trading volumes or infrequent updating of the quotation), a **theoretical pricing** of the security must be taken into consideration.

BTPS 2 ½ 12/01/2032 REGS Corp | Settings | Request Access
17:21:08 ALLX Mode Overlay Axes Split Bid/Offer
Spreads vs DBR 0 08/15/2031 REGS Co @CBBT Derived spreads are not available.
Edit Filters

PCS	Firm Name	Bid Px / Ask Px	Bid Yld / Ask Yld	BSz(MM) x ASz(MM)
MSG1	MSG Quotes	87.440 / 87.530	4.050 / 4.038	x 25
BVAL	BVAL (Score: 10)	87.701 / 87.725	4.016 / 4.013	x
	Last Trade	87.615	--	.17
MILA	MILAN EXCH	87.6500 / 87.9900	4.022 / 3.978	.001 x .155

ENIIM 2 ¾ PERP €↑76.170 +.144 75.627 / 76.713 .000 / .000
As of 09 Sep --x-- Source BGN

ENIIM 2 ¾ PERP REGS Corp | Settings | Request Access
17:16:40 ALLX Mode Overlay Axes Split Bid/Offer
Spreads vs DBR 0 02/15/2030 REGS Co @CBBT Derived spreads are not available.
Edit Filters

PCS	Firm Name	Bid Px / Ask Px	Bid Yld / Ask Yld	BSz(M) x ASz(M)
MSG1	MSG Quotes	75.920 / 76.670	6.899 / 6.747	x
BVAL	BVAL (Score: 9)	75.912 / 76.276	6.901 / 6.827	x
EXCH	EXCHANGE TRADED	75.280 / 76.830	7.031 / 6.715	x
	Last Trade	76.167	--	--
EBNP	BNP PARIBAS eTRADI...	75.750 / 76.750	6.934 / 6.731	3,000 x 3,000
CBBA	CBBT TCA Adjusted	75.930 / 76.480	6.897 / 6.786	1,000 x 1,000
AEMA	ADAMANT ELEC MKT A...	/	6.716 / 6.704	1,000 x 1,000
DZCT	DZ Bank Credit	76.006 / 76.306	6.882 / 6.821	x
LBNP	BNP PARIBAS	75.750 / 76.750	6.934 / 6.731	x
MZLN	MIZUHO EMEA	75.750 / 76.450	6.934 / 6.792	x
IXEP	IBOXX-EURO CORP CO...	75.64371 / 76.75086	6.956 / 6.731	x
BGN	BLOOMBERG GENERIC ...	75.627 / 76.713	6.959 / 6.739	x
HSCT	HSBC	75.625 / 76.625	6.960 / 6.756	x
BBT3	CBBT - FUTURES CLO...	75.62 / 76.69	7.088 / 6.865	x
HVBT	UniCredit Bank AG	75.576 / 76.786	6.970 / 6.724	x
IMIC	INTESA SANPAOLO IM...	75.500 / 76.750	6.985 / 6.731	x
BRLN	BERLIN EXCHANGE	75.280 / 76.830	7.031 / 6.715	x
GERM	GERMAN EXCHANGE	75.280 / 76.830	7.031 / 6.715	x

Figure I.48 Bond Pricing. From top to bottom: Case A, B and C. Source: Bloomberg®

Regulators have introduced the concept of **Fair Value Level** with the aim to provide an indication about the goodness of the price reported in a statement, therefore:

- FVL 1 means that the value can be directly inferred from the market participants (cases A and B).
- FVL 2 means that it is necessary to use a mathematical model for the estimation of the fair value, but all its inputs can be objectively deduced from the market.
- Otherwise, FVL 3 must be applied.

The **discounted future cash flows valuation model** is used for pricing bonds. All future cash flows are computed in accordance with the bond indenture and discounted at the time of valuation (typically at time 0) with an **appropriate** discount factor. The discount factor is estimated starting from the yield curve associated with the issuer, if directly available, or derived from the risk-free rates term structure (zero-curve or spot-curve), to which a risk premium is added. Such risk premiums taken into account for pricing a security are linked to **creditworthiness** and **illiquidity**. When a term structure stripped from the yields of the issuer's bonds is listed, it can directly be used for the construction of the discount curve. In case it is not available, the CDS (Credit Default Swap) curve linked to the issuer is commonly used as a proxy for estimating creditworthiness.

As known, CDS are derivatives that allow to transfer the counterparty risk of default to a third party, the protection seller. The latter receives a periodic flow of payments from the protection buyer and undertakes to repay the par-value, in the event of a credit event, and to receive the defaulted security in return. The amount of the payments constitutes a sort of insurance premium and it is a proxy of how much an issuer is considered reliable. CDS are quoted in basis points and generally senior and subordinate curves are available for each counterparty.

	Curve Id	3M	6M	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y	50Y
11)	I40	0.787	1.834	2.242	2.866	3.337	3.505	3.888	3.896	4.086	4.225	4.372	4.508	4.392	4.491	4.348	4.329	3.971

EUR Italy Sovereign Curve | Actions | 98) Chart | Export | Settings
X-Axis Tenor | Y-Axis Mid YTM | Currency None | PCS BGN
09/30/22 | MM/DD/YY | MM/DD/YY | Last 1D Modify
Values and Members | Values | Members | Constituents

	Tenor	Description	Price	Yield	Source	Update
11)	3M	BOTS 0 11/30/22 Corp	99.876	0.787	BGN	09/30/22
12)	6M	BOTS 0 03/31/23 Corp	99.102	1.834	BGN	09/30/22
13)	1Y	BOTS 0 09/14/23 Corp	97.897	2.242	BGN	09/30/22
14)	2Y	BTPS 0 08/15/24 Corp	94.869	2.866	BGN	09/30/22
15)	3Y	BTPS 1.2 08/15/25 Corp	94.280	3.337	BGN	09/30/22
16)	4Y	BTPS 0 08/01/26 Corp	87.634	3.505	BGN	09/30/22
17)	5Y	BTPS 2.65 12/01/27 Corp	94.429	3.888	BGN	09/30/22
18)	6Y	BTPS 0 ½ 07/15/28 Corp	82.730	3.896	BGN	09/30/22
19)	7Y	BTPS 2.8 06/15/29 Corp	92.757	4.086	BGN	09/30/22
20)	8Y	BTPS 1.65 12/01/30 Corp	82.645	4.225	BGN	09/30/22
21)	9Y	BTPS 0.95 12/01/31 Corp	74.700	4.372	BGN	09/30/22
22)	10Y	BTPS 2 ½ 12/01/32 Corp	84.136	4.508	BGN	09/30/22
23)	15Y	BTPS 0.95 03/01/37 Corp	63.921	4.392	BGN	09/30/22
24)	20Y	BTPS 1.8 03/01/41 Corp	67.016	4.491	BGN	09/30/22
25)	25Y	BTPS 2.7 03/01/47 Corp	75.941	4.348	BGN	09/30/22
26)	30Y	BTPS 2.15 09/01/52 Corp	64.207	4.329	BGN	09/30/22
27)	50Y	BTPS 2.15 03/01/2072 Corp	61.288	3.971	BGN	09/30/22

Figure I.49 EUR Italy Sovereign Yield Curve. Source: Bloomberg®

Figure I.50 Credit Agricole Senior and Subordinated CDS Curves. Source: Bloomberg®

Typically, CDS premiums are added to the zero-rates of the risk-free discount curve, and this kind of risk adjustment is called **Z-spread**. If the issuer of a bond does not have a listed credit curve, then two approaches may to be used:

- the averages of the sector spreads (**spread curves**) broken down by rating and therefore by seniority.
- the yields of similar listed bonds used as a proxy (**comparables**).

The premium for the illiquidity of a bond is more complicated to consider and there are still no universally accepted models. Bloomberg® recently proposed a mathematical approach related to the probability of success with which traders are able to close a trade on the secondary market.

Spread Type Z-Spd · Date Range 04/18/22 - 10/14/22 6M

Comparable Bonds View — Difference In Comparable Z-Spreads Over 6 Months

Security	Price	Yield	Spread	Diff	Low	Range (◆ Avg ● Now)	High	Avg	+/- ...	#SDs
Avg of Comparables		5.93	245	-65	-65		-33	-48	-17	-2.8
12) AAPL 4 3/8 05/13/45	87.72	5.31	180							
13) MSFT 4 7/8 12/15/43	94.30	5.33	177	3	-11		18	1	2	0.3
14) TXN 3 7/8 03/15/39	83.66	5.39	172	8	5		26	16	-8	-1.8
15) CRM 2.7 07/15/41	67.65	5.48	189	-9	-12		10	0	-9	-1.7
16) MSFT 3 3/4 02/12/45	80.86	5.21	170	10	3		25	13	-3	-1.0
17) TXN 4.1 08/16/52	82.96	5.23	191	-11	-11		0	-5	-6	-2.0

Figure I.51 Comparable Analysis. Source: Bloomberg®

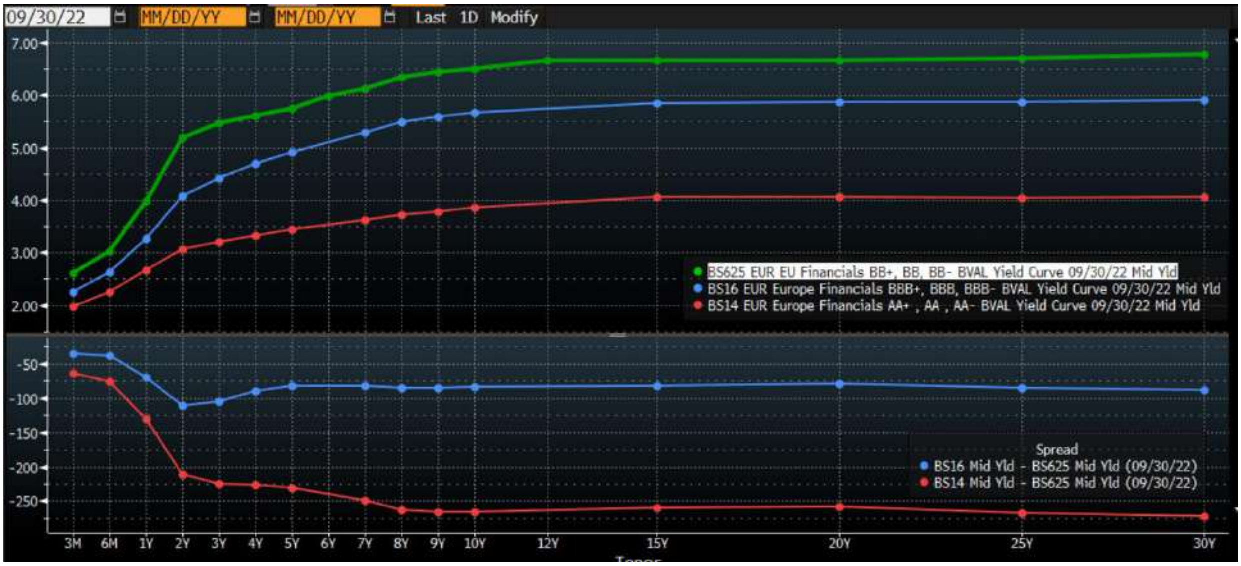


Figure I.52 Sector Spreads: EUR EU Financials yield curve. Source: Bloomberg®

As known, the theoretical price of a bond is based on the discounted cash flow method and the equilibrium concept. Let us review a few examples. The simplest security to consider is a zero-coupon bond which pays a single cash flow, CF_t, at the end of period t. The price of this bond, denoted by $B_{0,t}$, is equal to the value of the single discounted cash flow:

$$B_{0,t} = \frac{CF_t}{(1+k)^t} \text{ (Eq. I.39)}$$

Where CF_t is the cash flow received at the end of period t and k is the appropriate discount rate.

Let us calculate the price of a zero-coupon bond that will pay € 1,000 in 4 years, assuming a discount rate of 8%? Then, the price, if it expires in 6 years, always assuming a discount rate of 8%?

$$B_{0,t=4} = \frac{1000}{(1.08)^4} = € \ 735.03, \ \ B_{0,t=6} = \frac{1000}{(1.08)^6} = € \ 630.17$$

In the previous example, the discount rate was assumed to be constant regardless of the maturity of the security. In a real market context, this is interpolated starting from the zero-rates curve (spot curves), therefore it is necessary to index the discount by adding the time subscript. Thus, the formula has to be adjusted to use different discount rates for the different maturities:

$$B_{0,t} = \frac{CF_t}{\left(1+R_{0,t}\right)^t} \text{ (Eq. I.40)}$$

Let us calculate the price of a zero-coupon bond that pays € 1,000 in 4 years, assuming a spot rate ($R_{0,4}$) of 5%? Then, the price if it expires in 6 years, assuming a spot rate ($R_{0,6}$) of 6%?

$$B_{0,4} = \frac{CF_4}{\left(1+R_{0,4}\right)^4} = \frac{1000}{(1.05)^4} = € \ 822.70, \ \ B_{0,6} = \frac{CF_6}{\left(1+R_{0,6}\right)^6} = \frac{1000}{(1.06)^6} = € \ 704.96$$

This concept can be extended to a security that pays a coupon interest, i.e. a coupon-bearing bond. We know that a bond that pays out coupons can be seen as a series of cash flows that can be replicated by a portfolio of zero-coupon bonds. For example, if Investor A has a straight bond that expires in 4 years, with a nominal value of € 1,000 and a coupon rate of 6%then Investor B can replicate this security by buying the following zero-coupon bonds:

- One that expires in a year and pays € 60 at maturity.
- One that expires in two years and pays € 60 at maturity.
- One that expires in three years and pays € 60 at maturity.
- One that expires in four years and pays € 1,060 at maturity.

Investor B receives exactly the same future cash flows as A and therefore the price of the security must be equal to that of the equivalent zero-coupon portfolio. If this were not the case, arbitrage opportunities would arise: the cheaper of the two would be bought, the other would be sold short, its cash flows would be reconstructed synthetically (cash flow matching) and the differential of the two would be earned as a profit without any risk

(risk-free profit). We can conclude that the price of a bond can be computed using the following equation:

Bond Price = Price of replicating zero-coupon bond portfolio *(Eq. I.41)*

Since the price of the portfolio is equal to the sum of the prices of all the zero-coupon bonds, the value of any coupon-bearing bond is the sum of the discounted values of all its payments.

$$P = \sum_{t=1}^{T} B_{0,t} = \sum_{t=1}^{T} \frac{CF_t}{(1+R_{0,t})^t} = \frac{CF_1}{(1+R_{0,1})^1} + \frac{CF_2}{(1+R_{0,2})^2} + \cdots + \frac{CF_T}{(1+R_{0,T})^T} \quad \text{(Eq. I.42)}$$

Where CF_t is the cash flow received at the end of period t (coupon or repayment), T is the number of years remaining before maturity (time to maturity) and $R_{0,t}$ is the spot rate of the reference term structure (risk -free, risk-free + risk premium or issuer/sector curve).

Let us examine an example, and calculate the price of a straight bond that expires in 4 years, at par (100) supposing its annual coupons are equal to 2% p.a. The spot rates interpolated from the risk-free zero-curve are: $R_{0,1} = 0.5\%$, $R_{0,2} = 0.75\%$, $R_{0,3} = 0.9\%$, $R_{0,4} = 1.5\%$

$$P = \sum_{t=1}^{T=4} B_{0,t} = \sum_{t=1}^{4} \frac{CF_t}{(1+R_{0,t})^t} = \frac{2}{(1.005)^1} + \frac{2}{(1.0075)^2} + \frac{2}{(1.009)^3} + \frac{102}{(1.015)^4}$$

Time [years]	0	1	2	3	4	4
CFs [%]	0	2	2	2	2	100
Discount Rate	-	0.5%	0.75%	0.9%	1.5%	1.5%
Discount Factor	1	0.99502	0.98517	0.97348	0.94218	0.94218
NPV [%]	0	1.99005	1.97033	1.94696	1.88437	94.2184

The risk-free price of the bond is 102.01%.

Let us now add more information. Specifically, the security is a subordinated corporate bond, issued by a bank with a low creditworthiness, therefore the risk-free price cannot be considered as a good approximation to the fair value of the bond.

In this case, given that the issuer has neither listed CDS nor actively contributed prices of comparables on the secondary market, we decide to use a generic sector spread for the evaluation.

The average subordinated spread (S_t) curve (expressed in basis points) for a bank issuer with a BB rating is: $S_1 = 800\ bps$, $S_2 = 850\ bps$, $S_3 = 895\ bps$ and $S_4 = 945\ bps$.

Since the time to maturity of the bond is 4 years, the spread $S_{T=4} = 945\ bps$ is used to estimate the NPV.

The adjusted-risk price of the bond is thus 72.27%.

Time [years]	0	1	2	3	4	4
CFs [%]	0	2	2	2	2	100
Risk Free Disc. Rate	-	0.50%	0.75%	0.90%	1.5%	1.5%
Spread	-	9.45%	9.45%	9.45%	9.45%	9.45%
Risk Adj. Disc. Rate	-	9.95%	10.20%	10.35%	10.95%	10.95%
Discount Factor	1	0.90950	0.82345	0.74419	0.65992	0.65992
NPV [%]	0	1.81901	1.64690	1.48838	1.31984	65.9919

If the bond reimbursement is not "at par", but "at premium" or "at discount", the pricing methodology remains the same.

We now calculate the price of a straight bond that expires in 4 years, with 1,000 of principal and an annual coupon equal to 5% p.a. We also know that this the security will redeem with a 2% premium, or 102%. We first proceed to calculate the interpolated spot rates, which are: $R_{0,1} = 7\%$, $R_{0,2} = 8\%$, $R_{0,3} = 8.5\%$, $R_{0,4} = 9\%$.

$$P = \sum_{t=1}^{T=4} B_{0,t} = \sum_{t=1}^{4} \frac{CF_t}{\left(1 + R_{0,t}\right)^t} = \frac{50}{(1.07)^1} + \frac{50}{(1.08)^2} + \frac{50}{(1.085)^3} + \frac{50 + 1020}{(1.09)^4}$$

Time [years]	0	1	2	3	4	4
CFs	0	50	50	50	50	1020
Discount Rate	-	7%	8%	8.5%	9%	9%
Discount Factor	1	0.93458	0.85734	0.78291	0.70843	0.70843
NPV	0	46.72897	42.86694	39.14540	35.42126	722.59372

We reach the theoretical value of the bond, USD 886.76.

If the security pays infra-annual coupons, the same formula can be applied, but the financial quantities in the reference time period have to be appropriately "rescaled". The formula is the usual one:

$$P = \sum_{t=1}^{T} B_{0,t} = \sum_{t=1}^{T} \frac{CF_t}{\left(1 + R_{0,t}\right)^t} = \frac{CF_1}{\left(1 + R_{0,1}\right)^1} + \frac{CF_2}{\left(1 + R_{0,2}\right)^2} + \cdots + \frac{CF_T}{\left(1 + R_{0,T}\right)^T}$$

We assume a semi-annual frequency, and we have:
CF_t is the cash-flow (coupon or principal) received at the end of period t, (in this case the semester).

T is the number of periods (i.e. semesters) that remains until maturity.
$R_{0,T}$ is the rate of return for lending money from time 0 to the end of semester t, or the spot rate proportioned to the time period with the formula: $i_m = (1 + i_A)^{\frac{1}{m}} - 1$.

Let us take as an example a bond that expires in 2 years and pays a semi-annual coupon at an interest rate of 3% p.a. on a reference notional equal to € 100, the bond repaying at par. The spot rates, expressed using the traditional annual basis, are:

$R_{0,6M} = 5\%$, $R_{0,1Y} = 6\%$, $R_{0,1.5Y} = 6.5\%$, $R_{0,2Y} = 7.5\%$ (Rates expressed on **annual basis**)

Let us remember to convert the financial figures in order to compare them in the semi-annual period. Thus, the half-yearly Cash Flow is equal to $\frac{C}{2} = 1.5\%$.
In order to convert rates expressed on an annual basis into a half-yearly basis, we have to use the formulas for equivalent rates. As a result, the converted discount rates are the following:

$$R_{0,6M} = (1 + 0.05)^{\frac{1}{2}} - 1 = 2.4695\%$$
$$R_{0,1Y} = (1 + 0.06)^{\frac{1}{2}} - 1 = 2.9563\%$$
$$R_{0,1.5Y} = (1 + 0.065)^{\frac{1}{2}} - 1 = 3.1988\%$$
$$R_{0,2Y} = (1 + 0.07)^{\frac{1}{2}} - 1 = 3.6822\%$$

Rates are now expressed on a **semi-annual basis** and we can implement our calculation:

$$P = \sum_{t=1}^{4} \frac{CF_t}{\left(1 + R_{0,t}\right)^t} = \frac{1.5}{(1.024695)^1} + \frac{1.5}{(1.029563)^2} + \frac{1.5}{(1.031988)^3} + \frac{100 + 1.5}{(1.036822)^4}$$

The theoretical bond price is 92.075%.

Note that the same result could be obtained by directly using the discount rates on an annual basis but expressing the time periods in year fractions. The formula would then be:

$$P = \frac{1.5}{(1.05)^{0.5}} + \frac{1.5}{(1.06)^1} + \frac{1.5}{(1.065)^{1,5}} + \frac{100 + 1.5}{(1.075)^2} = 92.075$$

Time [years]	0	1	2	3	4	4
CFs [%]	0	1.5	1.5	1.5	1.5	100
Discount Rate	-	2.4695%	2.9563%	3.1988%	3.6822%	3.6822%
Discount Factor	1	0.97590	0.94340	0.90986	0.86533	0.86533
NPV [%]	0	1.46385	1.41509	1.36479	1.29800	86.53326

Time [years]	0	0.5	1	1.5	2	2
CFs [%]	0	1.5	1.5	1.5	1.5	100
Discount Rate	-	5%	6%	6.5%	7.5%	7.5%
Discount Factor	1	0.97590	0.94340	0.90986	0.86533	0.86533
NPV [%]	0	1.46385	1.41509	1.36479	1.29800	86.53326

Let us now consider a real market case. The aim is to understand if the market price for a traded security is close enough to the theoretical price.

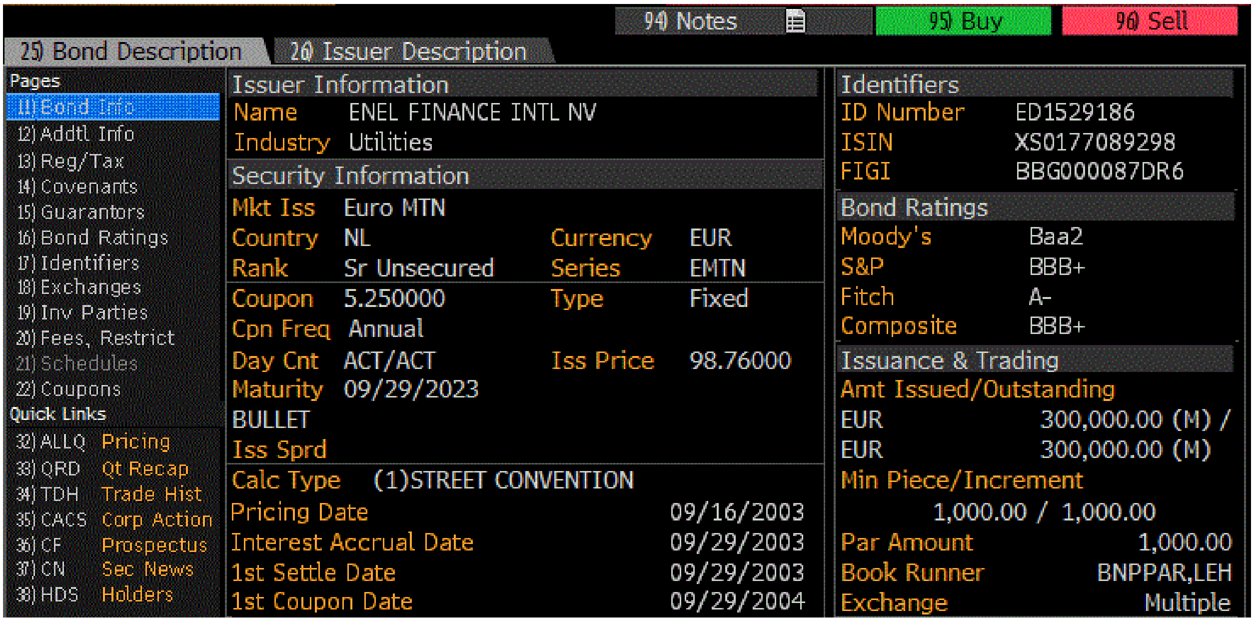


Figure I.53 Bullet bond pricing. Source: Bloomberg®

T	0.5	1	2	3	4	5	7	10
Spread	16.8	19	30	49	71	89	137	171

Table I.10 Enel CDS Senior. Reference Date: 2019/02/25. Source: Bloomberg®

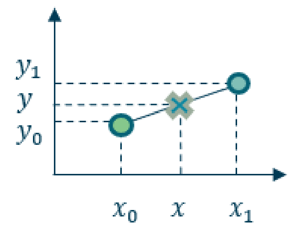
T	Date	Zero Rate
0.50137	27/08/2019	-0.23000%
0.586301	27/09/2019	-0.22804%
0.671233	28/10/2019	-0.22660%
0.753425	27/11/2019	-0.22363%
0.835616	27/12/2019	-0.22040%
0.920548	27/01/2020	-0.21752%
1.005479	27/02/2020	-0.21384%
1.084932	27/03/2020	-0.20998%
1.169863	27/04/2020	-0.20577%
1.252055	27/05/2020	-0.20136%
1.342466	29/06/2020	-0.19598%
1.419178	27/07/2020	-0.19194%
1.50411	27/08/2020	-0.18694%
2.005479	26/02/2021	-0.15456%
3.010959	28/02/2022	-0.06937%
4.008219	27/02/2023	0.02771%
5.008219	27/02/2024	0.13020%
6.010959	27/02/2025	0.23706%

Figure I.54 Interest rates and CDS term structure for bond pricing. Source: Bloomberg®

Eval Date	25/02/2019			Z-spread	81.65607886	
29/09/2018	Coupon	T	Risk Free Zero Rate	Adj. Zero Rate	Discount Factor	NPV
29/09/2019	5.25	0.591780822	-0.2279%	0.5886%	0.996532933	5.231797897
29/09/2020	5.25	1.592339261	-0.1812%	0.6353%	0.989966245	5.197322786
29/09/2021	5.25	2.592153285	-0.1049%	0.7117%	0.98178474	5.154369883
29/09/2022	5.25	3.592060233	-0.0128%	0.8038%	0.971653422	5.101180465
29/09/2023	105.25	4.592004381	0.0875%	0.9041%	0.95951254	100.9886948
					Dirty Price	121.6733659
					Accr. Interest	2.143150685
					Clean Price	119.5302152

Using the data shown in the table, we can proceed:

T is the year fraction between the payment date and the evaluation.



Z-Spread is linearly interpolated from the CDS spread.

$$y = y_0 + (x - x_0)\frac{y_1 - y_0}{x_1 - x_0}$$

$$Z_{\text{spread}} = 71 + (4.592004 - 4)\frac{89 - 71}{5 - 4} = 81.656 \text{ bps},$$

and Zero-rates are linearly interpolated from the spot curve:

$$R_{0,0.5918} = -0.22804 + (0.5928 - 0.5863)\frac{-0.2266 - (-0.22804)}{0.6712 - 0.5863} = -0.22795\%$$

$$\text{Adj. ZeroRates} = \text{Risk Free Zero Rate} + \text{Spread} = -0.22795\% + 0.8166\% = 0.58865\%$$

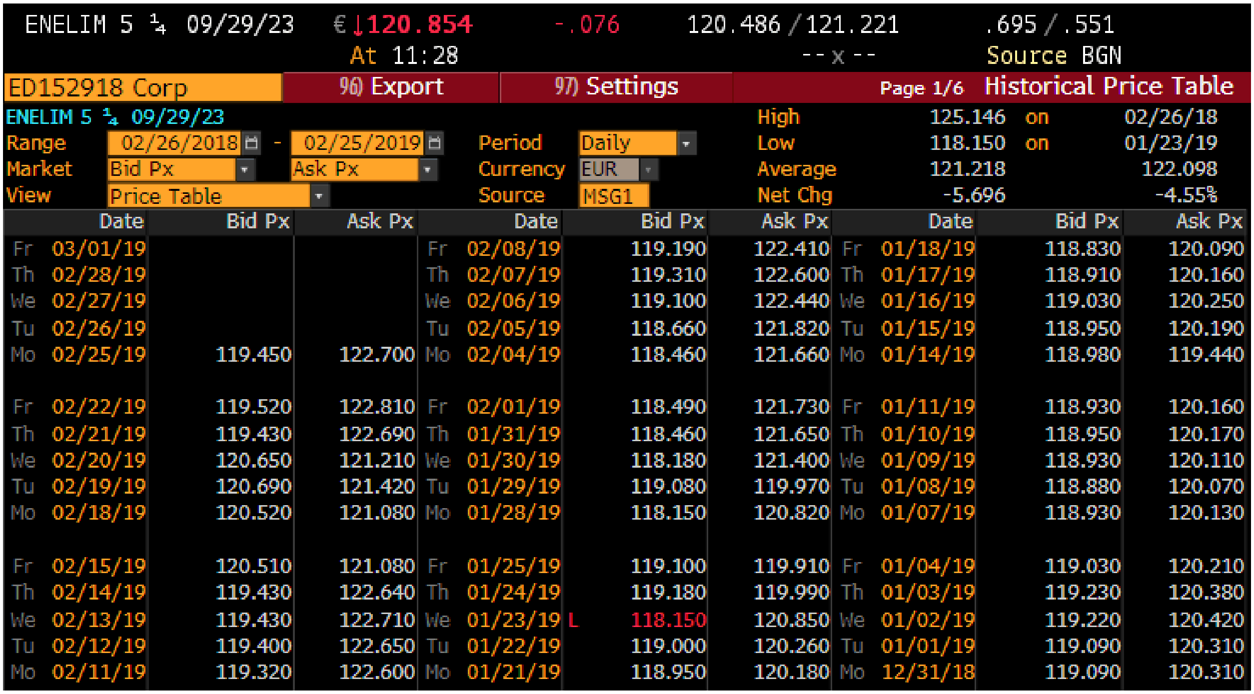
$$\text{Discount Factor} = \frac{1}{(1 + \text{Adj.Zero Rate})^T} = \frac{1}{(1 + 0.0058865)^{0.5918}} = 0.99653$$

$$\text{NPV(CashFlow)} = \text{CashFlow} \cdot DF = 5.25 \cdot 0.99653 = 5.2318$$

We then apply the same procedure for the other four future cash-flows, and the sum of these discounted amounts returns the gross price of the bond, equal to € 121.67337.
Given that the valuation date does not correspond with a payment date, we have to subtract the accrued interest from the gross price taking into account the proper day count convention, in this example ACT/ACT.

$$\text{Accrued Interest} = 5.25 \cdot 0.4082192 = 2.143151$$

$$\text{Clean Price} = 121.67337 - 2.143151 = 119.53022$$



ENELIM 5 ¼ 09/29/23 €↓120.854 -.076 120.486 / 121.221 .695 / .551
At 11:28 -- x -- Source BGN

ED152918 Corp | 96) Export | 97) Settings | Page 1/6 Historical Price Table

ENELIM 5 ¼ 09/29/23

Range 02/26/2018 - 02/25/2019 | Period Daily | High 125.146 on 02/26/18
Market Bid Px, Ask Px | Currency EUR | Low 118.150 on 01/23/19
View Price Table | Source MSG1 | Average 121.218 122.098
Net Chg -5.696 -4.55%

Date	Bid Px	Ask Px	Date	Bid Px	Ask Px	Date	Bid Px	Ask Px
Fr 03/01/19			Fr 02/08/19	119.190	122.410	Fr 01/18/19	118.830	120.090
Th 02/28/19			Th 02/07/19	119.310	122.600	Th 01/17/19	118.910	120.160
We 02/27/19			We 02/06/19	119.100	122.440	We 01/16/19	119.030	120.250
Tu 02/26/19			Tu 02/05/19	118.660	121.820	Tu 01/15/19	118.950	120.190
Mo 02/25/19	119.450	122.700	Mo 02/04/19	118.460	121.660	Mo 01/14/19	118.980	119.440
Fr 02/22/19	119.520	122.810	Fr 02/01/19	118.490	121.730	Fr 01/11/19	118.930	120.160
Th 02/21/19	119.430	122.690	Th 01/31/19	118.460	121.650	Th 01/10/19	118.950	120.170
We 02/20/19	120.650	121.210	We 01/30/19	118.180	121.400	We 01/09/19	118.930	120.110
Tu 02/19/19	120.690	121.420	Tu 01/29/19	119.080	119.970	Tu 01/08/19	118.880	120.070
Mo 02/18/19	120.520	121.080	Mo 01/28/19	118.150	120.820	Mo 01/07/19	118.930	120.130
Fr 02/15/19	120.510	121.080	Fr 01/25/19	119.100	119.910	Fr 01/04/19	119.030	120.210
Th 02/14/19	119.430	122.640	Th 01/24/19	119.180	119.990	Th 01/03/19	119.230	120.380
We 02/13/19	119.430	122.710	We 01/23/19 L	118.150	120.850	We 01/02/19	119.220	120.420
Tu 02/12/19	119.400	122.650	Tu 01/22/19	119.000	120.260	Tu 01/01/19	119.090	120.310
Mo 02/11/19	119.320	122.600	Mo 01/21/19	118.950	120.180	Mo 12/31/18	119.090	120.310

Figure I.55 Market quotes for the analyzed bond

The convenience of working in terms of year fractions is evident when the valuation date of the bond is placed between two payment dates, or when the frequency of the security payments is not regular. In fact, certain bonds have an irregular coupon payment date, typically on the first or the last coupon payment date. If it is a standard floating bond, then the pricing model remains the same. The future coupons are estimated starting from the implicit forward rates of the reference interest rate curve between two consecutive payment dates. Typically, an **additive margin** (expressed in basis points) can be added to them and/or multiplied by a factor (multiplier or **leverage**). The discounting phase takes place using the usual procedure.
Please also note that the same pricing procedure can be adopted for estimating the fair value of a sinkable bond. In this case, the cash-flows are constituted by the coupon interest, and a part of notional that is repaid before the expiration date.

As we have seen, the price of a bond is directly linked to the incoming cash-flows (coupon and repayment) and inversely linked to the discount rate. We can define the Current Yield as the ratio between the annual coupon paid and the market price of the bond, i.e. the Clean Price or Net Price:

Current Yield = Annual Coupon / Net Price *(Eq. I.43)*

When the coupon rate is fixed, the Current Yield is inversely proportional to the price of the bond. Such figure cannot be considered satisfactory for comparing two bonds though, for several reasons.
First of all, given that zero coupon bonds have no coupon by definition, the current yield will be zero for all these securities.
Secondly, it does not take into account the time to maturity of the bond. Then, it also does not consider the frequency of coupons and the rewards deriving from their reinvestment. Lastly, it does not consider the difference between the redemption value and the purchase price.

A little adjustment to the previous formula can be made, by considering the accrued interest, thus the formula becomes:

Annual Coupon / Gross Price *(Eq. I.44)*

Unfortunately, this method inherits most of the previous drawbacks.
The only figure that allows to avoid the criticisms raised against the previous ratios is the effective rate of return, also called **yield to maturity** (YTM).
The YTM is the discount rate that equates the present value of the future cash flows of the bond with its market value.
In the case of a bond that has just paid the coupon and is characterized by a regular frequency of payments, we have:

$$P = \sum_{t=1}^{T} \frac{CF_t}{(1+YTM)^t} = \frac{CF_1}{(1+YTM)^1} + \frac{CF_2}{(1+YTM)^2} + \cdots + \frac{CF_T}{(1+YTM)^T} \quad \textit{(Eq. I.45)}$$

Where:

P is the market price of the bond including the accrued interest.

CF_t is the cash flow received at the end of time period t (coupons and/or repayment).

T is the residual life of the bond (i.e. the time to maturity).

Let us make a numerical example, in which the YTM is unknown:

$$109 = \frac{5}{(1+YTM)^1} + \frac{5}{(1+YTM)^2} + \frac{5}{(1+YTM)^3} + \frac{5}{(1+YTM)^4} + \frac{105}{(1+YTM)^5}$$

In this case, the coupon has just been paid thus the Accrued Interest is zero, therefore, the Dirty Price is equal to the Clean Price.

The table shows the steps for computing YTM.

			YTM guess [C]
Period	Year [A]	CFs [B]	**4.50%**
t1	1	5	4.7847
t2	2	5	4.5786
t3	3	5	4.3815
t4	4	105	88.0489
		NPV	101.7938
		MKT Price	109
		gap	7.2062

Goal Seeking →

			YTM
Period	Year	CFs	**2.602%**
t1	1	5	4.8732
t2	2	5	4.7496
t3	3	5	4.6291
t4	4	105	94.7472
		NPV	108.9991
		MKT Price	109
		gap	0.0008

NPV row = [B]/([1+[C]])^[A], NPV = SUM(NPV rows)

$$112.5 = \frac{6}{(1+YTM)^{0.25}} + \frac{6}{(1+YTM)^{1.25}} + \frac{6}{(1+YTM)^{2.25}} + \frac{6}{(1+YTM)^{3.25}} + \frac{106}{(1+YTM)^{4.25}}$$

			YTM guess
Period	Year	CFs	**4.50%**
t1	0.25	6	5.9343
t2	1.25	6	5.6788
t3	2.25	6	5.4342
t4	3.25	6	5.2002
t5	4.25	106	87.9147
		NPV	110.1623
		MKT Price	108
		Accrued	9/12*6=4.5
		Full Price	112.5
		gap	2.3377

Goal Seeking →

			YTM
Period	Year	CFs	**3.915%**
t1	0.25	6	5.9427
t2	1.25	6	5.7188
t3	2.25	6	5.5034
t4	3.25	6	5.2961
t5	4.25	106	90.0391
		NPV	112.4999
		MKT Price	108
		Accrued	4.5
		Full Price	112.5
		gap	0.0001

Let us introduce a new, and increasingly popular concept, i.e. the **Greenium.** The term was introduced as a consequence of the increasing importance of the **ESG** (Environmental, Social, and Governance) factors in the financial world. It refers to the phenomenon that occurs when an investor has to pay a higher price for a "sustainable" financial instrument compared to an equivalent "non-green" bond. Let us present the calculation of the yield of the following Green Bond:

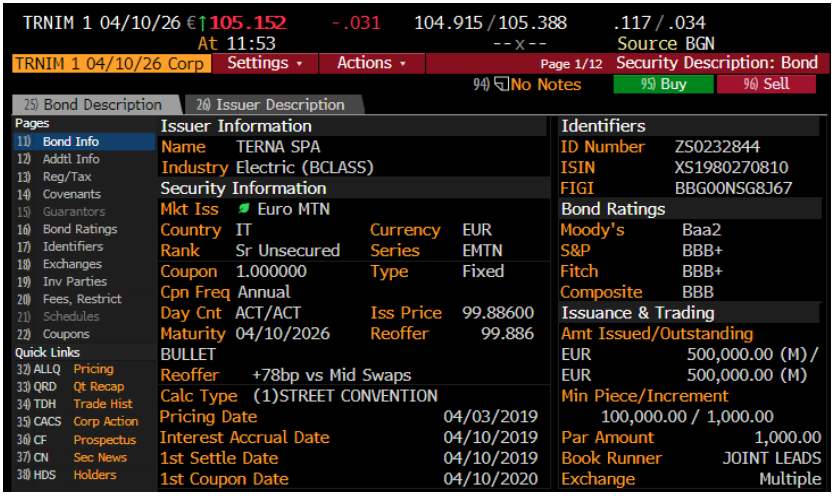


Figure I.56 Green bond analysis. Source: Bloomberg®

Eval Date	09/03/2020		YTM	0.1191%	
04/10/2020	Coupon	T	Rate	Discount Factor	NPV
04/10/2021	1	0.6	0.1191%	0.9993	0.9993
04/10/2022	1	1.5985	0.1191%	0.9981	0.9981
04/10/2023	1	2.5982	0.1191%	0.9969	0.9969
04/10/2024	1	3.5988	0.1191%	0.9957	0.9957
04/10/2025	1	4.5985	0.1191%	0.9945	0.9945
04/10/2026	101	5.5984	0.1191%	0.9936	100.3293
				Dirty Price	105.3139
				MKT Price	104.915
				Accr. Interest	0.3989
				MKT Dirty Price	105.3139
				Gap	-2.894E-06

Bid Market Price = 104.915, Reference Date: 3rd September 2020

The Green Bond yield is equal to 0.1191%. We now compare this yield with a similar "brown" (i.e., non-green) bond issued by the same Company, with the aim to presume the value of the greenium.

Figure I.57 Bullet bond analysis. Source: Bloomberg®

Eval Date	09/03/2020		YTM	0.2034%	
03/03/2020	Coupon	T	Rate	Discount Factor	NPV
03/03/2021	1.6	0.4959	0.2034%	0.9989	1.5984
03/03/2022	1.6	1.4945	0. 2034%	0.9970	1.5951
03/03/2023	1.6	2.4942	0. 2034%	0.9949	1.5919
03/03/2024	1.6	3.4948	0. 2034%	0.9929	1.5887
03/03/2025	1.6	4.4945	0. 2034%	0.9909	1.5855
03/03/2026	101.6	5.4943	0. 2034%	0.9889	100.4718
				Dirty Price	108.4314
				MKT Price	107.627
				Accr. Interest	0.8044
				MKT Dirty Price	108.4314
				Gap	-4.593E-05

Bid Market Price = 107.627, Reference Date: 3rd September 2020.

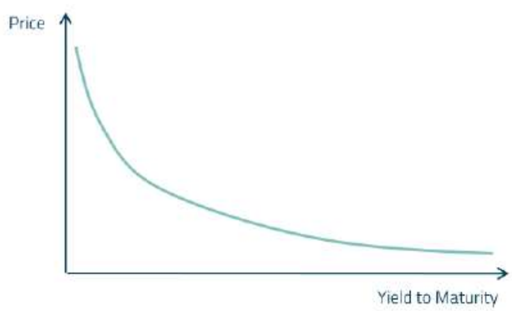
The discrepancy between the yield for a standard bond and a green bond is around 8 basis points. This estimation is close to the typical greenium value reported by the scientific financial literature.
Let us now add the concept of reinvestment to our analysis. The total return realized from holding a bond in a given period of time can be broken down into three components:

Total return = Price return + Coupon return + Reinvestment return *(Eq. I.46)*

Examining each of these components, we know that the price component is closely linked to changes in the market quote, and that the coupon component is the periodic remuneration of interest in the form of coupons. The third component, i.e., the reinvestment component is linked to the income generated by the reinvestment of all cash flows previously received (it constitutes the "interest on interest"). Let us further analyze the first term, the Price return, which can in turn be broken down into two components:

Price return = Price return due to yield change + Amortisation of premium/ discount *(Eq. I.47)*

The first term comes from the fact that a minimal change in the spot rates used in the discounting process will be instantly reflected on the value of the security. While the second contribution comes from the fact that the price of a bond will always converge to its face value. As known, the current yield is linked to the bond price according to an inversely proportional relation, with the obvious exception of zero-coupon bonds. We now add that there is also an inverse law between price and yield to maturity, although this relationship is not so straightforward.



The YTM is the IRR (Internal Rate of Return) of the bond, but even so, YTM can only be a proxy for total return. Firstly, because reinvestments are not actually made at the YTM rate, they are made at market rate. Then, reinvestment risk is also particularly relevant in long bonds with high coupons. Using YTM as a proxy for expected return implies that a single constant rate is assumed for lending and borrowing money, regardless of maturity. It is important to highlight that yield to maturity and total return are two different concepts though.

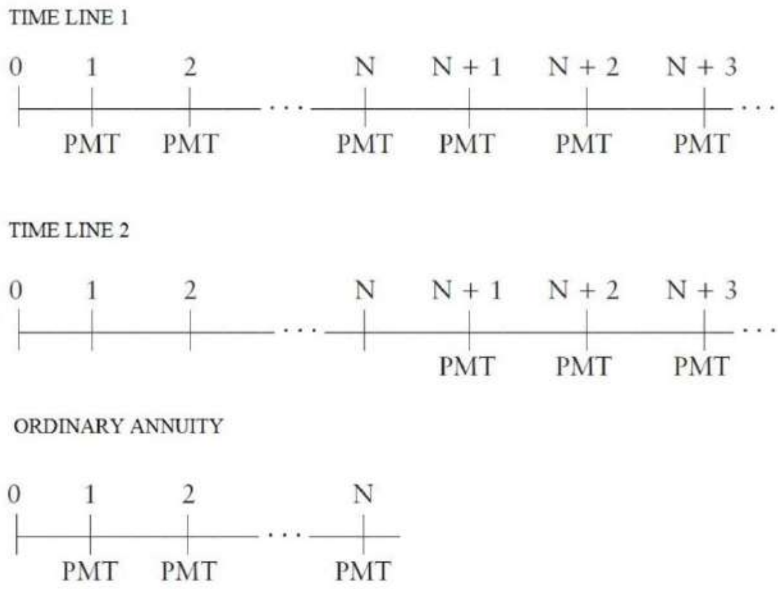
We will clarify through an example. An investor buys a bond that matures in 4 years, with an annual coupon of 5%, bought at par at year 0, therefore YTM=5%. Let us assume that interest rates in subsequent years are falling; thus, the coupon interest received in year 1 can be re-invested for the next 3 at 4%, the coupon collected in year 2 can be invested for the following 2 years at a rate of 2.5%, and the coupon received in year 3 can be reinvested at a rate of 1% for the last year. We want to determine the total return realized in 4 years from the investment. If we assume an initial investment of € 100, the final capital in 4 years can be calculated as follows::

$$5*1.04^3 + 5*1.025^2 + 5*1.01 + 105 = € 120.927$$

The total return on investment (i.e. the realized return) is therefore equal to:

$$100(1+r)^4=120.927 \rightarrow r= (120.927/100)^{1/4} -1 = 4.865\%$$

Annuity and Perpetual bond pricing



Let us now consider the following geometric progression where A is a positive constant below 1 and X is the sum of the progression:

$$X = \sum_{t=1}^{\infty} A^t \text{ (Eq. I.48)}$$

We notice that, as t increases, the term A^t becomes smaller and smaller given that $A < 1$.

For instance, if we set $A = 0.5$, then:

$$X = \sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots, X \rightarrow 1$$

The empirical test confirms the asymptotic convergence of the geometric progression to the quantity: $X = \sum_{t=1}^{\infty} A^t = \frac{A}{1-A}$.

In the previous case: $X = \sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

Now let us consider a perpetuity characterized by a constant payment PMT and a discount rate I.

The Present Value, PV, of this perpetuity is equal to:

$$PV = \sum_{t=1}^{\infty} \frac{PMT}{(1+I)^t} \text{ (Eq. I.49)}$$

This can be rewritten as a geometric progression:

$$PV = PMT \sum_{t=1}^{\infty} \frac{1}{(1+I)^t} = PMT \sum_{t=1}^{\infty} \left(\frac{1}{1+I}\right)^t \text{ (Eq. I.50)}$$

In this last equation, the summation turns out to be a geometric progression with $A = \frac{1}{1+I}$

$$\sum_{t=1}^{\infty}\left(\frac{1}{1+I}\right)^{t} = \frac{\left(\frac{1}{1+I}\right)}{1-\left(\frac{1}{1+I}\right)} = \frac{\left(\frac{1}{1+I}\right)}{\left(\frac{1+I-1}{1+I}\right)} = \left(\frac{1}{1+I}\right)\left(\frac{1+I}{I}\right) = \frac{1}{I} \rightarrow PV = \frac{PMT}{I} \text{ (Eq. I.51)}$$

Now we consider a timeline for a perpetuity starting at time 1 and a perpetuity starting at time N+1.

Note that if we subtract the second timeline from the first, we obtain the time line for an ordinary annuity characterized by N payments with a fixed amount denominated with PMT.

Therefore the present value of an ordinary annuity is equal to the discounted value of the first timeline minus the present value of the second timeline.

The formula for the Present Value of the first timeline, which is a perpetuity, is

$$PV_{TIME\ LINE\ 1} = \frac{PMT}{I}$$

If we apply the same formula for the second timeline, we obtain the value of the discounted payments at time N. In order to find the Present Value of the second timeline, we must further discount the perpetuity at time 0, applying $\frac{1}{(1+I)^N}$ as the discount factor:

$$PV_{TIME\ LINE\ 2} = \frac{PMT}{I}\frac{1}{(1+I)^N}$$

By subtracting $PV_{TIME\ LINE\ 1}$ from $PV_{TIME\ LINE\ 2}$, we obtain the Present Value of an ordinary annuity, PVA :

$$PVA = PV_{TIME\ LINE\ 1} - PV_{TIME\ LINE\ 2} = \frac{PMT}{I} - \frac{PMT}{I}\frac{1}{(1+I)^N} = PMT\left[\frac{1}{I} - \frac{1}{I(1+I)^N}\right]$$

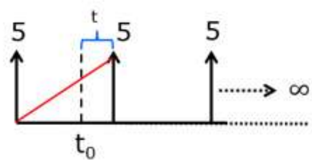
The future value of an ordinary annuity, FVA, is equal to the present value (PVA) compounded for N periods:

$$FVA = PVA(1+I)^N = PMT\left[\frac{1}{I} - \frac{1}{I(1+I)^N}\right](1+I)^N = PMT\left[\frac{(1+I)^N}{I} - \frac{1}{I}\right] = PMT\left[\frac{(1+I)^N - 1}{I}\right]$$

Previous formulas can be usefully implemented for pricing perpetual bonds.

$$Perpetual\ Bond\ Price = \frac{C}{R} \text{ (Eq. I.52)}$$

Where C is the coupon and R is the yield.



Let us consider, as an example, a perpetual bond, that pays an annual coupon in 3 months.

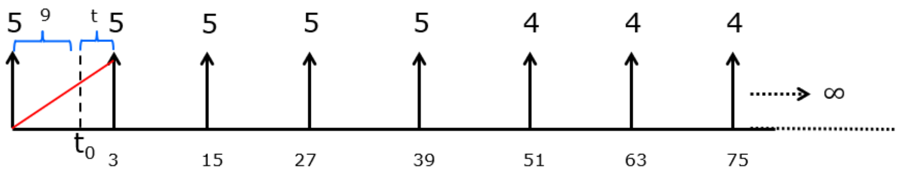
a) as a first case, we suppose that the bond has a coupon of 5% and a yield of 4%, and we calculate the gross and the clean price.

$$Gross\ Price = \frac{1}{(1+R)^t} \times \frac{C}{R} + C\frac{1}{(1+R)^t} = \frac{1}{(1+R)^t}\left[\frac{C}{R} + C\right] = \frac{1}{(1+0.04)^{\frac{3}{12}}}\left[\frac{0.05}{0.04} + 0.05\right] = 128.732\%$$

$$Clean\ Price = Gross\ Price - Accrued = 128.732\% - 5\%\frac{9}{12} = 124.982\%$$

b) as a second case, we now suppose that the perpetual bond pays a coupon of 5%, $C_1 = 5\%$, for the next four coupon dates (i.e. in 3, 15, 27 and 39 months) and after these four payments, it pays a fixed perpetual amount of 4%, $C_2 = 4\%$.

We calculate the gross and the clean price.



$$Gross\ Price = \frac{C_1}{(1+R)^{\frac{3}{12}}} + \frac{C_1}{(1+R)^{\frac{15}{12}}} + \frac{C_1}{(1+R)^{\frac{27}{12}}} + \frac{C_1}{(1+R)^{\frac{39}{12}}} + \frac{C_2}{R} \times \frac{1}{(1+R)^{\frac{39}{12}}}$$

$$= \frac{0.05}{(1.04)^{0.25}} + \frac{0.05}{(1.04)^{1.25}} + \frac{0.05}{(1.04)^{2.25}} + \frac{0.05}{(1.04)^{3.25}} + \frac{0.04}{0.04} \times \frac{1}{(1.04)^{\frac{39}{12}}} = 106.72\%$$

$$Clean\ Price = Gross\ Price\ - Accrued\ Interest = 106.72 - 5 \times \frac{9}{12} = 102.97\%$$

We know that the return from holding a bond in a portfolio for a given period can be broken down into two components.
The first is the change in the market value of the security (comparison between the sale price and the purchase price), and the second is the cash flows received as interest on the bond, as well as the interest on such cash flows (if reinvested).
As known, different market factors impact one or both of these aspects. If we want to measure the risk of a bond, i.e., measure the impact of the market factors on the characteristic return of the security, we can use the concept of YTM, starting from the analysis of the pricing formula of a security, all the input data to which the fair value is sensitive are present:

$$P = \sum_{t=1}^{T} \frac{CF_t}{(1+YTM)^t} = \frac{CF_1}{(1+YTM)^1} + \frac{CF_2}{(1+YTM)^2} + \cdots + \frac{CF_T}{(1+YTM)^T} \quad \textit{(Eq. I.53)}$$

Clearly, the price of a typical **fixed income security** moves in the opposite direction compared to the change in interest rates: if rates rise (/fall), the price of the security decreases (/increases).

We can define the **systemic risk** of a bond as the volatility in the total return due to an instantaneous fluctuation in the interest rate.
Generally speaking, in the past, bonds have been considered as safe investments: interest rates were stable for long periods, and the financial strategies associated with these instruments were quite conservative.
In more recent years, the increase in interest rate volatility and the anomalies of the interest rates term structures observed on the market have accentuated the importance of better considering and quantifying the risk on this asset class.

In particular, two other kinds of risk associated with bonds are the price risk and the reinvestment risk.
The **price risk** is by far the biggest risk an investor faces, although (for bonds repaying at par at maturity), the investor who plans to hold the bond till maturity is not interested in monitoring the change in the bond price throughout its life.
On the other hand, if the investor plans to sell the security before maturity, then, an increase in the interest rate will lead to a principal loss.
Reinvestment risk, as the name indicates, is defined as the variability of income from the reinvestment of coupon proceeds caused by a change in interest rates.
If, for example, interest rates fall, the cash flows received during this period will be reinvested at a lower rate.
Those two risks, price risk and investment risk act in opposite directions: if interest rates rise, the market price of the bond decreases, but, at the same time, the proceeds deriving from the coupons can be invested at a higher rate.
Thus, there is a trade-off between the two functionals that must be monitored on an ongoing basis by the quantitative analysts.

A typical strategy based on equalizing and therefore canceling these two risks is called immunization, a topic covered at the end of the teaching unit.

What happens if there is an instantaneous change in the bond yield?

We also examine what-if scenarios in the next pages setting the security price formula in function of its parameters.

Here are the tables and the relevant data for each of them:

Table I.11: Face= € 1,000; C=6%; yield= 5%; New yield =5.5%; T parameter.

Table I.12: Face= € 1,000; C=6%; yield= 5%; New yield =4.5%; T parameter.

Table I.13: Face= € 1,000; Bond Maturity = 10 years; yield= 5%; New yield =5.5%; Coupon parameter.

Table I.14: Face= € 1,000; Bond Maturity = 10 years; yield= 5%; New yield =4.5%; Coupon parameter.

Table I.15: Face= € 1,000; Bond Maturity = 10 years; Coupon= 5%; yield parameter (+0,5%).

Table I.16: Face= € 1,000; Bond Maturity = 10 years; Coupon= 5%; yield parameter (-0,5%).

DF_new	DF_old	Time	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Bond 6	Bond 7	Bond 8	Bond 9	Bond 10	Bond 11	Bond 12	Bond 13	Bond 14	Bond 15	Bond 16	Bond 17	Bond 18	Bond 19	Bond 20
0.9479	0.9524	1	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.8985	0.9070	2	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.8516	0.8638	3	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.8072	0.8227	4	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.7651	0.7835	5	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.7252	0.7462	6	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.6874	0.7107	7	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60
0.6516	0.6768	8	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60
0.6176	0.6446	9	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60
0.5854	0.6139	10	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60
0.5549	0.5847	11	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60
0.5260	0.5568	12	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60
0.4986	0.5303	13	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60
0.4726	0.5051	14	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60
0.4479	0.4810	15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60
0.4246	0.4581	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60
0.4024	0.4363	17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60
0.3815	0.4155	18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60
0.3616	0.3957	19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60
0.3427	0.3769	20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060
		Old Price	1009.52	1018.59	1027.23	1035.46	1043.29	1050.76	1057.86	1064.63	1071.08	1077.22	1083.06	1088.63	1093.94	1098.99	1103.80	1108.38	1112.74	1116.90	1120.85	1124.62
		New Price	1004.74	1009.23	1013.49	1017.53	1021.35	1024.98	1028.41	1031.67	1034.76	1037.69	1040.46	1043.09	1045.59	1047.95	1050.19	1052.31	1054.32	1056.23	1058.04	1059.75
		ΔP/P	-0.47%	-0.92%	-1.34%	-1.73%	-2.10%	-2.45%	-2.78%	-3.10%	-3.39%	-3.67%	-3.93%	-4.18%	-4.42%	-4.64%	-4.86%	-5.06%	-5.25%	-5.43%	-5.60%	-5.77%

Table I.11 What-if Scenario: Face= € 1,000; C=6%; yield= 5%; New yield =5.5%; T parameter

In case of a positive change in the market yield, long-dated bonds have a higher price sensitivity than shorter-dated bonds.

DF_new	DF_old	Time	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Bond 6	Bond 7	Bond 8	Bond 9	Bond 10	Bond 11	Bond 12	Bond 13	Bond 14	Bond 15	Bond 16	Bond 17	Bond 18	Bond 19	Bond 20
0.9569	0.9524	1	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.9157	0.9070	2	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.8763	0.8638	3	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.8386	0.8227	4	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.8025	0.7835	5	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.7679	0.7462	6	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60	60
0.7348	0.7107	7	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60	60
0.7032	0.6768	8	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60	60
0.6729	0.6446	9	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60	60
0.6439	0.6139	10	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60	60
0.6162	0.5847	11	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60	60
0.5897	0.5568	12	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60	60
0.5643	0.5303	13	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60	60
0.5400	0.5051	14	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60	60
0.5167	0.4810	15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60	60
0.4945	0.4581	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60	60
0.4732	0.4363	17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60	60
0.4528	0.4155	18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60	60
0.4333	0.3957	19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060	60
0.4146	0.3769	20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1060
		Old Price	1009.52	1018.59	1027.23	1035.46	1043.29	1050.76	1057.86	1064.63	1071.08	1077.22	1083.06	1088.63	1093.94	1098.99	1103.80	1108.38	1112.74	1116.90	1120.85	1124.62
		New Price	1014.35	1028.09	1041.23	1053.81	1065.85	1077.37	1088.39	1098.94	1109.03	1118.69	1127.93	1136.78	1145.24	1153.34	1161.09	1168.51	1175.61	1182.40	1188.90	1195.12
		DELTA P/P	0.48%	0.93%	1.36%	1.77%	2.16%	2.53%	2.89%	3.22%	3.54%	3.85%	4.14%	4.42%	4.69%	4.95%	5.19%	5.43%	5.65%	5.86%	6.07%	6.27%

Table I.12 What-if Scenario: Face= € 1,000; C=6%; yield= 5%; New yield =4.5%; T parameter

In case of a negative change in market yield, long-dated bonds have a higher price sensitivity than shorter-dated bonds.

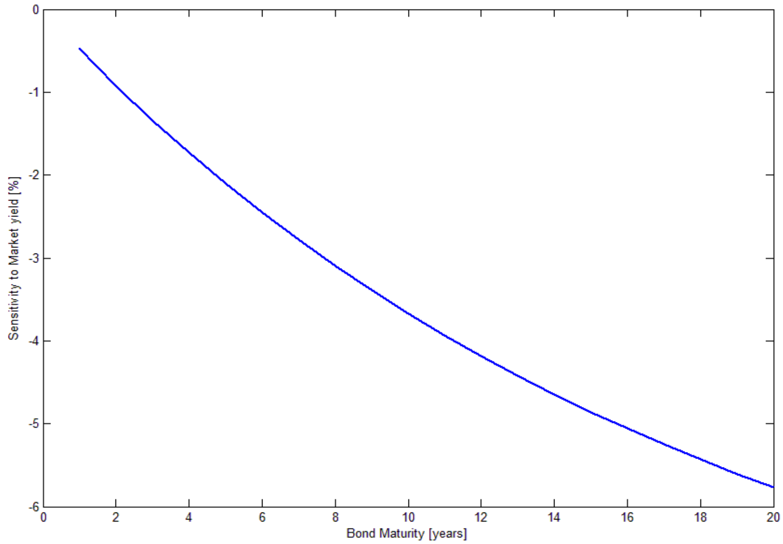


Figure I.58 What-if Scenario: Face= € 1,000; C=6%; yield= 5%; New yield =5.5%; T parameter

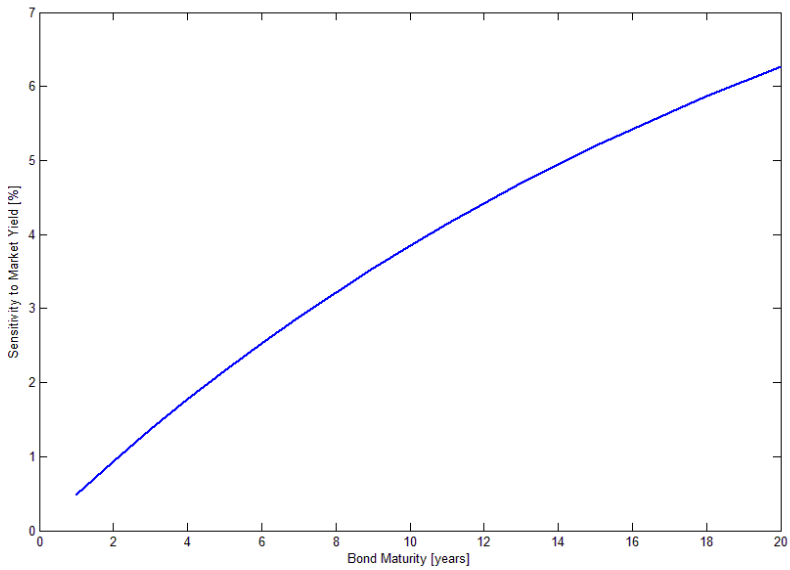


Figure I.59 What-if Scenario: Face= € 1,000; C=6%; yield= 5%; New yield =4.5%; T parameter

		Coupon	0.50%	1.00%	1.50%	2.00%	2.50%	3.00%	3.50%	4.00%	4.50%	5.00%	5.50%	6.00%	6.50%	7.00%	7.50%	8.00%	8.50%	9.00%	9.50%	10.00%
DF_new	DF_old	Time	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Bond 6	Bond 7	Bond 8	Bond 9	Bond 10	Bond 11	Bond 12	Bond 13	Bond 14	Bond 15	Bond 16	Bond 17	Bond 18	Bond 19	Bond 20
0.9479	0.9524	1	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.8985	0.9070	2	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.8516	0.8638	3	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.8072	0.8227	4	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.7651	0.7835	5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.7252	0.7462	6	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.6874	0.7107	7	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.6516	0.6768	8	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.6176	0.6446	9	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.5854	0.6139	10	1005	1010	1015	1020	1025	1030	1035	1040	1045	1050	1055	1060	1065	1070	1075	1080	1085	1090	1095	1100
		Old Price	652.52	691.13	729.74	768.35	806.96	845.57	884.17	922.78	961.39	1000.00	1038.61	1077.22	1115.83	1154.43	1193.04	1231.65	1270.26	1308.87	1347.48	1386.09
		New Price	623.12	660.81	698.49	736.18	773.87	811.56	849.25	886.94	924.62	962.31	1000.00	1037.69	1075.38	1113.06	1150.75	1188.44	1226.13	1263.82	1301.51	1339.19
		DELTA P	-29.40	-30.32	-31.24	-32.16	-33.09	-34.01	-34.93	-35.85	-36.77	-37.69	-38.61	-39.53	-40.45	-41.37	-42.29	-43.21	-44.13	-45.05	-45.97	-46.89
		DELTA P/P	-4.51%	-4.39%	-4.28%	-4.19%	-4.10%	-4.02%	-3.95%	-3.88%	-3.82%	-3.77%	-3.72%	-3.67%	-3.63%	-3.58%	-3.54%	-3.51%	-3.47%	-3.44%	-3.41%	-3.38%

Table I.13 What-if Scenario: Face= € 1,000; T = 10 years; yield= 5%; New yield =5.5%; Coupon parameter

		Coupon	0.50%	1.00%	1.50%	2.00%	2.50%	3.00%	3.50%	4.00%	4.50%	5.00%	5.50%	6.00%	6.50%	7.00%	7.50%	8.00%	8.50%	9.00%	9.50%	10.00%
DF_new	DF_old	Time	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Bond 6	Bond 7	Bond 8	Bond 9	Bond 10	Bond 11	Bond 12	Bond 13	Bond 14	Bond 15	Bond 16	Bond 17	Bond 18	Bond 19	Bond 20
0.9569	0.9524	1	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.9157	0.9070	2	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.8763	0.8638	3	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.8386	0.8227	4	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.8025	0.7835	5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.7679	0.7462	6	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.7348	0.7107	7	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.7032	0.6768	8	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.6729	0.6446	9	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0.6439	0.6139	10	1005	1010	1015	1020	1025	1030	1035	1040	1045	1050	1055	1060	1065	1070	1075	1080	1085	1090	1095	1100
		Old Price	652.52	691.13	729.74	768.35	806.96	845.57	884.17	922.78	961.39	1000.00	1038.61	1077.22	1115.83	1154.43	1193.04	1231.65	1270.26	1308.87	1347.48	1386.09
		New Price	683.49	723.05	762.62	802.18	841.75	881.31	920.87	960.44	1000.00	1039.56	1079.13	1118.69	1158.25	1197.82	1237.38	1276.95	1316.51	1356.07	1395.64	1435.20
		DELTA P	30.97	31.92	32.88	33.83	34.79	35.74	36.70	37.65	38.61	39.56	40.52	41.47	42.43	43.38	44.34	45.29	46.25	47.20	48.16	49.11
		DELTA P/P	4.75%	4.62%	4.51%	4.40%	4.31%	4.23%	4.15%	4.08%	4.02%	3.96%	3.90%	3.85%	3.80%	3.76%	3.72%	3.68%	3.64%	3.61%	3.57%	3.54%

Table I.14 What-if Scenario: Face= € 1,000; T = 10 years; yield= 5%; New yield =4.5%; Coupon parameter

For a given maturity, with respect to a positive change in the market yield, bonds with a lower coupon have a higher price sensitivity than those with a higher coupon. Clearly, a zero-coupon bond has the greatest volatility.

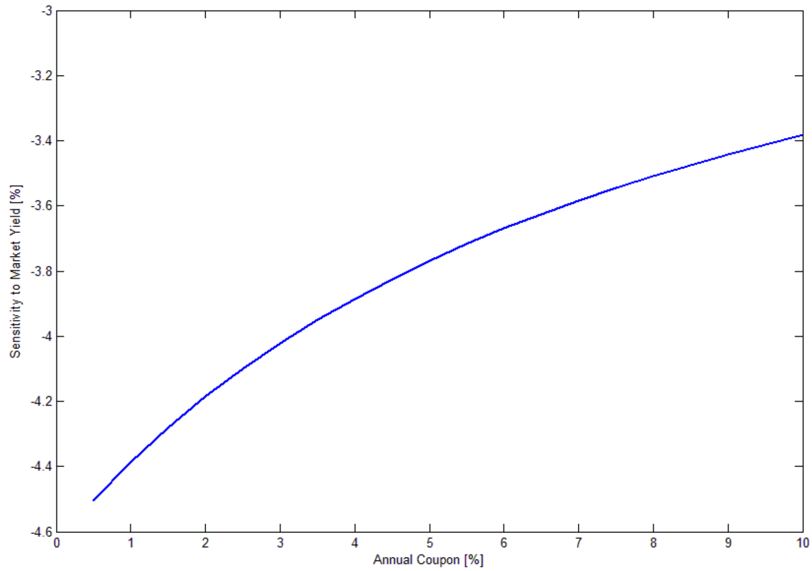


Figure I.60 What-if Scenario: Face= € 1,000; T = 10 years; yield= 5%; New yield =5.5%; Coupon parameter

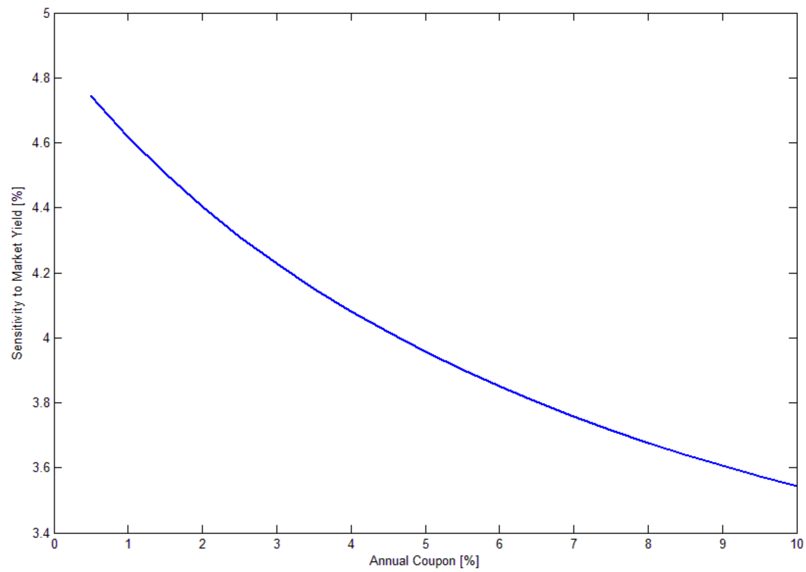


Figure I.61 What-if Scenario: Face= € 1,000; T = 10 years; yield= 5%; New yield =4.5%; Coupon parameter

Old yield	0.50%	1.00%	1.50%	2.00%	2.50%	3.00%	3.50%	4.00%	4.50%	5.00%	5.50%	6.00%	6.50%	7.00%	7.50%	8.00%	8.50%	9.00%	9.50%	10.00%
New yield	1.00%	1.50%	2.00%	2.50%	3.00%	3.50%	4.00%	4.50%	5.00%	5.50%	6.00%	6.50%	7.00%	7.50%	8.00%	8.50%	9.00%	9.50%	10,00%	10.50%
Time	**Bond 1**	**Bond 2**	**Bond 3**	**Bond 4**	**Bond 5**	**Bond 6**	**Bond 7**	**Bond 8**	**Bond 9**	**Bond 10**	**Bond 11**	**Bond 12**	**Bond 13**	**Bond 14**	**Bond 15**	**Bond 16**	**Bond 17**	**Bond 18**	**Bond 19**	**Bond 20**
1	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
2	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
3	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
4	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
5	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
6	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
7	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
8	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
9	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
10	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050
Old Price	1437.87	1378.85	1322.78	1269.48	1218.80	1170.60	1124.75	1081.11	1039.56	1000.00	962.31	926.40	892.17	859.53	828.40	798.70	770.35	743.29	717.45	692.77
New Price	1378.85	1322.78	1269.48	1218.80	1170.60	1124.75	1081.11	1039.56	1000.00	962.31	926.40	892.17	859.53	828.40	798.70	770.35	743.29	717.45	692.77	669.19
DELTA P	-59.02	-56.08	-53.30	-50.68	-48.20	-45.85	-43.64	-41.55	-39.56	-37.69	-35.91	-34.23	-32.64	-31.13	-29.70	-28.34	-27.06	-25.84	-24.68	-23.58
DELTA P/P	-4.10%	-4.07%	-4.03%	-3.99%	-3.95%	-3.92%	-3.88%	-3.84%	-3.81%	-3.77%	-3.73%	-3.70%	-3.66%	-3.62%	-3.59%	-3.55%	-3.51%	-3.48%	-3.44%	-3.40%

DFs	**0.5%**	**1.0%**	**1.5%**	**2.0%**	**2.5%**	**3.0%**	**3.5%**	**4.0%**	**4.5%**	**5.0%**	**5.5%**	**6.0%**	**6.5%**	**7.0%**	**7.5%**	**8.0%**	**8.5%**	**9.0%**	**9.5%**	**10.0%**	**10.5%**
1	0.995	0.9901	0.9852	0.9804	0.9756	0.97087	0.9662	0.9615	0.9569	0.95238	0.94787	0.9434	0.93897	0.93458	0.93023	0.92593	0.92166	0.91743	0.91324	0.90909	0.904977
2	0.9901	0.9803	0.9707	0.9612	0.9518	0.9426	0.9335	0.9246	0.9157	0.90703	0.89845	0.89	0.88166	0.87344	0.86533	0.85734	0.84946	0.84168	0.83401	0.82645	0.818984
3	0.9851	0.9706	0.9563	0.9423	0.9286	0.91514	0.9019	0.889	0.8763	0.86384	0.85161	0.83962	0.82785	0.8163	0.80496	0.79383	0.78291	0.77218	0.76165	0.75131	0.741162
4	0.9802	0.961	0.9422	0.9238	0.906	0.88849	0.8714	0.8548	0.8386	0.8227	0.80722	0.79209	0.77732	0.7629	0.7488	0.73503	0.72157	0.70843	0.69557	0.68301	0.670735
5	0.9754	0.9515	0.9283	0.9057	0.8839	0.86261	0.842	0.8219	0.8025	0.78353	0.76513	0.74726	0.72988	0.71299	0.69656	0.68058	0.66505	0.64993	0.63523	0.62092	0.607
6	0.9705	0.942	0.9145	0.888	0.8623	0.83748	0.8135	0.7903	0.7679	0.74622	0.72525	0.70496	0.68533	0.66634	0.64796	0.63017	0.61295	0.59627	0.58012	0.56447	0.549321
7	0.9657	0.9327	0.901	0.8706	0.8413	0.81309	0.786	0.7599	0.7348	0.71068	0.68744	0.66506	0.64351	0.62275	0.60275	0.58349	0.56493	0.54703	0.52979	0.51316	0.497123
8	0.9609	0.9235	0.8877	0.8535	0.8207	0.78941	0.7594	0.7307	0.7032	0.67684	0.6516	0.62741	0.60423	0.58201	0.5607	0.54027	0.52067	0.50187	0.48382	0.46651	0.449885
9	0.9561	0.9143	0.8746	0.8368	0.8007	0.76642	0.7337	0.7026	0.6729	0.64461	0.61763	0.5919	0.56735	0.54393	0.52158	0.50025	0.47988	0.46043	0.44185	0.4241	0.407136
10	0.9513	0.9053	0.8617	0.8203	0.7812	0.74409	0.7089	0.6756	0.6439	0.61391	0.58543	0.55839	0.53273	0.50835	0.48519	0.46319	0.44229	0.42241	0.40351	0.38554	0.368449

Table I.15 What-if Scenario: Face= € 1,000; Bond Maturity = 10 years; Coupon= 5%; yield parameter (+0,5%)

For a given maturity, bonds with a low yield (**low yield bonds**) are characterized by greater price volatility than those with a high yield (**high yield bonds**).

Old yield	0.50%	1.00%	1.50%	2.00%	2.50%	3.00%	3.50%	4.00%	4.50%	5.00%	5.50%	6.00%	6.50%	7.00%	7.50%	8.00%	8.50%	9.00%	9.50%	10.0%
New yield	0.00%	0.50%	1.00%	1.50%	2.00%	2.50%	3.00%	3.50%	4.00%	4.50%	5.00%	5.50%	6.00%	6.50%	7.00%	7.50%	8.00%	8.50%	9.00%	9.50%
Time	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Bond 6	Bond 7	Bond 8	Bond 9	Bond 10	Bond 11	Bond 12	Bond 13	Bond 14	Bond 15	Bond 16	Bond 17	Bond 18	Bond 19	Bond 20
1	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
2	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
3	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
4	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
5	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
6	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
7	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
8	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
9	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
10	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050	1050
Old Price	1437.87	1378.85	1322.78	1269.48	1218.80	1170.60	1124.75	1081.11	1039.56	1000.00	962.31	926.40	892.17	859.53	828.40	798.70	770.35	743.29	717.45	692.77
New Price	1500.00	1437.87	1378.85	1322.78	1269.48	1218.80	1170.60	1124.75	1081.11	1039.56	1000.00	962.31	926.40	892.17	859.53	828.40	798.70	770.35	743.29	717.45
DELTA P	62.13	59.02	56.08	53.30	50.68	48.20	45.85	43.64	41.55	39.56	37.69	35.91	34.23	32.64	31.13	29.70	28.34	27.06	25.84	24.68
DELTA P/P	4.32%	4.28%	4.24%	4.20%	4.16%	4.12%	4.08%	4.04%	4.00%	3.96%	3.92%	3.88%	3.84%	3.80%	3.76%	3.72%	3.68%	3.64%	3.60%	3.56%

DFs	0%	0.5%	1.0%	1.5%	2.0%	2.5%	3.0%	3.5%	4.0%	4.5%	5.0%	5.5%	6.0%	6.5%	7.0%	7.5%	8.0%	8.5%	9.0%	9.5%	10.0%
1	1	0.995025	0.990099	0.985222	0.980392	0.97561	0.970874	0.966184	0.961538	0.956938	0.952381	0.947867	0.943396	0.938967	0.934579	0.930233	0.925926	0.921659	0.917431	0.913242	0.909091
2	1	0.990075	0.980296	0.970662	0.961169	0.95181	0.942596	0.933511	0.924556	0.91573	0.907029	0.898452	0.889996	0.881659	0.873439	0.865333	0.857339	0.849455	0.84168	0.834011	0.826446
3	1	0.985149	0.97059	0.956317	0.942322	0.9286	0.915142	0.901943	0.888996	0.876297	0.863838	0.851614	0.839619	0.827849	0.816298	0.804961	0.793832	0.782908	0.772183	0.761654	0.751315
4	1	0.980248	0.96098	0.942184	0.923845	0.90595	0.888487	0.871442	0.854804	0.838561	0.822702	0.807217	0.792094	0.777323	0.762895	0.748801	0.73503	0.721574	0.708425	0.695574	0.683013
5	1	0.975371	0.951466	0.92826	0.905731	0.88385	0.862609	0.841973	0.821927	0.802451	0.783526	0.765134	0.747258	0.729881	0.712986	0.696559	0.680583	0.665045	0.649931	0.635228	0.620921
6	1	0.970518	0.942045	0.914542	0.887971	0.8623	0.837484	0.813501	0.790315	0.767896	0.746215	0.725246	0.704961	0.685334	0.666342	0.647962	0.63017	0.612945	0.596267	0.580117	0.564474
7	1	0.96569	0.932718	0.901027	0.87056	0.84127	0.813092	0.785991	0.759918	0.734828	0.710681	0.687437	0.665057	0.643506	0.62275	0.602755	0.58349	0.564926	0.547034	0.529787	0.513158
8	1	0.960885	0.923483	0.887711	0.85349	0.82075	0.789409	0.759412	0.73069	0.703185	0.676839	0.651599	0.627412	0.604231	0.582009	0.560702	0.540269	0.520669	0.501866	0.483824	0.466507
9	1	0.956105	0.91434	0.874592	0.836755	0.80073	0.766417	0.733731	0.702587	0.672904	0.644609	0.617629	0.591898	0.567353	0.543934	0.521583	0.500249	0.47988	0.460428	0.441848	0.424098
10	1	0.951348	0.905287	0.861667	0.820348	0.7812	0.744094	0.708919	0.675564	0.643928	0.613913	0.585431	0.558395	0.532726	0.508349	0.485194	0.463193	0.442285	0.422411	0.403514	0.385543

Table I.16 What-if Scenario: Face= € 1,000; Bond Maturity = 10 years; Coupon= 5%; yield parameter (-0,5%)

For a given maturity, bonds with a low yield (**low yield bonds**) are characterized by greater price volatility than those with a high yield (**high yield bonds**).

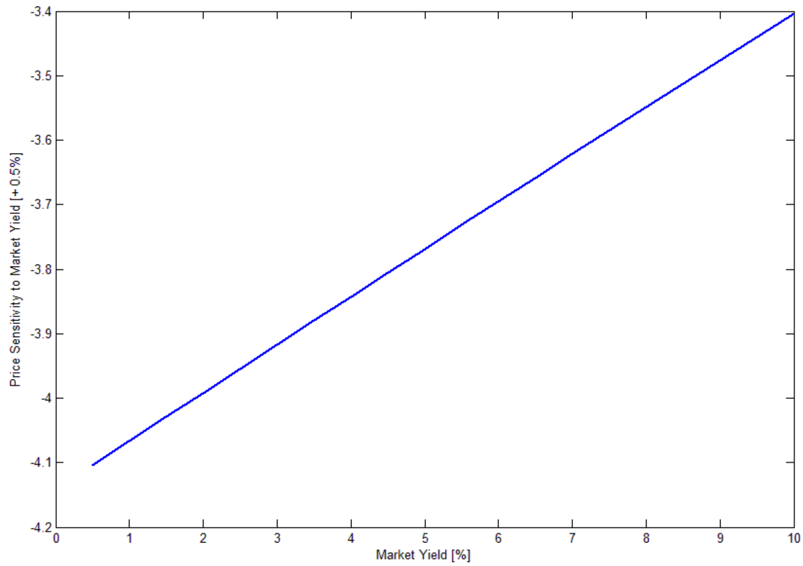


Figure I.62 What-if Scenario: Face= € 1,000; T = 10 years; Coupon= 5%; yield parameter (+0,5%)

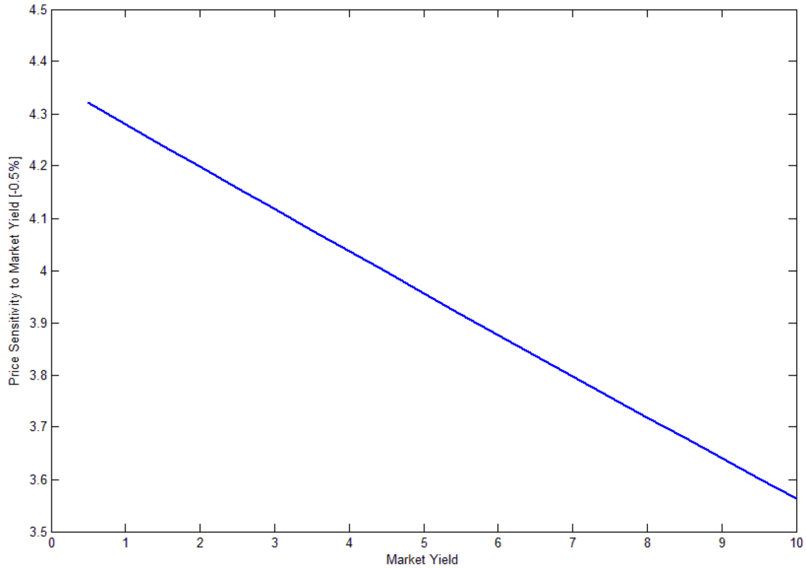


Figure I.63 What-if Scenario: Face= € 1,000; T = 10 years; Coupon= 5%; yield parameter (-0,5%)

When a bond has an amortization plan (i. e. it is a sinkable bond), it is characterized by a lower volatility than a "bullet" security with the same financial characteristics. The fact that the principal is partially repaid during the life of the security reduces its risk and such risk reduction is consequently reflected in a more stable quotation.

Again when the bond is callable, it shows a lower price volatility than one with the same financial characteristics, but without options. This empirical evidence can be theoretically explained by the fact that a bond with the possibility of being repaid ahead of time gives the possibility of reducing its time to maturity.

Our analysis highlights that **price volatility** is not a symmetrical phenomenon: given the market value of a bond, a decrease in the market yield does not produce the same effect on the price of the bond as an identical increase in the yield market. In short, price volatility is neither linear nor symmetrical.

The considerations made so far are not sufficient to develop a suitable and synthetic measure of the risk of a bond, since the sensitivity of the price variation was analysed comparing bonds with the same financial characteristics, and varying one market factor at a time. Now, we define certain indicators to quantify the risk associated with a bond and to compare securities with different financial characteristics. The most common proxies for quantifying risk are:

- Time to maturity (**TTM**).
- Weighted average maturity (**WAM**).
- Weighted average cash flow (**WACF**).

Along with those proxies, we also define the most suitable and complete quantitative measures as follows:

- Duration (**DUR**).
- Modified Duration (**MOD DUR**).
- Convexity (**CONV**).

The **Time to maturity** represents the number of years remaining before the bond matures. Securities with a longer maturity date are assumed to be riskier than those issued with a shorter maturity date because the investor has to wait longer before the money lent is returned (principal reimbursement), and also due to the greater sensitivity to interest rate fluctuations. It is a very vague indicator though, since it does not take into account any cash flows received before maturity. As an example, let us consider a bond with a 10% coupon, on one hand, and a zero coupon on the other, both bonds expiring in 10 years. According to this weak proxy, both should have the same risk since they have the same maturity; but after 5 years, the former will have already recovered half of its initial investment (which can be reinvested at higher rates in the event of growth in interest rates), while the latter will not have paid anything to the investor yet, since the revenues are in terms of principal appreciation.

As we have seen in the what-if analysis, there is no linear relationship between time to maturity and price volatility, so a 30-year bond is not three times as risky as a 10-year one.

The **weighted average maturity** (WAM) or average life is the weighted average of the maturities in which the repayment of the notional takes place.

$$WAM = \sum_{t=1}^{T} \frac{Principal\ paid\ at\ time\ t}{Total\ principal\ to\ be\ repaid} \times t \quad (Eq.\ I.54)$$

This figure does not consider coupons, but only principal repayments. Thus, for bullet bonds, the average life of a security is identical to the time to maturity (WAM = TTM), but it does prove optimal for the quantification of risk for sinking fund bonds and for mortgage backed securities.

Let us present a practical example, in which we calculate the WAM of a 6% coupon, 10-year sinking fund, priced at par (€100) at a yield of 6%. The amortization plan withdraws 20% of the bond annually, starting from the sixth year. Coupon interest is paid every six months. The WAM focuses exclusively on the cash flows due to the principal which in our case occur on an annual basis starting from the sixth year.

$$WAM = \frac{20\ € \times 6}{100\ €} + \frac{20\ € \times 7}{100\ €} + \frac{20\ € \times 8}{100\ €} + \frac{20\ € \times 9}{100\ €} + \frac{20\ € \times 10}{100\ €} = 8\ years$$

As we said, the WAM in this case is a weak approximation for the bond's risk: although it considers the temporal distribution of the repayments of the notional over time, it ignores the payment of the coupons. Therefore, two sinkable bonds with the same amortization plan could have the same average life (and therefore the same "risk") despite one paying a 9% coupon and the other a 3% coupon.

Let us then introduce the **Weighted average cash flow** (WACF), calculated in a similar way to weighted average maturity, but in this case the formula considers all cash flows, both coupon and principal:

$$WACF = \sum_{t=1}^{T} \frac{Cash\ flow\ paid\ at\ time\ t}{Total\ cash\ flows\ to\ be\ paid} \times t \quad (Eq.\ I.55)$$

For example, the WACF of a bond that has a face value of USD 1000, a 4-year maturity and a coupon rate of 5% is equal to:

$$WACF = \frac{50\ USD \times 1}{1200\ USD} + \frac{50\ USD \times 2}{1200\ USD} + \frac{50\ USD \times 3}{1200\ USD} + \frac{1050\ USD \times 4}{1200\ USD} = 3.75\ years$$

The main drawback of using WACF is that cash flows are considered on a nominal basis and not on a discounted basis: therefore, the time value of money is not taken into consideration. For the reasons stated thus, TTM, WAM and WACF cannot be considered as valid indicators of the intrinsic risk of a bond.

Bond Type	Relationship
Coupon bearing bullet bond	WACF < WAM = TTM
Sinkable bond	WACF < WAM < TTM
Zero-coupon bond	WACF = WAM = TTM

Table I.17 WACF-WAM-TTM relationship for different bonds

The concept of **Duration** as a measure of the intrinsic risk associated with a bond was proposed in 1938 by Frederick R. Macaulay. Duration (or DUR) can be interpreted as an advanced version of the WACF: the DUR of a series of cash flows is equal to the time average over which the cash flows occur: the weight of each monetary input is calculated using the respective present value.

$$DUR = \sum_{t=1}^{T} \frac{PV(CF_t)}{Price} \times t = \sum_{t=1}^{T} w_t \times t \quad \text{(Eq. I.56)}$$

If all cash flows are discounted at the YTM of the bond, k, the weight associated with each cash flow is equal to: $w_t = \frac{CF_t/(1+k)^t}{P}$ and the complete formula for Duration is:

$$DUR = \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \times t = \frac{1}{P} \sum_{t=1}^{T} \frac{CF_t}{(1+k)^t} \times t =$$

$$= \frac{1}{P}\left[\frac{CF_1}{(1+k)^1} \times 1 + \frac{CF_2}{(1+k)^2} \times 2 + \frac{CF_3}{(1+k)^3} \times 3 + \cdots + \frac{CF_T}{(1+k)^T} \times T\right] \quad \text{(Eq. I.57)}$$

Where CF_t is the cash flow (coupon or principal) received at date t, T is the time to maturity, k is the discount rate equal to the market yield and P is the fair-value of the bond (i.e. market price or present value of all future payments).
If a reliable market price is available, this value should be used in the formula; otherwise the theoretical one has to be used. In this last case, the formula for calculating the Duration becomes:

$$DUR = \frac{\sum_{t=1}^{T} \frac{t \times CF_t}{(1+k)^t}}{P} = \frac{\sum_{t=1}^{T} \frac{t \times CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}} \quad \text{(Eq. I.58)}$$

Duration is measured in years. In the case of a zero-coupon bond, since it has no payments during its life, DUR is simply the present value of its future cash flow multiplied by its TTM and divided by the price.
Since the price is the present value of the final cash flow, the Macaulay duration of a zero coupon is equal to its TTM.

We present an example, of a bond maturing in 10 years and paying an annual coupon of 7%. Its YTM is equal to k=9% and its Macaulay Duration is 7.32 years.

DUR can be represented by plotting cash flows as a function of time. In the following figure, the height of each white bar represents the money inflow (column [B]), while the lower portion in blue is the current value (column [C]).

If we thought of these PV values as weights with mass and placed them on the horizontal line, the duration would be the fulcrum (i.e. the centre of gravity).

t [Years]	CF	PV(CF)	CF weight	Time weighted by CF weight
[A]	[B]	$[C]=[B]/(1+k)^t$	[D]=[C]/P	[E]=[A] x [D]
1	7	6.422018349	0.073676838	0.073676838
2	7	5.891759953	0.067593429	0.135186859
3	7	5.40528436	0.062012321	0.186036962
4	7	4.958976477	0.056892037	0.227568149
5	7	4.549519704	0.05219453	0.260972648
6	7	4.173871288	0.047884889	0.287309337
7	7	3.829239714	0.043931091	0.307517639
8	7	3.513063958	0.040303753	0.322430028
9	7	3.222994457	0.036975921	0.332783285
10	107	45.19795634	0.518535191	5.185351906
	Price	87.1646846	**Duration**	7.318833649

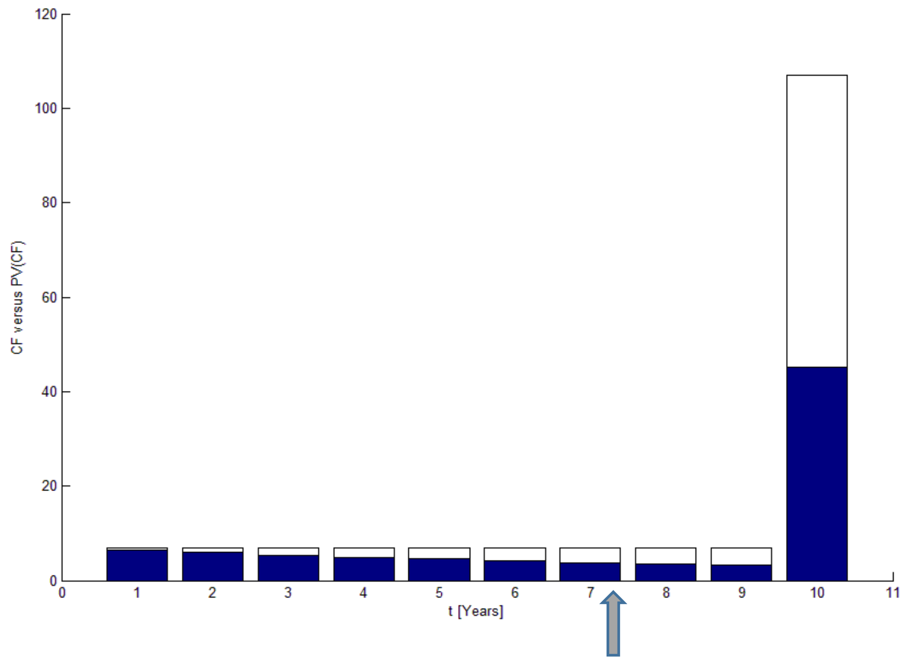


Figure I.64 Duration as a fulcrum of discounted Cash Flows

Let us make an example of a zero-coupon that expires in 10 years and has a face value of €1000, currently traded at €558.39. Its yield to maturity is 6% and its Macaulay Duration is equal to the time to maturity. This check is immediate as it is sufficient to apply the formula:

$$Macaulay\ Duration = \frac{\sum_{t=1}^{T} \frac{t \times CF_t}{(1+k)^t}}{P} = \frac{\frac{10 \times 1000}{(1.06)^{10}}}{558.39} = 10\ years = TTM$$

This methodology also remains unchanged for securities which envisage infra-annual coupon payments (for example quarterly or half-yearly) or have an amortization plan.

As an example, let us consider a sinkable bond with an annual coupon of 7% p.a., a six-monthly payment frequency, a residual life of 10 years and a nominal value of USD 1,000. The security has an amortization plan which provides for the gradual repayment of the notional amount: starting from the sixth year and with an annual frequency, the principal is repaid by 20% p.a. If the market yield is 6%, as shown in the following table, the price of the bond is €1,056.5653 and the Macaulay duration is 6.28 years.

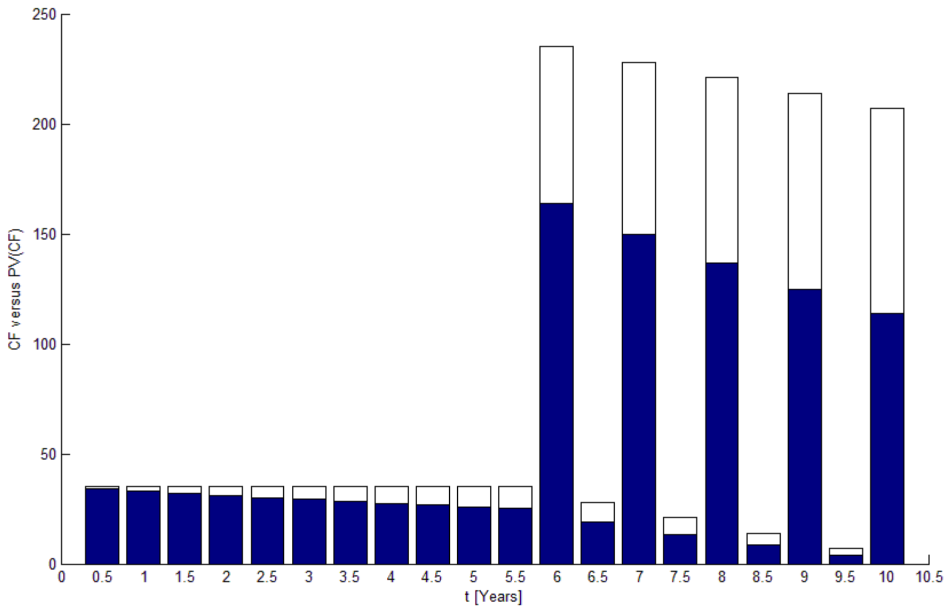


Figure I.65 Duration of a Sinkable bond

t [Years]	CF	DF	PV(CF)	CF weight	Time weighted by CF weight
[A]	[B]	[C]=exp(-k*[A])	[D]=[B]*[C]	[E]=[D]/Price	[F]=[A] x [E]
0.5	35	0.970445534	33.96559367	0.032147177	0.016073589
1	35	0.941764534	32.96175868	0.031197085	0.031197085
1.5	35	0.913931185	31.98759148	0.030275072	0.045412607
2	35	0.886920437	31.04221529	0.029380308	0.058760616
2.5	35	0.860707976	30.12477917	0.028511989	0.071279972
3	35	0.835270211	29.2344574	0.027669332	0.083007996
3.5	35	0.810584246	28.37044861	0.02685158	0.093980529
4	35	0.786627861	27.53197514	0.026057996	0.104231982
4.5	35	0.763379494	26.7182823	0.025287865	0.113795395
5	35	0.740818221	25.92863772	0.024540496	0.12270248
5.5	35	0.718923733	25.16233067	0.023815215	0.130983681
6	235	0.697676326	163.9539366	0.155176334	0.931058002
6.5	28	0.677056874	18.95759249	0.01794266	0.116627288
7	228	0.65704682	149.8066749	0.141786474	0.992505318
7.5	21	0.637628152	13.39019118	0.01267332	0.095049903
8	221	0.618783392	136.7511296	0.129429883	1.035439068
8.5	14	0.600495579	8.406938103	0.007956856	0.067633274
9	214	0.582748252	124.708126	0.118031626	1.062284636
9.5	7	0.565525439	3.958678071	0.003746742	0.035594052
10	207	0.548811636	113.6040087	0.10752199	1.075219901

The Macaulay Duration considers all the variables that influence the volatility of a bond's price: all the cash flows, the yield to maturity, as well as the current market price of the bond. In terms of interest rate risk, it is indifferent for an investor to invest in a coupon-bearing bond and in a zero-coupon with a maturity date equal to the duration of the security that pays coupons. Using the Macaulay Duration, it is implicitly assumed that all cash flows are discounted (and reinvested) at the same discount rate k, equal to the YTM.

In real markets though, the term structure of rates is not flat, and each cash flow should be discounted with the appropriate spot rate $R_{0,t}$. An extension of the concept of Duration, which employs the entire term structure of rates, has been proposed by Fischer and Weil:

$$DUR_{FW} = \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \times t = \frac{1}{P} \sum_{t=1}^{T} \frac{CF_t \times t}{(1+R_{0,t})^t} = \frac{1}{P}\left[\frac{CF_1 \times 1}{(1+R_{0,1})^1} + \frac{CF_2 \times 2}{(1+R_{0,2})^2} + \cdots + \frac{CF_T \times T}{(1+R_{0,T})^T}\right]$$ *(Eq. I.59)*

The Macaulay Duration is equal to the time, expressed in years, at which the total value of a bond is not sensitive to changes in interest rates.

Let us now consider a bond with a 10-year residual life, a notional amount of € 100 and a coupon of 10%. The current market yield is 7% for all maturities and the Macaulay duration is 7.068 years.

$$DUR = \frac{\sum_{t=1}^{T} \frac{t \times CF_t}{(1+k)^t}}{P} = \frac{\sum_{t=1}^{T} \frac{t \times CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}} = \frac{\frac{1 \times 10}{(1.07)^1} + \frac{2 \times 10}{(1.07)^2} + \frac{3 \times 10}{(1.07)^3} + \cdots + \frac{10 \times 110}{(1.07)^{10}}}{\frac{10}{(1.07)^1} + \frac{10}{(1.07)^2} + \frac{10}{(1.07)^3} + \cdots + \frac{110}{(1.07)^{10}}} = 7.068$$

The bond price is equal to the denominator: $P_{k=7\%} = \frac{10}{(1.07)^1} + \frac{10}{(1.07)^2} + \cdots + \frac{110}{(1.07)^{10}} = 121.07$

We assume that the market yield halves to 3.5%. In this new context, the bond price becomes: $P_{k=3.5\%} = \frac{10}{(1.035)^1} + \frac{10}{(1.035)^2} + \frac{10}{(1.035)^3} + \cdots + \frac{110}{(1.035)^{10}} = 154.06$. This extreme change in the rate leads to a capital gain of € 32.99.

T=10 years; C=10% p.a.; notional € 100 k=7% → k=3.5%

t [Years]	CF	PV(CF)	CF weight	Time weighted by CF weights
[A]	[B]	[C]=[B]/(1+k)^t	[D]=[C]/Price	[E]=[A] x [D]
1	10	9.345794393	0.077192838	0.077192838
2	10	8.734387283	0.072142839	0.144285679
3	10	8.162978769	0.067423214	0.202269643
4	10	7.62895212	0.06301235	0.2520494
5	10	7.129861795	0.058890047	0.294450233
6	10	6.663422238	0.055037427	0.33022456
7	10	6.227497419	0.051436847	0.360057932
8	10	5.820091046	0.04807182	0.38457456
9	10	5.439337426	0.044926935	0.404342411
10	110	55.91842213	0.461865683	4.618656828
	Price	121.0707446	**Duration**	7.068104085

t [Years]	CF	PV(CF)
[A]	[B]	[C]=[B]/(1+k)^t
1	10	9.661835749
2	10	9.335107004
3	10	9.019427057
4	10	8.714422277
5	10	8.419731669
6	10	8.135006443
7	10	7.859909607
8	10	7.594115562
9	10	7.337309722
10	110	77.98106951
	Price	154.0579346

We calculate the bond price for each year remaining before the maturity and we assume a market yield of 7% (before) and a market rate of 3.5% (after) with the aim to illustrate the working principle of duration.

The capitalized value for each year of the reinvested interest is also computed at the market rate of the paid coupon, according to the formula:

$$\textit{Future value of reinvested coupons in year } t = \sum_{i=1}^{t} C_i \times (1+k)^{t-i} \quad \text{(Eq. I.60)}$$

Table of discount factors between t1 and t2 with a yield equal to 7%.

$DF_{t1,t2}$	0	1	2	3	4	5	6	7	8	9
1	0.9346	0	0	0	0	0	0	0	0	0
2	0.8734	0.9346	0	0	0	0	0	0	0	0
3	0.8163	0.8734	0.9346	0	0	0	0	0	0	0
4	0.7629	0.8163	0.8734	0.9346	0	0	0	0	0	0
5	0.7130	0.7629	0.8163	0.8734	0.9346	0	0	0	0	0
6	0.6663	0.7130	0.7629	0.8163	0.8734	0.9346	0	0	0	0
7	0.6227	0.6663	0.7130	0.7629	0.8163	0.8734	0.9346	0	0	0
8	0.5820	0.6227	0.6663	0.7130	0.7629	0.8163	0.8734	0.9346	0	0
9	0.5439	0.5820	0.6227	0.6663	0.7130	0.7629	0.8163	0.8734	0.9346	0
10	0.5083	0.5439	0.5820	0.6227	0.6663	0.7130	0.7629	0.8163	0.8734	0.9346

Table of discount factors between t1 and t2 with a yield equal to 3.5%.

DFt1,t2	0	1	2	3	4	5	6	7	8	9
1	0.9662	0	0	0	0	0	0	0	0	0
2	0.9335	0.9662	0	0	0	0	0	0	0	0
3	0.9019	0.9335	0.9662	0	0	0	0	0	0	0
4	0.8714	0.9019	0.9335	0.9662	0	0	0	0	0	0
5	0.8420	0.8714	0.9019	0.9335	0.9662	0	0	0	0	0
6	0.8135	0.8420	0.8714	0.9019	0.9335	0.9662	0	0	0	0
7	0.7860	0.8135	0.8420	0.8714	0.9019	0.9335	0.9662	0	0	0
8	0.7594	0.7860	0.8135	0.8420	0.8714	0.9019	0.9335	0.9662	0	0
9	0.7337	0.7594	0.7860	0.8135	0.8420	0.8714	0.9019	0.9335	0.9662	0
10	0.7089	0.7337	0.7594	0.7860	0.8135	0.8420	0.8714	0.9019	0.9335	0.9662

Determination of the bond price as the valuation date varies using a flat market yield of 7%.

Val. Date		0	1	2	3	4	5	6	7	8	9	10
Year	CF	121.0707	119.5457	117.9139	116.1679	114.2996	112.3006	110.1616	107.8729	105.4241	102.804	100
1	10	9.345794	0	0	0	0	0	0	0	0	0	
2	10	8.734387	9.345794	0	0	0	0	0	0	0	0	
3	10	8.162979	8.734387	9.345794	0	0	0	0	0	0	0	
4	10	7.628952	8.162979	8.734387	9.345794	0	0	0	0	0	0	
5	10	7.129862	7.628952	8.162979	8.734387	9.345794	0	0	0	0	0	
6	10	6.663422	7.129862	7.628952	8.162979	8.734387	9.345794	0	0	0	0	
7	10	6.227497	6.663422	7.129862	7.628952	8.162979	8.734387	9.345794	0	0	0	
8	10	5.820091	6.227497	6.663422	7.129862	7.628952	8.162979	8.734387	9.345794	0	0	
9	10	5.439337	5.820091	6.227497	6.663422	7.129862	7.628952	8.162979	8.734387	9.345794	0	
10	110	55.91842	59.83271	64.021	68.50247	73.29764	78.42848	83.91847	89.79277	96.07826	102.804	

Determination of the bond price as the valuation date varies using a flat market yield of 3.5%.

Val. Date		0	1	2	3	4	5	6	7	8	9	10
Year	CF	154.0579	149.45	144.6807	139.7445	134.6356	129.3478	123.875	118.2106	112.348	106.2802	100
1	10	9.661836	0	0	0	0	0	0	0	0	0	
2	10	9.335107	9.661836	0	0	0	0	0	0	0	0	
3	10	9.019427	9.335107	9.661836	0	0	0	0	0	0	0	
4	10	8.714422	9.019427	9.335107	9.661836	0	0	0	0	0	0	
5	10	8.419732	8.714422	9.019427	9.335107	9.661836	0	0	0	0	0	
6	10	8.135006	8.419732	8.714422	9.019427	9.335107	9.661836	0	0	0	0	
7	10	7.85991	8.135006	8.419732	8.714422	9.019427	9.335107	9.661836	0	0	0	
8	10	7.594116	7.85991	8.135006	8.419732	8.714422	9.019427	9.335107	9.661836	0	0	
9	10	7.33731	7.594116	7.85991	8.135006	8.419732	8.714422	9.019427	9.335107	9.661836	0	
10	110	77.98107	80.71041	83.53527	86.45901	89.48507	92.61705	95.85865	99.2137	102.6862	106.2802	

It is interesting to note that close to the seventh year, the total value (price + reinvested coupons) is approximately the same for both scenarios: the Macaulay Duration was in fact equal to 7.07.

The following figure shows the bond prices using a bar plot and the total investment value. We indicate the data referring to pricing with the yield equal to 3.5% in red, and the one relating to the yield of 7% in blue.

	YTM = 7 %			YTM = 3.5 %		
Val. Date	Bond Price	Fut. Value inv. coupons	Total Value	Bond Price	Fut. Value of inv. coupons	Total Value
0	121.0707446	0	121.0707446	154.0579346	0	154.057935
1	119.5456967	10	129.5456967	149.4499623	10	159.449962
2	117.9138955	20.7	138.6138955	144.680711	20.35	165.030711
3	116.1678682	32.149	148.3168682	139.7445359	31.06225	170.806786
4	114.299619	44.39943	158.699049	134.6355946	42.14942875	176.785023
5	112.3005923	57.5073901	169.8079824	129.3478404	53.62465876	182.972499
6	110.1616338	71.53290741	181.6945412	123.8750149	65.50152181	189.376537
7	107.8729481	86.54021093	194.4131591	118.2106404	77.79407508	196.004716
8	105.4240545	102.5980257	208.0220802	112.3480128	90.5168677	202.864880
9	102.8037383	119.7798875	222.5836258	106.2801932	103.6849581	209.965151
10	100	138.1644796	238.1644796	100	117.3139316	217.313932

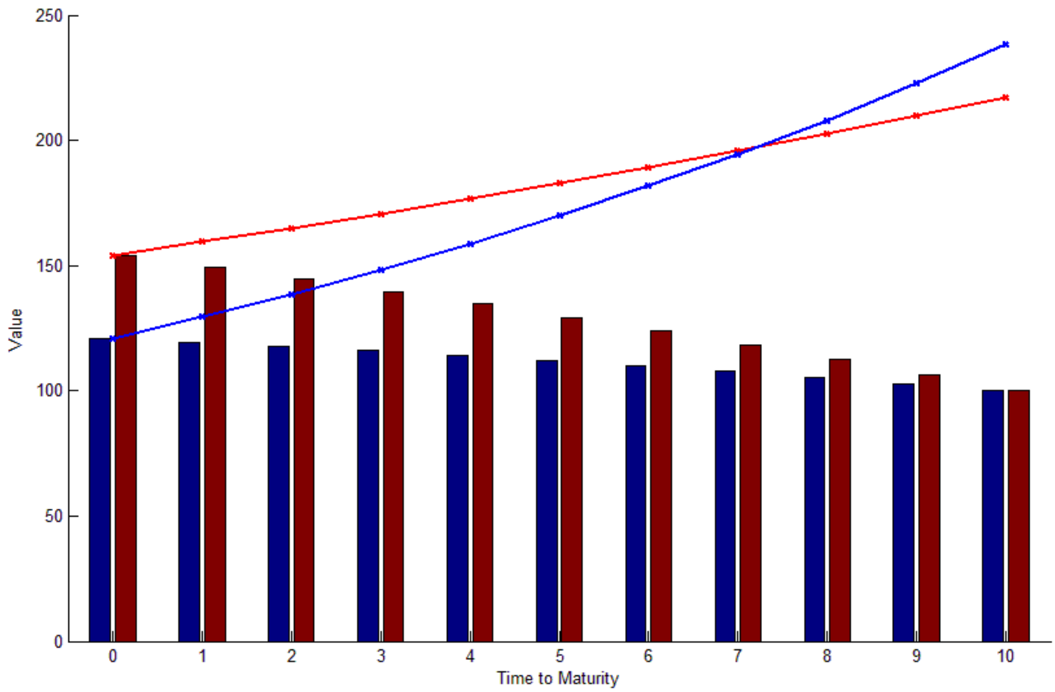


Figure I.66 Bond Risk measures: duration

Duration is used as a method to approximate price changes using market yield changes. The relationship, mathematically derived starting from the analytical development of the first derivative of the bond price with respect to the discount rate k, is the following:

$$\frac{\Delta P}{P} = -\frac{D}{1+k} \times \Delta k \quad \textit{(Eq. I.61)}$$

The formula shows that the percentage change in the price of a bond due to a change in the interest rate can be approximated by its Macaulay Duration.
The term $D^{MOD} = \frac{D}{1+k}$ is called modified duration or sensitivity.

$$\Delta P = -D^{MOD} \times \Delta k \times P \quad \textit{(Eq. I.62)}$$

Let us consider an example of a bond with a nominal value of €1,000 maturing in 4 years and offering an annual coupon of 7%. The market yield is 7% and the duration is 3.67 years.
Let us examine what happens if the market yield increases by 50 bps (7 →7.5%) and by 200 bps:

$$\frac{\Delta P}{P} = -\frac{3.67}{1+0.07} \times 0.005 = -1.71\%$$

The duration approximation would predict a decrease of 1.71% from the current price.
To express the value of the bond in absolute terms, we estimate the price of the bond with the current market yield:

$$P = \frac{60€}{1.07} + \frac{60€}{1.07^2} + \frac{60€}{1.07^3} + \frac{1060€}{1.07^4} = 966.13\ €$$

Faced with an increase in the market yield from 7% to 7.5%, the price of the bond, according to the modified duration approach, would decrease by 1.71%, i.e., it would assume a value of € 949.61. In order to understand how good this approximation is, we reprice the bond with a yield of 7.5%

$$P = \frac{60€}{1.075} + \frac{60€}{1.075^2} + \frac{60€}{1.075^3} + \frac{1060€}{1.075^4} = 949.76\ €$$

However, the approximation cannot be considered good for Δk=2%. In this case, the duration approach would propose a decrease of 6.85% (i.e. a price of €899.95), while the one calculated using the discounted cash flows method with a yield of 9% would give a fair value of €902.81. From this example, we can deduce that the approximation of the DUR for calculating the change in the price of a bond is reasonable only for a small change in the market yield (Δk not large).

Let us investigate how accurate the modified duration approximation is. Let us consider the following baseline scenario: 6%, 10-year bonds priced at par, the current market yield is 6% and the Macaulay duration is 7.8 years.

T	CF	4.75%	5.00%	5.25%	5.50%	5.75%	6.00%	6.25%	6.50%	6.75%	7.00%	7.25%
1	60	57.2792	57.1429	57.0071	56.8720	56.7376	56.6038	56.4705	56.3380	56.2060	56.0747	55.9440
2	60	54.6818	54.4218	54.1635	53.9071	53.6526	53.3998	53.1487	52.8995	52.6520	52.4063	52.1622
3	60	52.2022	51.8303	51.4618	51.0968	50.7353	50.3771	50.0223	49.6709	49.3227	48.9778	48.6361
4	60	49.8351	49.3621	48.8948	48.4330	47.9766	47.5256	47.0799	46.6393	46.2040	45.7737	45.3484
5	60	47.5753	47.0116	46.4559	45.9081	45.3680	44.8355	44.3104	43.7928	43.2824	42.7791	42.2829
6	60	45.4179	44.7729	44.1386	43.5147	42.9012	42.2976	41.7039	41.1200	40.5456	39.9805	39.4246
7	60	43.3584	42.6409	41.9369	41.2462	40.5685	39.9034	39.2508	38.6103	37.9818	37.3649	36.7595
8	60	41.3922	40.6104	39.8451	39.0959	38.3626	37.6447	36.9419	36.2538	35.5801	34.9205	34.2746
9	60	39.5153	38.6765	37.8575	37.0578	36.2767	35.5139	34.7688	34.0411	33.3303	32.6360	31.9577
10	1060	666.4469	650.7480	635.4550	620.5564	606.0411	591.8984	578.118	564.689	551.603	538.850	526.420
	New Price	1097.704	1077.217	1057.216	1037.688	1018.6201	1000	981.815	964.055	946.709	929.764	913.210
	Approx. Price	1091.981	1073.585	1055.189	1036.792	1018.396	1000	981.603	963.207	944.811	926.415	908.018
	Delta Price	5.72321	3.63244	2.02762	0.89568	0.2239074	0	0.21199	0.84830	1.89764	3.34909	5.19202

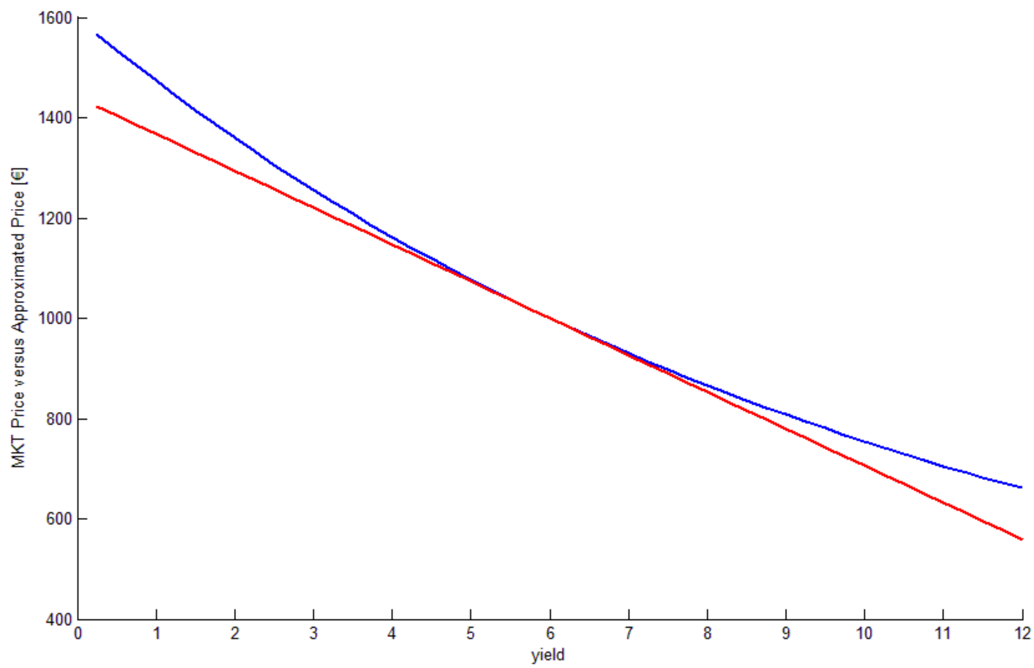


Figure I.67 Bond Value Versus Approximated Price

We implemented a study, comparing the change in the market price with the one approximated by the formula of the modified duration, as the market yield changes.
The exact market price is calculated using the discount of future cash flows and it is compared with the price estimated using D^{MOD}.
The blue line in the graph represents the market price, the red line shows the price linearly approximated by the modified duration formula.
The price/yield graph shows that the slope of the red line is equal to the price duration, i.e. it is equal to the quantity $D^P = \frac{D}{1+k} \times P = -\frac{\Delta P}{\Delta k}$.
In mathematical terms, the price duration is the first derivative of the price/yield curve. The modified duration approximation considers the relationship between price and yield as a linear function.
In reality though, the function that links price to yield is a convex curve; therefore, the duration does not take into account this convexity (in fact it is an approximation of the first order of the Taylor polynomial with respect to the price/yield function).
Consequently, the error term becomes larger when the yield (and consequently the price) moves away from current market levels: the further the new yield is from the starting yield-to-maturity, Δk, the greater the error that will be committed in the approximation.
We can thus draw the following conclusions:

- Duration is an instantaneous value that varies continuously over time, as are the inputs for its estimation.

- Duration does not take into account the asymmetry present in price volatility.

-The modified duration approximation always underestimates the new price.

-The accuracy of the approximation depends on the convexity existing at the time of measurement in the price/yield relationship.

- DUR should not be used to approximate a price change when considering a large change in market return.

In order to improve the approximation, the second order term can be added to the linear term, defined as convexity.
In mathematical terms, this term can be quantified as:

$$Convexity = C = \frac{1}{2} \times \frac{1}{P} \times \frac{1}{(1+k)^2} \times \sum_{t=1}^{T} \frac{t \times (t+1) \times CF_t}{(1+k)^t} \quad \textit{(Eq. I.63)}$$

Let us see an example, considering a bond with a 10-year maturity, a notional amount of €100, and a 5% annual coupon. Current market yield is 5.5% and we want to calculate its convexity.

$$C = \frac{1}{2} \times \frac{1}{P} \times \frac{1}{(1+k)^2} \times \sum_{t=1}^{T} \frac{t \times (t+1) \times CF_t}{(1+k)^t} = \frac{1}{2} \times \frac{1}{96.23} \times \frac{1}{(1{,}055)^2} \times 7902.04 = 36.89$$

t	Cash Flow	Present Value	PV*t*(t+1)
1	5	4.739336493	9.478672986
2	5	4.492262079	26.95357247
3	5	4.258068321	51.09681985
4	5	4.036083717	80.72167433
5	5	3.825671769	114.7701531
6	5	3.626229165	152.3016249
7	5	3.437184043	192.4823064
8	5	3.257994353	234.5755934
9	5	3.088146306	277.9331676
10	105	61.47021084	6761.723192
	Bond Price	96.23118709	7902.036777

t	CFt	PV(CFt)	PV*t	PV*t*(t+1)
1	70	65.58605828	65.5860582	131.172116
2	70	61.45044343	122.900886	368.702660
3	70	57.5756052	172.726815	690.907262
4	70	53.94509998	215.780399	1078.902
5	70	50.54352101	252.717605	1516.30563
6	70	47.35643307	284.138598	1988.97018
7	70	44.37031113	310.592177	2484.73742
8	70	41.57248302	332.579864	2993.21877
9	70	38.95107563	350.559680	3505.59680
10	1070	557.8516005	5578.51600	61363.6760
	Bond Price	1019.202631	7686.09809	76122.1889

The quadratic term is added to the approximated price formula:

$$\Delta P = -D^{MOD} \times P \times \Delta k + C \times P \times (\Delta k)^2 \rightarrow \frac{\Delta P}{P} = -D^{MOD} \times \Delta k + C \times (\Delta k)^2$$

The price convexity is defined as: $C^P = C \times P$

$$\Delta P = -D^P \times \Delta k + C^P \times (\Delta k)^2$$

Now let us consider another example, i.e., a bond maturing in 10 years, face value= €1,000, coupon of 7% annually and currently trading at 102%. The market yield is thus equal to 6.73%. We want to assess how the price changes if the market yield were to increase by 200bps.

$$D = \sum_{t=1}^{T} \frac{PV(CF_t) \times t}{P} = \frac{7686.098}{1019{,}203} = 7.54$$

$$C = \frac{1}{2} \times \frac{1}{P} \times \frac{1}{(1+k)^2} \times \sum_{t=1}^{T} \frac{t \times (t+1) \times PV(CF_t)}{(1+k)^t} = \frac{1}{2} \times \frac{1}{1{,}0673^2} \times \frac{76122{,}19}{1019{,}203} = 32.78$$

If there were a +2% increase in the market yield, the new market yield would be 8.73%, and the duration would predict a price change equal to:

$$\frac{\Delta P}{P} = -D \times \frac{\Delta k}{1+k} = -7.54 \times \frac{+0.02}{1.0673} = -14.13\%$$

The new bond price would then be 1019.203×(1−0.1413)= € 875.189.

Now considering the convexity, the second order approximation would be:

$$\frac{\Delta P}{P} = -D^{MOD} \times \Delta k + C \times (\Delta k)^2 = -0.1413 + 32.78 \times (0.02)^2 = -12.82\%$$

The new approximate price of the security would be equal to 1,019.203×(1−0.1282)= € 888.541.

By precisely calculating the NPV of the cash flows, i.e. using the market yield of 8.73%, the bond price would be € 887.64.

We can state that convexity is always a positive quantity and measures the rate of change in the slope of the price-yield curve with respect to changes in yield.

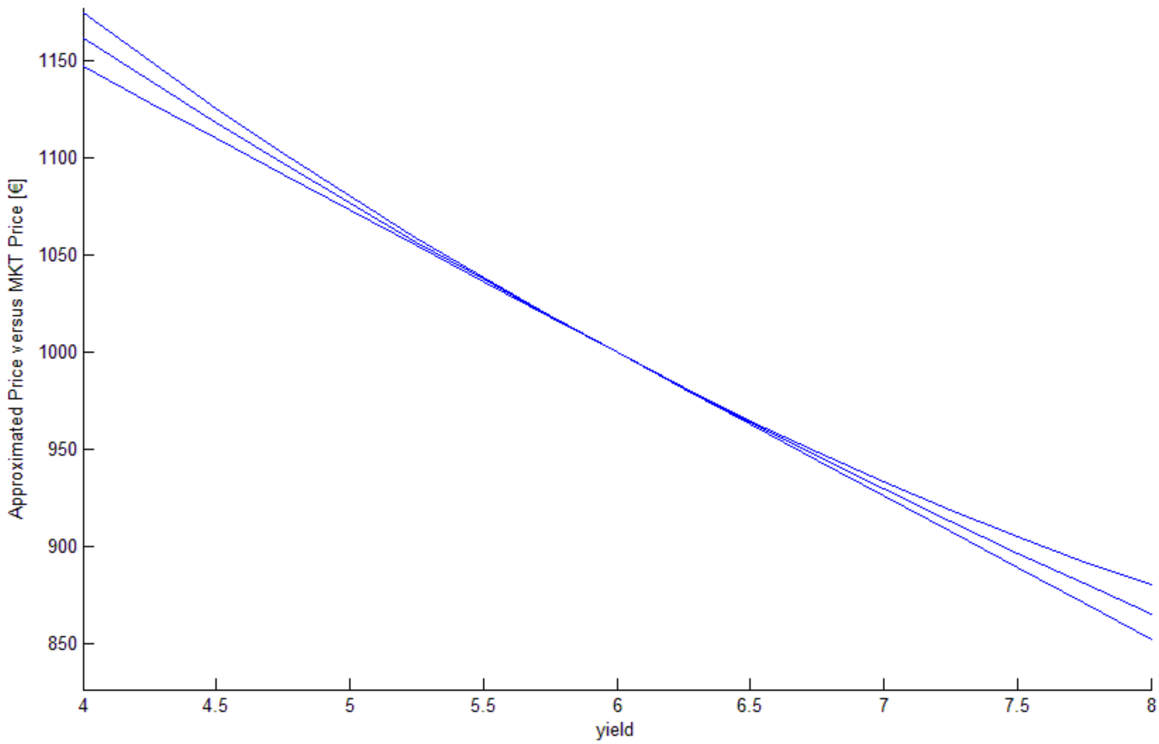


Figure I.68 Local approximation of the first order price with DUR and second order with the CONV

Let us make another example and compute the main risk measures associated with the following bond:

Figure I.69 Bond Risk Measures. Source: Bloomberg®

Valuation Date: 11/03/2020; MKT Clean Price: 103.916; YTM: 0.3816%; Current Coupon: 0.95; Previous Payment date: 09/15/2020; CF: 0.475; Accrued Interest: 0.064; MKT Dirty Price: 103.979.

Year Fraction	Payment Date	Cash Flow	Discount Factor	NPV	CF weight	t x weight
0.361643836	03/15/2021	0.4750	0.99862365	0.4743	0.004562	0.00165
0.865753425	09/15/2021	0.4750	0.99670827	0.4734	0.004553	0.003942
1.36040146	03/15/2022	0.4750	0.99483241	0.4725	0.004545	0.006183
1.864051095	09/15/2022	0.4750	0.99292603	0.4716	0.004536	0.008455
2.360027379	03/15/2023	0.4750	0.99105227	0.4707	0.004527	0.010685
2.863791923	09/15/2023	0.4750	0.98915271	0.4698	0.004519	0.012941
3.360700602	03/15/2024	0.4750	0.98728257	0.4690	0.00451	0.015157
3.864258347	09/15/2024	0.4750	0.98539101	0.4681	0.004501	0.017395
4.36040146	03/15/2025	0.4750	0.98353084	0.4672	0.004493	0.019591
4.864051095	09/15/2025	0.4750	0.98164613	0.4663	0.004484	0.021812
5.36018772	03/15/2026	0.4750	0.97979306	0.4654	0.004476	0.023992
5.863903011	09/15/2026	0.4750	0.97791526	0.4645	0.004467	0.026196
6.360027379	03/15/2027	0.4750	0.97606928	0.4636	0.004459	0.028359
6.863791923	09/15/2027	100.4750	0.97419843	97.8826	0.941367	6.461349
			Dirty Price	103.9792	**Duration**	6.6577
					Mod Dur	6.6324

Year Fraction	Payment Date	Cash Flow	Discount Factor	PV(CF)	PV*t	PV*t*(t+1)
0.361643836	03/15/2021	0.4750	0.998623653	0.4743	0.171544392	0.233582364
0.865753425	09/15/2021	0.4750	0.996708269	0.4734	0.409879209	0.764733537
1.36040146	03/15/2022	0.4750	0.994832405	0.4725	0.642851442	1.517387481
1.864051095	09/15/2022	0.4750	0.992926032	0.4716	0.879160807	2.517961472
2.360027379	03/15/2023	0.4750	0.991052273	0.4707	1.110982487	3.732931573
2.863791923	09/15/2023	0.4750	0.989152711	0.4698	1.345545584	5.198908159
3.360700602	03/15/2024	0.4750	0.987282567	0.4690	1.576031531	6.872601647
3.866995074	09/15/2024	0.4750	0.985380736	0.4681	1.809969665	8.809113444
3.866995074	03/15/2024	0.4750	0.985380736	0.4681	1.809969665	8.809113444
4.864051095	09/15/2025	0.4750	0.981646126	0.4663	2.268019035	13.2997795
5.362925303	03/15/2026	0.4750	0.979782841	0.4654	2.495888539	15.88115234
5.863903011	09/15/2026	0.4750	0.977915258	0.4645	2.723840106	18.69617431
6.360027379	03/15/2027	0.4750	0.976069276	0.4636	2.948717975	21.70264503
6.863791923	09/15/2027	100.4750	0.974198431	97.88	671.8457127	5283.25489
			Dirty Price	103.98		5391.290974

As shown in the table, we obtain a Duration of 6.66, a Modified Duration of 6.63, a Convexity of 25.73 (Bloomberg® Convexity: Convexity*2/100)
Yield Bump: 4.00%; MKT Current Price: 103.916; Current YTM: 0.3815%; Bumped YTM: 4.3815%.

	1st order	2nd order
ΔP/P	-26.530%	-22.4131%
Approx. Price	76.34749	80.625164
Exact Price	80.16	

Year Fraction	Payment Date	CF	DF	NPV
0.361643836	03/15/2021	0.4750	0.984611503	0.4677
0.865753425	09/15/2021	0.4750	0.963555181	0.4577
1.36040146	03/15/2022	0.4750	0.943331821	0.4481
1.864051095	09/15/2022	0.4750	0.923176491	0.4385
2.360027379	03/15/2023	0.4750	0.903749132	0.4293
2.863791923	09/15/2023	0.4750	0.884435172	0.4201
3.360700602	03/15/2024	0.4750	0.86578847	0.4112
3.864258347	09/15/2024	0.4750	0.847293278	0.4025
4.36040146	03/15/2025	0.4750	0.829456874	0.3940
4.864051095	09/15/2025	0.4750	0.811734609	0.3856
5.36018772	03/15/2026	0.4750	0.794646973	0.3775
5.863903011	09/15/2026	0.4750	0.77766627	0.3694
6.360027379	03/15/2027	0.4750	0.761296199	0.3616
6.863791923	09/15/2027	100.4750	0.745026591	74.8565
			Dirty Price	80.2196
			Accrued Int	0.0636
			Clean Price	80.1561

The duration of a bond portfolio is simply the weighted average of the individual bond durations:

$$Portfolio\ Duration = \sum_{i=1}^{N} w_i \times D_i \quad \textit{(Eq. I.64)}$$

w_i is the weight (in terms of market value) of the i-th security within the portfolio.
D_i is the duration of security i.
N is the number of securities included in the portfolio.

Similarly, the convexity of a bond portfolio is simply the weighted average of the individual bond convexities:

$$Portfolio\ Convexity = \sum_{i=1}^{N} w_i \times C_i \quad \textit{(Eq. I.65)}$$

Where w_i is the weight (in terms of market value) of the i-th security within the portfolio.
D_i represents the duration of security i.
N is the number of securities included in the portfolio.

The Modified Duration and the Convexity enable us to implement quick scenarios on potential changes in the interest rate, both at the individual security level and at the aggregate portfolio level. The analysis of Macaulay Duration allows to buy/sell bonds in order to immunize a given portfolio against changes in interest rates. We can state that **immunization** strategies are dynamic in nature and must theoretically be rebalanced with any significant change in interest rates.

From an ALM (Assets and Liability Management) perspective, a credit institution might be interested in freezing the surplus (S) given by the difference in market value between the bonds it has in its balance sheet assets (A) and those it has issued or bought to finance its claims (L) by a parallel shift in the term structure of interest rates. The equation to set off is:

$$\Delta S = \Delta A - \Delta L = \left[A \times \frac{D_A}{1+k} - L \times \frac{D_L}{1+k}\right] \times \Delta k \quad \textit{(Eq. I.66)}$$

Setting $\Delta S = 0$, we obtain:

$$A \times D_A = L \times D_L \quad \textit{(Eq. I.67)}$$

Let us for example consider an investor having a portfolio composed of Bond A and Bond b, as follows:

Bond A has a coupon of 0%, a maturity of 3 years, and a Face Value of CHF 200,000.

Bond B also has a coupon of 0%, a maturity of 7 years, and the Face Value here is CHF 300,000.

In a period of 5 years, the investor will have the opportunity to buy his grandparents' house for an amount of CHF 490,000.The current market yield is 3.5%. First of all, our investor wants to calculate the present value and duration of the portfolio, considering the asset side:

PV_A = 200,000/(1.0353) = CHF 180,388

PV_B = 300,000/(1.0357) = CHF 235,797

PV_{ASSET} = PV_A + PV_B = CHF 180,388 + CHF 235,797 = CHF 416,185

D_A = 3 years

D_B = 7 years

D_{ASSET} = w_A x D_A+w_B x D_B=3 x (180,388/416,185) + 7 x (235,797/416,185) = 5.27 years

Then, the investor wants to calculate the present value and duration of the portfolio on the liability side:

$PV_{LIABILITY}$ = 490,000/(1.035) = CHF 412,567

$D_{LIABILITY}$ = 5 years

Lastly, he wished to calculate the current surplus between assets and liabilities:

S=PV_{ASSET}–$PV_{LIABILITY}$=416,185-412,567= CHF 3,618

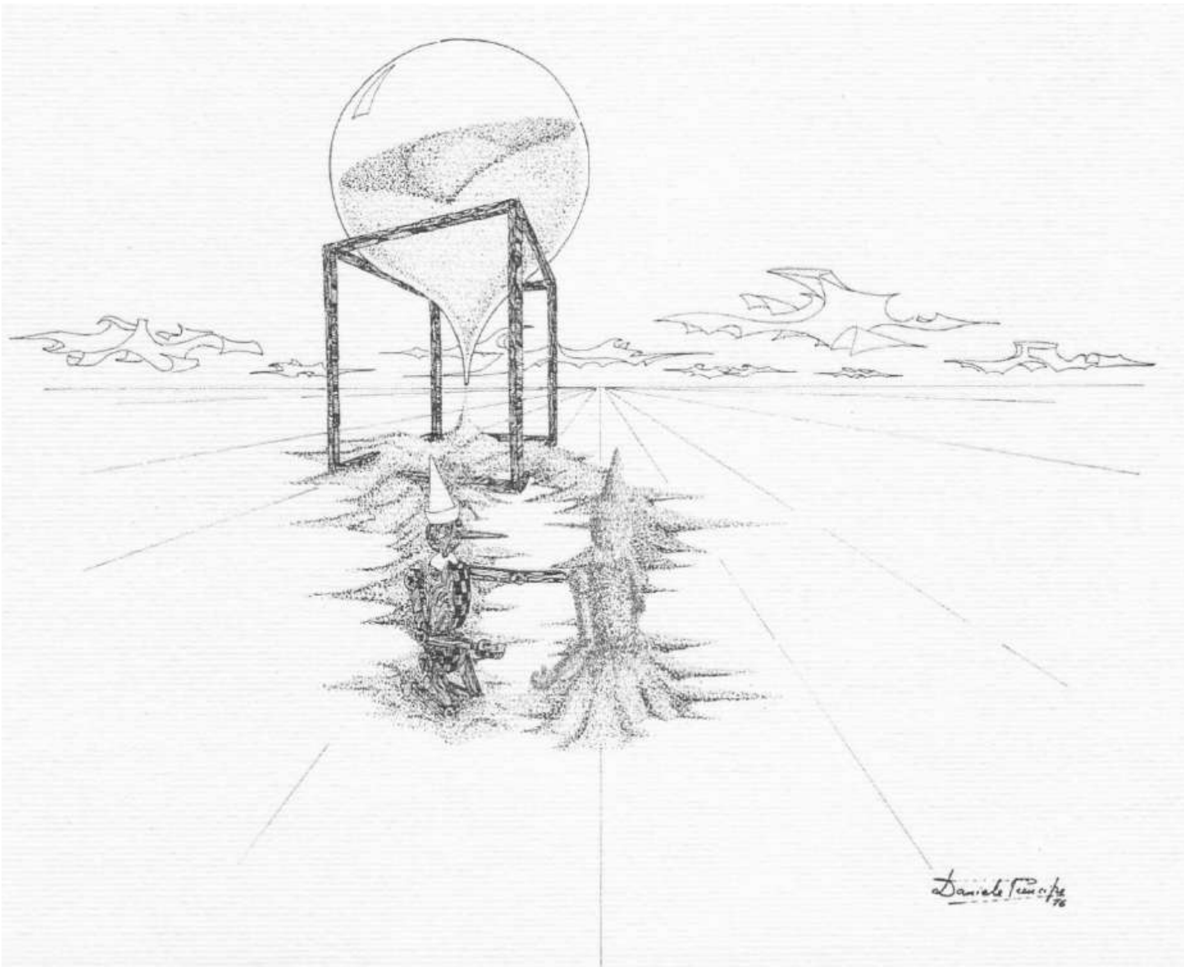
Our investor may also wonder when would it be best to buy the house, to immunize the surplus with parallel shifts in yield:

PV_{ASSET} x D_A = $PV_{LIABILITY}$ x D_L

D_L = (PV_{ASSET} x D_A) / $PV_{LIABILITY}$ = (416,185 * 5.27) / 412,567 = 5.32 years

FURTHER READINGS

Fabozzi F. J. – "The handbook of Fixed Income Securities" – McGraw Hill (2012).
Giribone P. G. – "Principi di analisi quantitativa finanziaria – Modulo: Fixed Income" – CARIGE Academy, MAREBA (2020).
Giribone P. G. – "Break the Rules! In che modo i tassi d'interesse negativi hanno condizionato la finanza quantitativa" – CARIGE Academy 2019, Seminario per Docenti Aziendali (2019).



PART II: FUTURES AND FORWARDS

Chapter 1 – Fundamentals

Basic definitions
Pay-off
Market mechanics
Marking to Market
The leverage effect
Quoting convention
Main futures markets

Chapter 2 – Quantitative Analysis

Valuation and analysis
Capital Asset Pricing Model (CAPM)
Net hedging pressure theory
Normal-backwardation and Contango
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Hedging strategies
Naive strategy
Basis and correlation risk
Minimum variance hedge ratio estimation

II.1 FUNDAMENTALS

Every market transaction consists of three moments: Trading, Clearing and Settlement. In a spot transaction, these three events follow one another in a short time. This implies that the payment of an agreed price and the delivery of the underlying asset takes place immediately or a few days after the transaction date. On the contrary, in a forward or a futures transaction, the trading phase and the clearing phase take place immediately, but the settlement will take place on the agreed future date. This implies that there is no exchange of money or goods at the time of trading and the payment of the agreed price and the delivery of the underlying asset will take place later.

A **forward contract** is a private agreement stipulated by two counterparties in which a seller agrees to deliver an amount or a prefixed quantity (**contract size**) of an asset (underlying asset i.e., shares, commodities, foreign currencies, bonds, etc…) at an agreed price (**forward price**) to the other counterparty (**buyer**) on a future **delivery date** (maturity) in accordance with an established agreement. Since the forward is an OTC contract, its terms can be customized to specific needs if both parties accept the conditions. It is important to understand that in a forward transaction, money is not usually transferred at the time of stipulation, as a forward is simply a commitment to make a future, not an immediate (spot) transaction. Although all contractual aspects are agreed and accepted at the time of stipulation, the asset delivery and/or any exchanges of money take place on the pre-established future date. Consequently, forward transactions have an intrinsic **counterparty risk**. This aspect generates the risk of observing forward transactions on the markets that take place exclusively between participants with the same credit risk because they can then trust each other, which poses a potential threat to the liquidity of the markets.

As an example, let us consider a large company in the agricultural sector that agrees today (i.e. at t=0) to deliver 7 tons of wheat to a multinational, in three months' time, at a unit price of 370 USD/ton. Payment and delivery will take place in three months. In this way the forward contract protects both counterparties from possible future price changes. The terms of the contract are negotiated over the counter and therefore not on organized markets or exchanges.

To effectively solve the problems related to counterparty risk, it is necessary to consider the futures market. A **futures contract** is a standardized agreement between two counterparties in which one party (**seller**) agrees to deliver to the other (**buyer**) an established amount or a quantity (**contract size**) of the underlying asset at a future date (maturity or **delivery date**) at an agreed price (**futures price**) to be paid on the delivery date. As opposed to forward contracts, all futures contractual terms are strictly standardized, including the quantity of underlying, the quality and the methods of delivery.

Let us consider as an example the New York Cotton Exchange which offers futures contracts on orange juice.

Ticker Symbol	OJ
Trading Unit	15,000 lbs. of orange juice solids (3% more or less)
Trading Hours	10:00 a.m. - 1:30 p.m. (NY Time)
Price Quotation	Prices quoted in cents and hundredths of a cent.
Trading Months	January, March, May, July, September, November
Minimum Fluctuation	5/100 of a cent per pound
Last Trading Day	14th business day prior to the last business day of the month
Basis Grade	U.S. Grade A with a Brix value of not less than 62.5 degrees
Delivery Points	Exchange licensed warehouse in Florida, New Jersey, Delaware and California
Delivery Methods	Drums or tanks, at the seller's discretion

Table II.1 Futures contracts on orange juice

Figure II.1 Futures contracts on orange juice: Description. Source Bloomberg® - DES module

Figure II.2 Futures contracts on orange juice: Prices and Open Interest. Source Bloomberg®

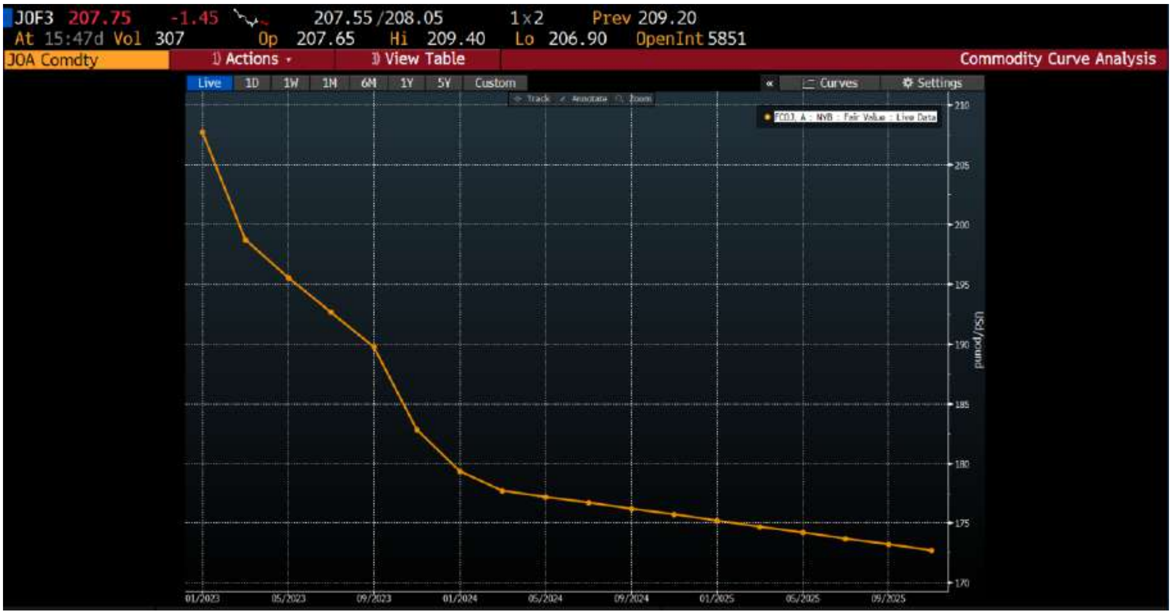

Figure II.3 Futures contracts on orange juice: curve. Source Bloomberg®

Figure II.4 Futures contracts on orange juice: quotes. Source Bloomberg®

Each futures exchange has its own Clearing House. The role of a Clearing House is to manage the deals, regulate deliveries and ensure compliance with the contractual terms between the counterparties involved in the transaction. When a buyer and a seller enter into a futures contract, they implicitly agree to the price, quantity, delivery date, and characteristics of the underlying asset in the contract. The Clearing House therefore assumes the obligation to buy the contract from the seller and sell it to the buyer. The buyer and the seller no longer have legal duties towards each other, in fact they never come into direct contact: if Trader A wants to sell and Trader B wants to buy, A will sell to the Clearing House and the Clearing House will sell to Trader B. Such intermediation reduces the risk of default for both traders and allows each trader to close the position independently of the other. In order to limit their counterparty risk and continue to guarantee financial integrity in the futures market, Clearing Houses have introduced a two-step methodology for refinancing deposits paid by participants: the first is the posting of initial margins, and the second is the marking to market of all contracts. The **Initial margin** to be posted by counterparties is typically expressed as a percentage of the contract value. The **marking to market** (MtM) procedure, on the other hand, consists of a daily calculation of the profits and losses associated with the open positions on derivatives. The task of the Clearing House is to offset the profits and losses relating to the account of each participant, with the corresponding payment of the margins. Thereby, the counterparty that has suffered a loss has the corresponding amount debited from its account opened with the Clearing House. The sum is automatically credited to the counterparty which has made a profit. In the event of losses, if the amount falls below the maintenance margin, the Clearing House requests the reinstatement of this margin. Unless the trader closes his position, profits and losses are subject to changes in quotes and are only paid out when the contract expires. These two mechanisms (initial margin and marking to market) are the main discriminating factors between a forward contract and a futures contract and Clearing Houses are also in charge of supervising the delivery of the underlying on the last trading day.

On the other hand, exchanges provide physical locations, computers, and methods for trading futures. As known, nowadays, most markets consist of electronic trading systems where trade orders are automatically matched and queued. Each match is executed according to the First in First Out (FIFO) logic. The buy order with the highest price is called the current bid price, the sell order with the lowest price is called the current ask price.

Feature	Futures Contract	Forward Contract
Trading	Regulated exchange	Over the Counter - OTC
Terms of contract	Standard	All terms are negotiable
Default risk	Generated by clearing house	Generated by each counterparty
Cash Flow	Daily cash flows linked to the change of the price (MtM) in order to guarantee execution	Generally not required
Pricing	Transparent	Opaque
Liquidity	Yes, provided by clearing house	In the absence of a secondary market, the contract cannot be traded before the maturity date

Table II.2 Futures versus Forwards contracts

On the futures market, there are two possible positions a trader can take i.e., a long position and a short position. A **"long" position** is the result of a purchase of futures contracts: the holder of a long position has the obligation to buy the underlying asset on the expiry date for the agreed price.
A **"short" position** results from a sale of futures contracts: the holder of a short position has the obligation to sell the underlying asset on the delivery date for the agreed price. In the jargon it is usually said that the trader who is long has bought the futures, while whoever has gone short has sold the futures. Clearly, it is only a figurative terminology because, as we have seen, there is no real sale of the contract, but a stipulation takes place which is the result of two counterparties entering into a mutual agreement. The calculation of the daily gain/loss is immediate and is given by the difference between the spot price and the one originally agreed, multiplied by the quantity purchased. While, at maturity, the profit/loss of the holder of a long position in futures is equal to:

Profit/Loss to Long=Contract Size x (Spot Price at maturity – Originally Agreed futures price) (*Eq. II.1*)

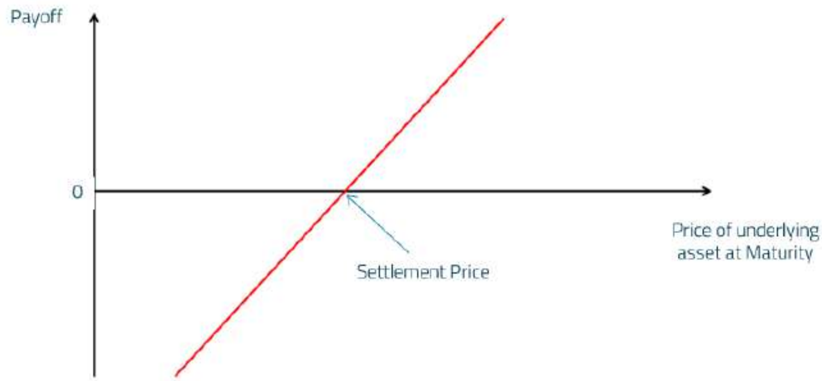

Figure II.5 Payoff of a long position in a futures contract

The payoff at maturity is the same as that obtained by an investor who would have directly bought the underlying at the price level, at the time of the stipulation of the futures contract. Theoretically, this statement is true, but it is incorrect from a cash flow perspective. In fact, buying a futures contract does not require the payment of the full price in advance (as in the case of direct purchase of the spot), but it is only necessary to pay the initial margin. Therefore, although the final payoff is the same, the return will be significantly different. On the other hand, at maturity, the profit/loss of the holder of a short position in futures is given by:

Profit/Loss to Short = Contract Size x (Originally Agreed futures price – Spot Price at maturity) (*Eq. II.2*)

As the price of the underlying rises above the agreed price (futures or settlement price), the payoff of a futures contract falls by the same amount. Whereas when the price falls below the settlement price, the value of the short position increases by the same amount.

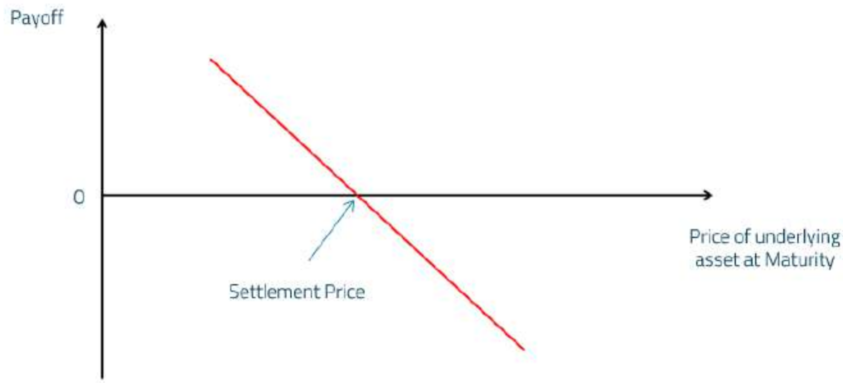

Figure II.6 Payoff of a short position in a futures contract

From the analysis of the equations and the payoffs, it is clear that the gains (/losses) for a trader who has a long position in a futures are equal to the losses (/gains) of a trader who has a short position. Thus, it can be said that the futures markets are a **zero-sum game**, i.e. the sum of the overall gains and losses on all positions is equal to zero, since each long position is neutralized from the corresponding short position.

Let us make an example and consider two counterparties entering a September currency futures contract in June that fixes the EUR/USD exchange rate at 1.5. The contract size is EUR 125,000, and let us assume that two business days before the third Wednesday of September, the EUR/USD spot rate is quoted at 1.492. We want to calculate the gains and losses.
At maturity, we know that the seller is expected to deliver EUR 125,000 to the buyer at the agreed price of 125,000 x 1.5 = USD 187,500. But, since the market price of this amount of EUR 125,000 is now equal to: 1,492 x 125,000 = EUR 186,500, the short position results in a gain of USD 1,000 and simultaneously the long position results in a loss of the same amount (USD 1,000).

A futures position is defined as **open** when the parties still have to buy/sell the underlying asset on the expiry date for the agreed price, in other words the parties are still exposed to the risk of price changes. One of the greatest advantages of standardized contracts lies in their ease of negotiation. Such liquidity allows a futures position to be closed or neutralized before its expiry.
The term "**Closing a position**" means making an equal but opposite trade to the one originally made. For example, if an investor is long in a futures contract, then he can go short to offset his current position at any time. This new contract does not necessarily have to take place with the same counterparty as the initial stipulation.
The net effect for the investor is that he no longer has open futures positions. Similarly, a trader who is short can go long on a futures with the same characteristics to offset his position.

Let us examine a practical example in which two counterparties enter a September currency futures contract in June that sets the USD/EUR exchange rate at 1.5 with a contract size of EUR 125,000. Let us then assume that, one month before expiry, a futures contract quotes USD/EUR at 1.492.
At that point, the futures buyer would like to close the position to limit the loss due to the exchange rate differential. So, the buyer should enter into another trade in which he sells (goes short) a USD/EUR futures contract at 1.492.
Then, at maturity the buyer will pay EUR 125,000 at the agreed price of EUR 125,000 x 1.5 USD/EUR = USD 187,500; and he will also receive EUR 125,000 at the market price of EUR 125,000 x 1.492 USD/EUR = USD 186,500.

The total loss, therefore, will be 187,500 – 186,500 = USD 1,000.

Note that (1.5-1.492) x EUR 125,000 = EUR 1,000.

Starting from the moment in which the position is closed, the loss remains constant, regardless of the level of the spot rate.

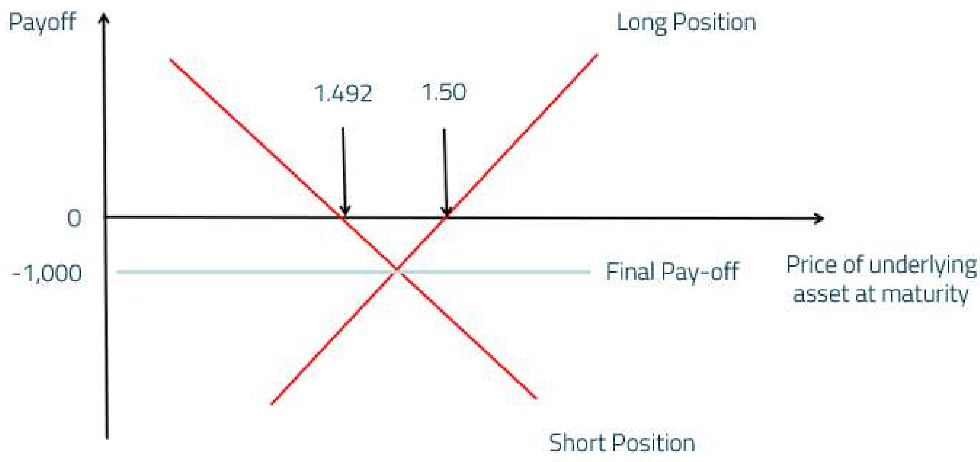

Figure II.7 Offset of a position in a futures/forward contract

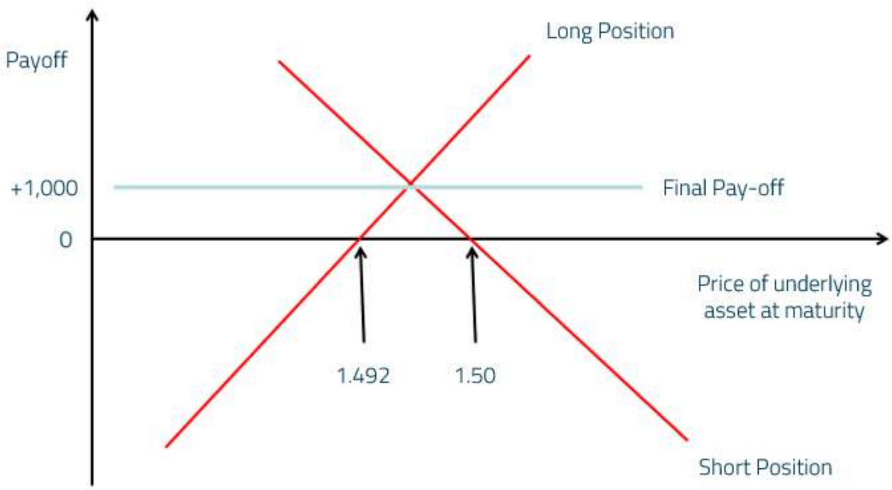

Figure II.8 Offset of a position in a futures/forward contract (specular side)

Implementing an offsetting trade like the one shown in the example is not normally possible with forward contracts, since they are highly customized, it is unlikely that a counterparty will accept the exact same conditions set and that a one-to-one closing of the existing position may occur.

It should also be noted that many exchanges may establish price limits in the trading of futures contracts in order to give the markets sufficient time to assimilate major events without causing significant price volatility and speculation. **Price limits** are normally defined in terms of the previous closing price +/- an amount of cents in cash per trading unit. Once the futures price has reached its daily price limit, trading at a higher price level cannot occur until the next trading day. Similarly, once the futures price has fallen below the daily limit, it is not possible to trade for a price level lower than the threshold throughout the trading day. Such imposition of limits on the price has the negative aspect that the trader is not able to close the futures position whenever he wants.

Certain regulated markets also allow to establish **position limits**, which regulate the maximum number of contracts that can be purchased by a single trader.

Another characteristic of Futures is that they are legally binding contracts: anyone holding an open position at expiration is obligated to receive (buy) / deliver (pay) the underlying. It should be highlighted though that most traders are not really interested in receiving the underlying at maturity and therefore will liquidate the open positions by entering into a transaction offsetting the existing one. The closed position can be then reopened (**rollover**) by re-proposing the existing strategy, before the liquidation of the position, by negotiating a new futures contract. In a few cases the delivery of the underlying takes place. In this case, it can be a **physical delivery** or the contract can be settled in cash (**cash delivery**). The **Physical Delivery** means the delivery, acceptance and payment of the physical underlying, and it occurs, for example, when financial futures are based on non-financial instruments. Typically, futures contracts that provide for the physical delivery of the asset are accompanied by choice options in favor of the buyer. For example, we mention the flexibility given by the choice of the place of delivery (**location option**), by the delivery time within a range of dates provided for in the contract (**timing option**). On the other hand, a **Cash Settlement** means honoring one's duties by means of a cash payment, which takes place on the futures expiry date. Typically, the cash balance is applied to futures having an intangible underlying (e.g., futures on financial indices) or when it is stipulated in a contractual clause. The choice to settle the obligations in cash allows to avoid delivery of the underlying and profits/losses are recorded due to the difference in price between the entry price and the exit price.

As we have seen above, **margins** and **marking to market** are two mechanisms used by exchanges and brokers in order to limit the impact of a potential default by the counterparty in a futures transaction.

Let us consider the case of an investor who contacts his agent wanting to buy a futures contract. Since each futures holder is exposed to a potential loss, the brokerage will require both counterparties (buyer and seller) to provide a **collateral** amount before opening a position. The collateral can be in the form of liquidity (cash) or constituted by securities with a high credit rating and it must be deposited in a dedicated account (**margin account**). The initial amount paid when entering into a futures contract is called the **initial margin**. Such amount mainly depends on the volatility of the price level of the underlying, but it typically ranges between 5% and 15% of the underlying value of the futures. It is used to address potential losses that could be incurred as a result of adverse price fluctuations. Clearly, at the end of each trading day, the futures price may have gone up or down, thus the margin account needs to be readjusted to reflect the investor's daily gain or loss on his open positions.

If there is a gain, the investor can withdraw any amount in excess of his initial margin, while in case of losses, the deposited margin is reduced. A **maintenance level** is set, to ensure there always is a margin. When the balance falls below the maintenance margin, then the investor receives a **margin call** and he has to increase the margin account up to the initial margin threshold in a short time, otherwise the broker will close his position. This daily procedure is called **marking to market**. In fact, it is as if a futures contract were closed and reopened every day at the market price registered on that day.

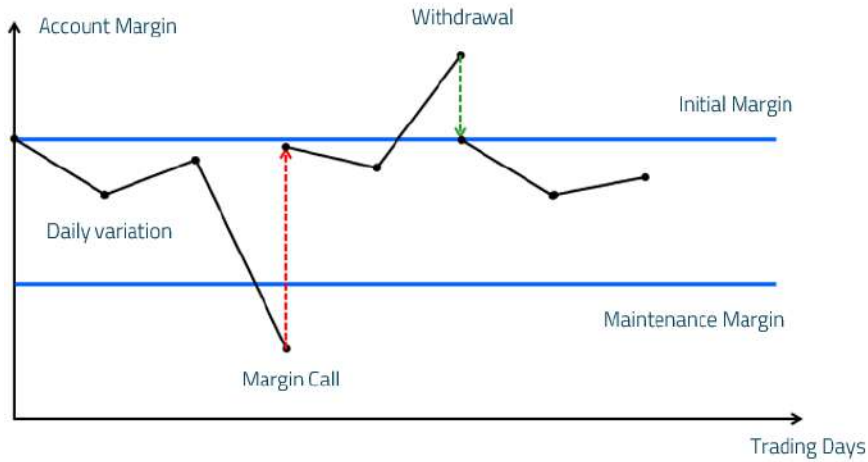

Figure II.9 Resettlement and margin variations

Practically, certain brokers allow investors to earn interest on the balance deposited in the margin account, whereas certain brokers may also accept securities such as government bonds or stocks as initial margin, but then, those assets are usually "discounted" compared to their market value (the so called "haircut" is applied on those assets). The standard minimum level for the initial and maintenance margin are defined by the exchange, but brokers can also ask for higher margins according to their customers.

Let us make an example (see Example 1 in Table II.3) and assume that on January 14, an investor buys a futures contract on COMEX (New York Commodity Exchange) for 100 ounces of gold to be delivered on February 1. The following table shows the margins and the adjustment to the marking to market. The initial margin is USD 5,063 and the maintenance margin is USD 3,750. If the margin account balance falls below the maintenance level threshold, the investor receives a margin call and after he posts the further amount, the balance would then align with the initial margin requirement level.
On the other hand, if the account balance is higher than the initial margin, the investor can withdraw the excess margin from his account.

Once:	100					
Trading Day	Futures Price [USD]	Gain/Loss [USD]	Initial margin [USD]	Cash withdrawal [USD]	Margin call [USD]	Margin account [USD]
01/14	808				5063	5063
01/15	807	-100	4963	0		4963
01/16	812	500	5463	400		5063
01/17	808	-400	4663	0		4663
01/20	810	200	4863	0		4863
01/21	807	-300	4563	0		4563
01/22	803	-400	4163	0		4163
01/23	799	-400	3763	0		3763
01/24	804	500	4263	0		4263
01/27	809	500	4763	0		4763
01/28	818	900	5663	600		5063
01/29	823	500	5563	500		5063
01/30	828	500	5563	500		5063
01/31	833	500	5563	5563		0
Net gain	25	2500				0

Table II.3 Marking to Market (Example 1)

We now present another example (see Example 2 in Table II.4), considering an investor buying two futures contracts on gold. The Initial margin is USD 5,063 for each contract (USD 10,126 in total). The maintenance margin is USD 3,750 (USD 7,500 for the overall position).
We assume that the contract was entered into on 1st June at USD 800 and closed on 22nd June at USD 784.60. The following table shows the performance of the margin account and the changes in the position of the account.
On 9th June, the balance drops to USD 54, therefore well below the maintenance margin. On that day, the investor receives a margin call from his broker for an additional margin to be integrated equal to USD 2,680 (so that the total balance readjusts to the initial margin of USD 10,126). The summary table assumes that such payment was made at the end of 10th June.
Later on, on 22nd June, the investor closes his position by going short by two contracts. The result of the whole strategy is a loss of USD 3,080.
We observe that the investor could have withdrawn money from the margin account on June 10, as the current balance was higher than the required margin.

Once:	100	**N. Futures contracts**	2		
Trading Day	Futures Price [USD]	Daily Gain/Loss [USD]	Cumulative gain/loss [USD]	Cash withdrawal [USD]	Margin call [USD]
	800		0	10126	
06/01	794	-1200	-1200	8926	
06/02	792.2	-360	-1560	8566	
06/03	796.4	840	-720	9406	
06/04	794.2	-440	-1160	8966	
06/05	793.4	-160	-1320	8806	
06/06	790.8	-520	-1840	8286	
06/09	**786.6**	**-840**	**-2680**	**7446**	**2680**
06/10	787.2	120	-2560	10246	
06/11	783.6	-720	-3280	9526	
06/12	785.4	360	-2920	9886	
06/15	774	-2280	-5200	7606	
06/16	774	0	-5200	7606	
06/17	776.2	440	-4760	8046	
06/18	777.4	240	-4520	8286	
06/19	782	920	-3600	9206	
06/22	784,6	520	-3080	9726	

Table II.4 Marking to Market (Example 2)

Initial margins and the marking to market are typical of futures contracts, not of forward contracts, in which case the settlement is made only once, at the expiry date. No cash flows are thus exchanged during the life of the forward contract, which means that the risk of default is strictly linked to the counterparty.

Let us compare the behavior of both contracts in a practical example (see Example 3 in Table II.5). We assume that, on January 14, an investor buys one futures and one forward contract, both written on 100 ounces of platinum, to be settled on February 1. For the sake of simplicity, we assume that forward prices are equal to spot prices and that there are no credit problems. The following table shows that the final gain in futures and forwards is identical, while the distribution of cash flows is very different between the two types of contracts. If the margin account pays an interest, the final profit will not only depend on the price of the underlying asset at maturity, but it will also be influenced by the balance on the deposit during the life of the contract. From this perspective, a futures contract can be viewed as a series of forward contracts, each lasting one day.

Once:	100		
Trading Day	Forward and Futures Price [USD/oz]	Variations in forward account [USD]	Variations in futures account [USD]
01/14	331	0	0
01/15	336	0	500
01/16	336	0	0
01/17	333	0	-300
01/20	333	0	0
01/21	338	0	500
01/22	339	0	100
01/23	342	0	300
01/24	344	0	200
01/27	340	0	-400
01/28	343	0	300
01/29	341	0	-200
01/30	337	0	-400
01/31	334	0	-300
Gain	3	300	300

Table II.5 Marking to Market (Example 3)

It should be highlighted that futures trading, unlike that of the underlying, allows participants to use leverage. Consequently, a small variation in price leads to a larger variation in terms of P&L (profit and loss).

Let us see a practical example involving the DAX futures, which are based on the DAX stock index, one of the most important stock indexes of the German Stock Exchange. To open a position in DAX futures, a margin deposit of EUR 15,000 per contract is required. Let us assume that the DAX is quoted at 7,200 points, and that the value of the futures amounts to EUR 180,000 (i.e., the contract size is 25 times the price level of the underlying). Thereby, if the DAX index increases by 1%, reaching 7,272 then the value of the futures position increases to 72 x EUR 25 = EUR 1,800, which translates into a 12% profit on invested capital (i.e. on the initial margin). Obviously, in the case of a decrease of the index to 1%, it would equally imply a 12% loss on the open position in futures.

On the real markets, most futures contract quotes follow the same conventions. Let us take the following table as an example:

COTTON (CTN) 50,000 lbs; cents per lb.							
Expiration	Open	High	Low	Settle	Change	Open Interest	Volume
Oct 22	67.48	68.50	67.20	67.53	0.37	4297	79
Dec 22	69.49	70.90	69.37	69.78	0.42	154533	8379
Mar 23	74.19	75.31	74.00	74.38	0.39	34544	1865
May 23	77.14	77.25	76.00	76.38	0.31	3612	220
Jul 23	79.05	79.20	78.00	78.31	0.32	7279	2823
Oct 23	77.14	77.25	76.00	76.38	0.34	116	-
Est. Vol. 11,088; prev. Vol. 9,848; open int. 218,270; -6,000							

Table II.6 Quoting conventions

The first line of Table II.6 describes the asset underlying the futures, the contract size and how it is quoted. The term **Expiration** indicates the month of expiration of the contract. The column **Open** contains the price of the first completed transaction (or, in its absence, the price of the first offer); the column **High** reports the highest ask or highest price at which the contract is traded; the column **Low** indicates the lowest bid or the lowest price at which the contract was traded; the column **Settlement Price** (or Settle) contains the closing price used to calculate margin account changes; the column **Net Change** (Change) reports the difference between the settlement price recorded on the current trading day and the previous one. The column **Open Interest** indicates the cumulative amount of all outstanding futures contracts (on one side only) i.e. the number of futures contracts which, at a certain point in time, have a long and a short position open at the same time. Lastly, the column **Volume** shows the number of contracts traded in one day.

The information in the last row of the table is the estimated volume (for all maturities and one side only) on the last and previous trading day, the open interest and lastly, the change from the previous trading day. The spot of the underlying is often shown. Clearly then, to obtain the price of a futures contract, the market quotation must be multiplied by the contract size. In the proposed example, each price has to be multiplied by 50,000 to obtain the value of a futures contract.

In addition to this, each futures contract has a different **tick size** and **tick value**. Those figures are generally not shown on the quotes on the screen as they are conventions reported in the contract specifications and they are adopted by the entire stock exchange in which they are traded. The **tick size** (or minimum price change) is the smallest increment by which the price can move. If, for example, the tick size is equal to 0.0001, it means that if the current price is equal to 1.2902, the smallest variations can be 1.2903 or 1.2901. The **tick value** (or minimum price value) is the amount of money per tick size. For example, if a contract has a tick value of USD 12.50, it means that for every 0.0001 increase/decrease in price the profit/loss of a trade will increase/decrease by USD 12.50.

By definition, a futures contract can only be traded on official exchanges (**futures exchanges**). Historically, trading took place in special spaces and by "open outcry", with the onset of technology nowadays trading takes place on multilateral electronic trading platforms. There are 13 futures exchanges registered in the USA and more than 50 scattered throughout the rest of the world.

The most important ones are: Chicago Board Options Exchange (CBOE), Chicago Mercantile Exchange (CME), International Monetary Market (IMM), Chicago Climate Exchange Intercontinental Exchange (ICE), New York Board of Trade (NYBOT), Kansas City Board of Trade (KCBT), Minneapolis Grain Exchange (MGEX), New York Mercantile Exchange (NYMEX), Philadelphia stock exchange (PHLX) …

Figure II.10 Trading room

In the Eurozone, the main pan-European markets are Eurex and Euronext. In addition to these, several countries have maintained their own domestic exchanges:

- Great Britain: International Petroleum Exchange (IPE) in London; London Metal Exchange (LME).
- Japan: Central Japan Commodity Exchange (C-COM), Osaka Securities Exchange, Tokyo Commodity Exchange (TCE), Tokyo Stock Exchange (TSE), Tokyo International Financial Foreign Exchange (TIFFE),
- Hong Kong: Hong Kong Exchanges and Clearing (HKEx)

The Exchanges do not trade futures themselves, but they provide the means for buyers and sellers to meet, they offer researches and quotes useful for taking decisions, as well as supervise trading and enforce regulations. Exchanges also monitor and encourage financial and ethical standards, and they typically provide daily statistics and historical series of the traded instruments.

Exchanges can be organized both as publicly listed companies, and as private companies/ organizations, either for profit or not.

Their earnings derive from the fees they collect on each exchange, from the clearing services they offer, as well as from the sale of real-time information and historical series, as mentioned above.

As we have seen, the underlyings of futures contracts vary from one exchange to another.
There are futures written on agricultural products and foods (wheat, soybeans, rice, sugar, coffee, cocoa, orange juice, potatoes, cattle), on precious and industrial metals (copper, aluminum, gold, silver), on oil and products derived from it (fuel oil, petrol) and on timber. Financial futures are also available on the main foreign currencies, on financial instruments linked to interest rates (short and long-term bonds), and on the most important stock indexes.

Figure II.11 CTM – Contract Exchange Menu: Categories. Source: Bloomberg®

Figure II.12 CTM – Contract Exchange Menu: Exchange A-I. Source: Bloomberg®

Figure II.13 CTM – Contract Exchange Menu: Exchange I-N. Source: Bloomberg®

Figure II.14 CTM – Contract Exchange Menu: Exchange N-Z. Source: Bloomberg®

Figure II.15 CTM – Contract Exchange Menu: Region North America, South America and Europe. Source: Bloomberg®

Figure II.16 CTM – Contract Exchange Menu: Region Europe and Asia/Pacific. Source: Bloomberg®

Figure II.17 CTM – Contract Exchange Menu: Region Europe and Asia/Pacific, Africa and Middle East. Source: Bloomberg®

FURTHER READINGS

Bruzda J. – “Examination of the Cost-of-Carry Formula for Futures Contracts on WIG20. Wavelet and Nonlinear Cointegration Analysis” – Faggini, M., Lux, T. (eds) Coping with the Complexity of Economics. New Economic Windows. Springer, Milano (2009).
Deep A. – “Optimal dynamic hedging using futures under a borrowing constraint” – Bank for International Settlements (BIS) Working Paper N. 109 (2002).
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I.2 QUANTITATIVE ANALYSIS

As known, on the maturity date of a futures contract, the futures price should be equal to the spot price, since the two assets become perfect substitutes for each other. We denote with $\mathbf{F_{t,T}}$ the market price at time t of the futures contract with maturity T, thus the following relation holds:

$$\mathbf{F_{T,T} = S_T} \text{ } (\textit{Eq. II.3})$$

Otherwise, an arbitrageur could buy the lower priced financial instrument and sell the higher priced instrument for an immediate risk-free profit. If the spot is overpriced, the arbitrageur buys the futures, brings the spot through the delivery process, and sells it. If, on the other hand, the futures contract is overpriced, the arbitrageur buys the spot, sells the futures and holds the spot till expiration. However, before the maturity ($t<T$) the spot price is not necessarily equal to the futures price. The difference between the two prices is called the basis:

$$\mathbf{B_{t,T} = F_{t,T} - S_t} \text{ } (\textit{Eq. II.4})$$

The basis can be positive, negative or zero, and it converges to zero as the expiration date of the contract approaches. Let us examine the factors that cause the difference between the spot and the futures prices. Financial literature indicates three models that can be used for pricing a futures contract, namely:

A. The CAPM (Capital Asset Pricing Model), in which the return is a function of market risk.
B. The net hedging pressure theory, in which returns are systematically biased in favor of a net speculative position versus a net hedging position.
C. The cost of carry model, in which arbitrage creates a link between the spot and the futures markets.

A. The **CAPM** (Capital Asset Pricing Model)

According to the CAPM model, the return on a security is a function of both its market risk exposure (beta) and the market risk premium. According to this framework, an analyst could try to estimate the beta of futures contracts and, consequently, their value. Unfortunately, the huge leverage implied in futures contracts invalidates some of the assumptions included in the CAPM working principle. In particular, the distribution of futures returns empirically tends to have fatter tails than those present in a normal distribution. In addition, some researchers have highlighted a very low beta parameter for futures contracts written on commodities and this fact suggests a weak relationship. For these considerations, CAPM should not be employed as a robust and reliable model for futures pricing.

B. The **net hedging pressure theory**

According to the unbiased expectation hypothesis, futures prices are unbiased predictors of the expected future spot price, therefore:

$$\mathbf{F_{t,T} = E(S_t)} \text{ } (\textit{Eq. II.5})$$

In real life however, an analyst might on average accept the absence of bias. This would imply that the theoretical profit expected from a futures position is equal to zero on average. But supply-demand factors may challenge the average absence of bias, as discussed by Keynes and Hicks. In short, let us consider the case of commodity futures. Keynes first suggested that the spot prices of a commodity are so volatile that a commodity producer will agree to sacrifice some of the returns in order to hedge against the risk of price fluctuations during its production period. Consequently, in a producer-dominated market, hedging pressure by the producer could mean that the future price of a futures contract on that commodity may have a discount to the spot price of the same commodity. The result is a forward curve characterized by a negative slope (downward sloping forward curve), and this situation is called **normal-backwardation**.

Normal backwardation results in a **positive roll yield** which means that an investor can:

1) Buy a futures contract at a lower price than the spot,

2) As soon as the contract expires, sell it for profit,

3) Re-establish the position in a new contract.

The antithetical hypothesis is called **contango,** and it is the case in which commodity consumers will take a long position in order to receive the commodity in the future at a guaranteed price and speculators will take the short position. Thus, if the net long hedging position exceeds the net short speculating position, futures prices will be overpriced compared to their true fair value, thereby encouraging speculators to sell futures. Consequently, the forward curve will assume a positive slope (**negative roll yield**).

The three theories we have examined (unbiased expectation hypothesis, normal-backwardation and contango situation) can be unified in the so-called **hedging pressure theory** or net hedging hypothesis:

$\mathbf{F_{t,T} < E(S_t)}$ when short hedgers outnumber long hedgers.

$\mathbf{F_{t,T} > E(S_t)}$ when long hedgers outnumber short hedgers.

$\mathbf{F_{t,T} = E(S_t)}$ when the number of long hedgers exactly equals the number of short hedgers.

C. The **cost of carry model**

Unlike the previous models, this one is based on the concept of **arbitrage-free pricing**, i.e., it is based on the assumption that futures contracts should be valued in such a way as to exclude profits from arbitrage between spot and futures markets. The result of this approach is that the fair price depends on several parameters such as the spot price of the underlying asset (S_t), the time to maturity of the futures (T-t), the time value of money ($R_{t,T}$), and the incoming/outgoing cash due to the underlying. It is the most commonly implemented model for the theoretical valuation of a Futures contract, consequently it will be discussed with greater detail.

The first case we discussed dealt with the pricing of a futures written on an asset that does not provide for income during the life of the contract. Within this category, we can mention shares that do not pay dividends, discount bonds (or zero coupon bonds) and precious metals (pure commodities). If we momentarily disregard margin calls, a futures contract is essentially a mechanism for buying an asset (underlying) now, for delivery at a future date at a prefixed price. To implement this strategy, the only requirement is to save enough money to be able to pay the future price of the asset when due. Since the future price is known, we can easily determine how much money is required by computing the present value. Alternatively, another obvious way to obtain the underlying asset at a future date is to borrow money today to buy it and then hold it to maturity. This trading method is called a "**cash and carry**" strategy. When such strategy is properly implemented, it will exactly match the cash flows of a futures strategy.
Since the two ways of operating lead to the same result (i.e. having the underlying at the future date), they must have the same implementation cost, otherwise arbitrage opportunities would arise. Therefore, equalizing the two implementation costs creates the link between futures and spot prices. In short, to synthetically construct a futures contract with the "cash and carry" strategy, a trader must:

1) First of all, buy the underlying asset by borrowing cash.
2) Then the underlying should be kept until the delivery date of the futures (carry).
3) Finally, on the due date, he should pay off the initial loan.

As we said above, if we disregard the margins, this strategy perfectly replicates the cash flows of a futures contract on its expiration date.

Futures written on a share with no pay-out	Net outflows today	Net inflows date T
Portfolio today (date t)		
1 Buy one share at cost S_t	$-S_t$	
2 Finance purchase of share by borrowing at risk-free rate $R_{t,T}$	$+S_t$	
3 Sell one forward contract, maturity T, forward price $F_{t,T}$	0	
TOTAL	0	
Portfolio at date T		
1 Value of share		$+S_T$
2 Repay borrowing		$-S_t \times (1+R_{t,T})$
3 Value of forward (Deliver the underlying asset at T to liquidate the futures contract)		$-(S_T - F_{t,T})$
TOTAL		$F_{t,T}-S_t \times (1+R_{t,T})$

Table II.7 Cash and carry strategy

We observe that the value of this portfolio is always known since it is independent from the underlying price assumed on date T: thus, there is no uncertainty. Therefore, to avoid any arbitrage opportunity, the final payoff must be zero because otherwise positive cash flows can be generated on date T, without any initial investment and without risk.

$$\mathbf{F_{t,T} - S_t \times (1 + R_{t,T}) = 0 \rightarrow F_{t,T} = S_t \times (1 + R_{t,T})} \quad (Eq.\ II.6)$$

$\mathbf{F_{t,T}}$ is the futures price on date t of a contract expiring on date T; S_t is the spot price of the underlying on date t and $\mathbf{R_{t,T}}$ is the risk-free interest rate for the period (T-t).

An important feature of financial assets compared to commodities is that they have no storage costs. If we consider a futures on a non-financial asset, such as gold, silver or corn, then the costs of storage must also be taken into account.

Futures on corn	Net outflows today	Net inflows date T
Portfolio today (date t)		
1 Buy one asset at cost S_t	$-S_t$	
2 Finance purchase of the asset by borrowing at risk-free rate $R_{t,T}$	$+S_t$	
3 Sell one forward contract, maturity T, forward price $F_{t,T}$	0	
TOTAL	0	
Portfolio at date T		
1 Value of asset		$+S_T$
2 Repay borrowing		$-S_t \times (1+R_{t,T})$
3 Pay storage costs		$-k(t,T)$
4 Value of forward (Deliver the underlying asset at T to liquidate the futures contract)		$-(S_T - F_{t,T})$
TOTAL		$F_{t,T}-S_t \times (1+R_{t,T})-k(t,T)$

Table II.8 Cash and carry strategy: Futures on corn

Again, the payoff on date T is already known on date t. Therefore, the strategy is risk-free and in order to avoid arbitrage, the payoff must be zero, which implies the value of the futures must be equal to:

$$\mathbf{F_{t,T} = S_t \times (1+R_{t,T}) + k(t,T)} \quad (Eq.\ II.7)$$

Cash and Carry arbitrage

Suppose we observe the market price of a futures contract $F^M_{t,T}$ overpriced compared to its theoretical value, $F^T_{t,T}$, that is: $F^M_{t,T} > F^T_{t,T}$. An investor could implement a cash and carry arbitrage strategy to profit from the price mismatch.

Reverse Cash and carry arbitrage

Suppose we observe the market price of a futures contract $F^M_{t,T}$ underpriced compared to its theoretical value, $F^T_{t,T}$, that is $F^M_{t,T} < F^T_{t,T}$. An investor could implement a reverse cash and carry arbitrage strategy to profit on the price mismatch.

Cash and carry arbitrage	Net outflow today	Net inflow date T
Portfolio today (date t)		
1 Buy one asset at cost S_t	$-S_t$	
2 Finance purchase of the asset by borrowing at risk-free rate $R_{t,T}$	$+S_t$	
3 Sell one futures contract, maturity T, market price $F^M_{t,T}$	0	
TOTAL	0	
Portfolio at date T		
1 Value of asset		$+S_T$
2 Repay borrowing		$-S_t \times (1+R_{t,T})$
3 Pay storage costs		$-k(t,T)$
4 Value of futures (Deliver the underlying asset at T to liquidate the futures contract)		$-(S_T - F^M_{t,T})$
TOTAL		$F^M_{t,T} - St \times (1+R_{t,T})-k(t,T) = F^M_{t,T} - F^T_{t,T}$

Table II.9 Cash and carry arbitrage

Reverse Cash and carry arbitrage	Net outflow today	Net inflow date T
Portfolio today (date t)		
1 Sell short underlying asset	$+S_t$	
2 Lend proceeds from short sale at risk free, Rt,T	$-S_t$	
3 Buy one futures contract, maturity T, price of FMt,T	0	
TOTAL	0	

Portfolio at date T		
1 Value of asset		$-S_T$
2 Receive proceeds of the loan		$+S_t \times (1+R_{t,T})$
3 Charge storage costs		$+k(t,T)$
4 Value of futures (Buy the underlying asset at T to liquidate the futures contract)		$(S_T - F^M_{t,T})$
TOTAL		$-F^M_{t,T} + S_t \times (1+R_{t,T})+k(t,T) = F^T_{t,T} - F^M_{t,T}$

Table II.10 Reverse Cash and carry arbitrage

A further case study relates to the pricing of a futures contract written on an asset characterized by known proceeds during its life. The logic is similar to the previous case, to avoid any arbitrage, there should be no difference for the investor between buying the futures and holding the position open until expiration or buying the spot and holding it until the futures expires. The only difference from the previous case is that if the investor buys the spot, he will receive a known income (dividends or coupon payments), while the holder of the futures contract will not receive anything.
The formula can be generalized as follows:

$$\mathbf{F_{t,T} = S_t \times (1+R_{t,T}) + k(t,T) - FV(revenues)} \quad (\textit{Eq. II.8})$$

Where $F_{t,T}$ is the futures price on date t of a contract expiring on date T.
S_t is the spot price of the underlying on date t.
$R_{t,T}$ is the risk-free interest rate for the period (T-t).
FV(revenues) is the future value of the underlying revenues.
k(t,T) is the future value of costs – carrying costs (insurance/storage).

Let us consider as an example, a hypothetical stock, paying a dividend of USD 1 per quarter. The spot price is S_t = USD 140 and the futures price for a one-year contract is $F^M_{0,1}$ = USD 148. The next dividend is paid in 3 months and the interest rate for lending and borrowing is assumed to be the same and equal to 10% p.a. The future value of the dividend stream is equal to:

$$FV(dividends)=1 \times (1+9/12 \times 0.10) + 1 \times (1+6/12 \times 0.10) + 1 \times (1+3/12 \times 0.10) + 1 = 4.15$$

Whereas the theoretical price of the futures is:

$$\begin{aligned} F^{TH}_{0,1} &= S_t \times (1+R_{t,T}) + k(t,T) - FV(revenues)= \\ &= 140 \times (1+0.10) + 0 - 4.15 = \\ &= 149.85 > 148 \end{aligned}$$

The theoretical price of the futures is higher than the price observed on the market. Thus a reverse cash and carry arbitrage strategy can be implemented.

Reverse Cash and carry arbitrage	Net outflow today	Net inflow date T
Portfolio today (date t)		
1 Sell short underlying asset	+140	
2 Lend proceeds from short sale at risk free, $R_{t,T}$	-140	
3 Buy one forward contract, maturity T, forward price of $F^{M}_{0,T}$= 148	0	
TOTAL	0	
Cash-Flows at each quarter		
1 Borrow $1 per quarter	+1	
2 Pay $1 per quarter	-1	
Portfolio at date T		
1 Value of asset		$-S_T$
2 Receive proceeds of the loan		154
3 Pay back loan		-4.15
4 Value of forward(Buy the underlying asset at T to liquidate the contract)		S_T - 148
TOTAL		1.85

Table II.11 Reverse Cash and carry arbitrage on a stock with cash-out

The difference between the theoretical and the observed value of the futures contract is equal to USD 1.85 and it is a pure risk free profit.

Practically, analysts observe that the futures price on the markets tends to be lower than the theoretical price. This does not always translate as an arbitrage opportunity though, because this discrepancy is caused by certain intrinsic limitations existing in the spot-futures price relationship. We can list the following considerations:

- the existence of transaction costs (commissions to brokers, fees, bid-ask spread),
- the market liquidity, i.e., the lower the liquidity, the higher the bid-ask spread,
- short selling is often restricted or prohibited,
- the borrowing rate is different from the lending rate (while in the previous model, the rate is the same),
- the amount of the dividends and the payment dates are a-priori unknown,
- the received/paid interest in the marking to market procedure should also be considered.

For these reasons, in real life, traders often set convenience thresholds (lower and upper bounds) within which there is an effective gain in applying arbitrage strategies.

Figure II.18 Contract Table SPX Index Futures. Reference Date: 21st December 2022. Source: Bloomberg®

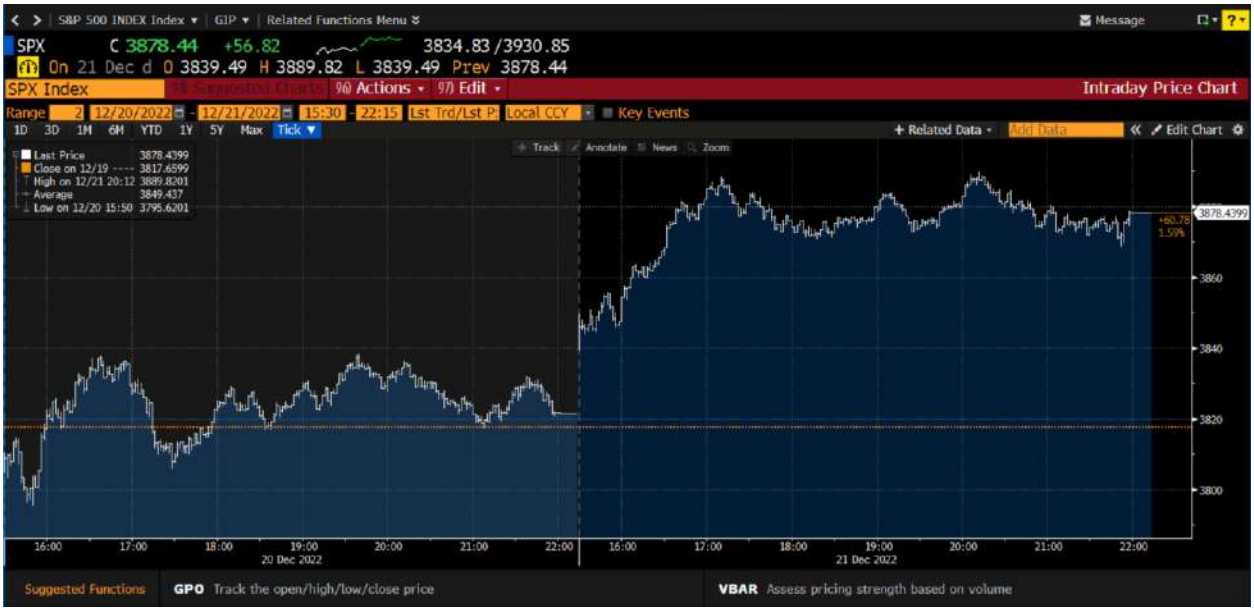

Figure II.19 SPX Index (Standard & Poor's 500 Index) – Intra Day. Source: Bloomberg®

Figure II.20 S&P500 Mini Futures Dec 23 description. Source: Bloomberg®


S&P 500 INDEX 3872.14 USD Bloomberg Mid As of 21-Dec-2022 21:51

1) Vol Table | 2) 3D Surface | 3) Term | 4) Skew | 5) Dividends | 6) Prices

Listed | Yields

Expiry	Exp Date	Risk Free	Impl Dvd	Impl (Yld)	BDVD Divs
31-Ma...	31 Mar 2023	4.619%	14.930	1.408%	N/A
21-Ap...	21 Apr 2023	4.667%	17.322	1.350%	N/A
28-Ap...	28 Apr 2023	4.680%	18.065	1.331%	N/A
19-Ma...	19 May 2023	4.716%	23.072	1.461%	N/A
31-Ma...	31 May 2023	4.733%	25.386	1.487%	N/A
16-Ju...	16 Jun 2023	4.752%	27.065	1.442%	N/A
30-Ju...	30 Jun 2023	4.766%	29.029	1.434%	N/A
21-Ju...	21 Jul 2023	4.783%	31.406	1.397%	N/A
18-Au...	18 Aug 2023	4.794%	36.753	1.445%	N/A
15-Se...	15 Sep 2023	4.801%	40.285	1.418%	N/A
29-Se...	29 Sep 2023	4.802%	42.229	1.413%	N/A
20-Oc...	20 Oct 2023	4.800%	44.617	1.389%	N/A
17-No...	17 Nov 2023	4.789%	49.342	1.406%	N/A
15-De...	15 Dec 2023	4.773%	54.060	1.420%	N/A
29-De...	29 Dec 2023	4.761%	54.133	1.369%	N/A
19-Ja...	19 Jan 2024	4.722%	55.772	1.335%	N/A
15-Ma...	15 Mar 2024	4.640%	64.126	1.344%	N/A

Figure II.21 Option Implied Dividend Yield and Risk Free rate. Source: Bloomberg®

Here is a practical example of an estimation of the fair value for the Future using the cost of carry model, with the following data:

Evaluation Date: 21st December 2022

Spot Level: USD 3878.44 (Close Price)

Futures Z3 Expiration: 15th December 2023

Time to Maturity: (12/15/2023 – 12/21/2022)/365 = 359/365 = 0.9835616

Risk free rate (r): 4.773% p.a.

Dividend Yield (q): 1.42% p.a.

Theoretical Value ($F^{TH}_{t,T}$): S x exp[(r-q) x T] = 3878.44 x exp[(4.773%-1.42%) x 0.9835616] = USD 4008.47

Market Value ($F^{M}_{t,T}$): USD 4006

Model Gap: USD 2

In this financial context, the term hedging means the creation of a position in the futures markets, as opposed to the physical market, with the aim of reducing or limiting the risks associated with price changes. The risk reduction is carried out symmetrically with respect to the current value of the position. Consequently, the trader who decides to protect his portfolio from the risk by hedging himself (**hedger**) will renounce part of the profit if the price change plays in his favor.

The hedging process using forward and futures contracts appears intuitively simple: if a trader has bought the spot, he will sell forward/futures contracts assuming a short position; similarly, if a trader has sold the spot (or needs a spot), he will buy futures/forward contracts (long position).

Let us clarify by means of an example, supposing that the Gold Company has 1,000 ounces of gold and wants to sell it in July; the current spot is USD 900 per ounce. The company wishes to hedge against gold price movements, therefore it decides to sell 1,000 ounces at the forward price of USD 940 per ounce. Let us show that the hedging strategy is good by considering two future scenarios.

The first scenario provides that the value of the spot on the desired date in July is equal to USD 1,000/ounce; while in the second scenario it is USD 850/ounce.

Scenario	Spot revenue	profit from forward transaction	total profit
1	USD 1,000,000	USD -60,000	USD 940,000
2	USD 850,000	USD +90,000	USD 940,000

Table II.12 Hedging strategies

In this case, the company implements a hedging strategy through an ad-hoc forward agreement, and, regardless of the future spot value assumed by the underlying, the total profit is unchanged.

An intermediary does not always have the opportunity and flexibility to enter into an OTC forward contract, so he will have to turn to the liquid futures market. In that case, the probability of finding exactly the instrument that meets the desired technical specifications is extremely rare, given the standard nature of futures contracts (but the upside is that they are much safer in terms of counterparty risk).

In order to understand the basic concepts of hedging, let us consider a spot asset with an initial value of S_0 and a futures contract written on such underlying with an initial value of $F_{0,T}$ and a contract size of k. Let us suppose we have a portfolio P which contains N_S spot assets. Ideally, we would like to add N_F futures to our portfolio P with the aim to neutralize its value. In mathematical notation, the change in value of the portfolio for the next period is given by:

$$\mathbf{\Delta V_P = N_S \times \Delta S + N_F \times k \times \Delta F} \quad (Eq.\ II.9)$$

Where ΔS is the unit price variation of the spot and ΔF is the unit price variation of the Futures. By setting the condition of $\Delta V_P = 0$, a perfect hedging condition is imposed.

$$N_S \times \Delta S + N_F \times k \times \Delta F = 0 \rightarrow N_F = -\frac{N_S}{k} \times \frac{\Delta S}{\Delta F} = -\frac{N_S}{k} \times HR \quad (Eq.\ II.10)$$

Where HR is called the hedge ratio and is equal to $\frac{\Delta S}{\Delta F}$.

We notice that the coverage ratio can be rewritten in terms of the market value of the current spot position:

$$HR = -\frac{N_F \times k \times S}{N_S \times S} = -\frac{N_F \times Futures\ contract\ size \times Spot\ Asset\ Price}{Market\ Value\ of\ Spot\ Position} \quad (Eq.\ II.11)$$

It follows that:

$$N_F = -HR \times \frac{Market\ Value\ of\ Spot\ Position}{Futures\ contract\ size \times Spot\ Asset\ price} \quad (Eq.\ II.12)$$

If there were a one-to-one relationship between the change in spot prices and the change in futures prices, i.e. if $\Delta S = \Delta F$, the above equation would become:

$$\begin{cases} HR = 1 \\ N_F = -\frac{N_S}{k} \end{cases} \quad (Eq.\ II.13)$$

Hedging the risk using this equation is called a **naive strategy**.

Let us now consider an example. An American firm wishes to hedge a EUR 25 million commitment due in November by taking a position on an IMM EUR futures contract due at the December delivery. EUR futures have a contract size of EUR 125,000. Since the firm's commitment equals a short position in euros, the firm will buy futures contracts. In accordance with the naive method, the company should buy 200 contracts:

$$N_F = -\frac{-25{,}000{,}000}{125{,}000} = 200$$

The naive method is simple to use and reasonably accurate for spot positions with extremely similar characteristics to the futures contracts. In real markets, however, perfect hedging (i.e. total elimination of risk) will only occur when the futures and the spot move proportionally over time and when the investor's time horizon exactly coincides with the expiry of the futures contract. Thus in a real context, the main barriers to perfect hedging are **basis risk** and quality or **correlation risk**. **Basis risk** arises when the maturity of the Futures contract (T) does not coincide with that of the time horizon to be hedged. As an example, if an investor had a long position in the spot market and a short position in the futures market, the hedged profit of his position would be given by:

$$\textbf{Hedged Profit} = (S_T - S_t) - (F_{T,T} - F_{t,T}) \quad (\textit{Eq. II.14})$$

If the holding period of the spot ends exactly when the futures expires, the time base T is zero since $S_T = F_{T,T}$

$$\textbf{Hedged Profit} = F_{t,T} - S_t \quad (\textit{Eq. II.15})$$

In this case, the covered profit is perfectly known in any time interval t; therefore, the hedge is effective.

Figure II.22 Basis variation over time

If the spot holding period ends at time $T_1<T$, the hedger will have to liquidate his futures position before its natural expiry at the price of $F^T{}_{1,T}$.

Therefore, the hedged profit is:

$$\textbf{Hedged Profit} = -(F_{T1,T} - F_{t,T}) + (S_{T1} - S_t) = (F_{t,T} - S_t) - (F_{T1,T} - S_{T1}) \quad (\textit{Eq. II.16})$$

$$\textbf{Hedged Profit} = \text{Basis}_t - \text{Basis}_{T1} \quad (\textit{Eq. II.17})$$

For a generic time t, the base T_1 is not known. Consequently, the final profit is uncertain and in this case the hedger can replace the price risk with the basis risk. To minimize delivery basis risk, an analyst may be tempted to choose the contract with the closest maturity to the investor's needs, but this is not necessarily the optimal strategy, thus the trader must always carefully analyze the trade-off between basis risk and liquidity risk, especially if the desired hedge is very long.

On the other hand, **correlation** or **"quality" risk** exists when the spot security/commodity/asset is not exactly homogeneous to that underlying the futures. Cross hedging occurs when the spot and the underlying of the futures are not equal.

As an example, let us consider an airline that is impacted by the future price of fuel. Since jet fuel futures are not actively traded, a trader will have to choose fuel oil or similar contracts. The two assets are similar, but they are not perfectly correlated (**asset mismatch**).

Now let us consider the case of an investor holding a portfolio of long positions in N_S spot assets covered by N_F futures with a contract size equal to k. The expected value of the profit/loss (expected P&L) on the hedged position is equal to:

$$\mathbf{E(Profit) = N_S \times E(\Delta S) + N_F \times k \times E(\Delta F)} \quad (\textit{Eq. II.18})$$

Since perfect hedging is not possible, the hedger's goal is to minimize the variance of his hedged profit in monetary terms. In mathematical terms, this means:

$$\mathbf{Var(Profit) = N_S^2 \times Var(\Delta S) + N_F^2 \times k^2 \times Var(\Delta F) + 2 \times N_F \times k \times N_S \times Cov(\Delta S, \Delta F)} \quad (\textit{Eq. II.19})$$

In order to minimize the variance, it is necessary to zero its first derivative with respect to N_F:

$$\frac{\partial Var(Profit)}{\partial N_F} = 0 \quad (\textit{Eq. II.20})$$

It follows that:

$$2 \times N_F \times k^2 \times Var(\Delta F) + 2 \times k \times N_S \times Cov(\Delta S, \Delta F) = 0 \quad (\textit{Eq. II.21})$$

This can be rewritten as:

$$N_F = \frac{N_S}{k} \times \frac{Cov(\Delta S, \Delta F)}{Var(\Delta F)} \quad (\textit{Eq. II.22})$$

The ratio $\frac{Cov(\Delta S, \Delta F)}{Var(\Delta F)}$ is called minimum-variance hedge ratio (HR).

$$HR = \frac{Cov(\Delta S, \Delta F)}{Var(\Delta F)} = \rho_{\Delta S, \Delta F} \times \frac{\sigma_{\Delta S} \times \sigma_{\Delta F}}{\sigma_{\Delta F}^2} = \rho_{\Delta S, \Delta F} \times \frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} \quad (\textit{Eq. II.23})$$

Where $\sigma_{\Delta S}$ is the standard deviation of ΔS, $\sigma_{\Delta F}$ is the standard deviation of ΔF, $\rho_{\Delta S, \Delta F}$ is the correlation coefficient between ΔS and ΔF. Thus, HR is the hedge ratio that minimizes the variance of the overall hedged position.

We now present a more detailed example on the minimum variance hedge ratio estimation, with the data shown on the table below. We proceed and calculate the coefficients seen above and reach an estimate for the hedge ratio.

Month (i)	Delta F for month (xi)	Delta S for month (yi)	xi^2	yi^2	xi yi
1	-0.0158	-0.0238	0.00024964	0.000566	0.000376
2	-0.0479	-0.0508	0.00229441	0.002581	0.002433
3	0.0053	-0.0202	0.00002809	0.000408	-0.00011
4	0.0881	0.0391	0.00776161	0.001529	0.003445
5	0.1052	0.0568	0.01106704	0.003226	0.005975
6	0.0017	-0.0416	0.00000289	0.001731	-7.1E-05
7	0.0141	0.0173	0.00019881	0.000299	0.000244
8	-0.0335	-0.1056	0.00112225	0.011151	0.003538
9	0.1139	0.0461	0.01297321	0.002125	0.005251
10	0.0278	0.0472	0.00077284	0.002228	0.001312
11	0.0302	-0.036	0.00091204	0.001296	-0.00109
12	0.0814	0.0199	0.00662596	0.000396	0.00162
13	0.0405	0.0447	0.00164025	0.001998	0.00181
14	-0.0036	-0.032	0.00001296	0.001024	0.000115
15	0.088	0.0923	0.007744	0.008519	0.008122
SUM	0.4954	0.0534	0.053406	0.039078	0.032977

Table II.13 Example of the minimum variance hedge ratio estimation

$$\sigma_{\Delta F} = \sqrt{\frac{\sum x_i^2}{n-1} - \frac{(\sum x_i)^2}{n(n-1)}} = \sqrt{\frac{0.053406}{15-1} - \frac{(0.4954)^2}{15(15-1)}} = 0.05144$$

$$\sigma_{\Delta S} = \sqrt{\frac{\sum y_i^2}{n-1} - \frac{(\sum y_i)^2}{n(n-1)}} = \sqrt{\frac{0.039078}{15-1} - \frac{(0.0534)^2}{15(15-1)}} = 0.052704$$

$$\rho_{\Delta S,\Delta F} = \frac{n\sum x_i \times y_i - \sum x_i \times \sum y_i}{\sqrt{[n\sum x_i^2 - (\sum x_i)^2]\times[n\sum y_i^2 - (\sum y_i)^2]}} = \frac{15\cdot 0.032977 - 0.4954\cdot 0.0534}{\sqrt{[15\cdot 0.053406 - 0.4954^2]\times[15\cdot 0.039078 - 0.0534^2]}} = 0.82237$$

$$\text{HR} = \rho_{\Delta S,\Delta F}\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} = 0.82237\frac{0.052704}{0.05144} = 0.842584$$

This means that futures contracts to buy/sell should be 84.26% of the "amount" of assets to be hedged. Practically, the number of futures contracts to be used for the hedge has to be approximated therefore the trader can only reach an approximation of the optimal hedge. The Ordinary Least Square regression (OLS) suggests that, as the hedge ratio is defined, then HR constitutes the beta coefficient of the following linear regression model:

$$\Delta S_t = \alpha^* + \beta^* \times \Delta F_t + \varepsilon_t \text{ (Eq. II.24)}$$

Where:

ΔS_t are the spot price variations.
ΔF_t are the futures prices variations.
ε_t is the residual with expectation equal to zero.
α^* and β^* are the coefficients estimated using a linear regression.

If the spot is hedged with several futures contracts of different nature, it is necessary to resort to a multiple linear regression.

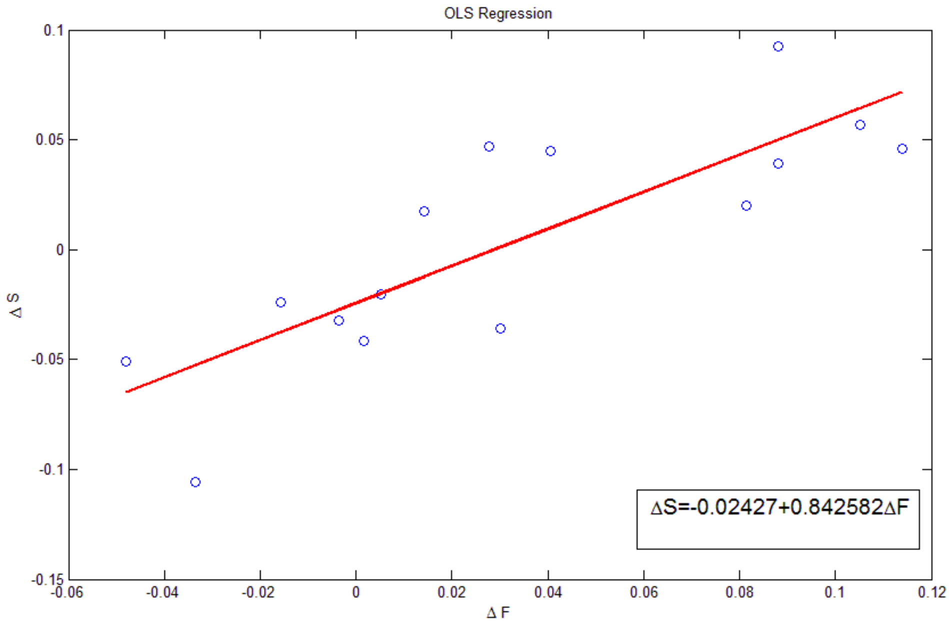

Figure II.23 OLS regression for the futures hedging

FURTHER READINGS

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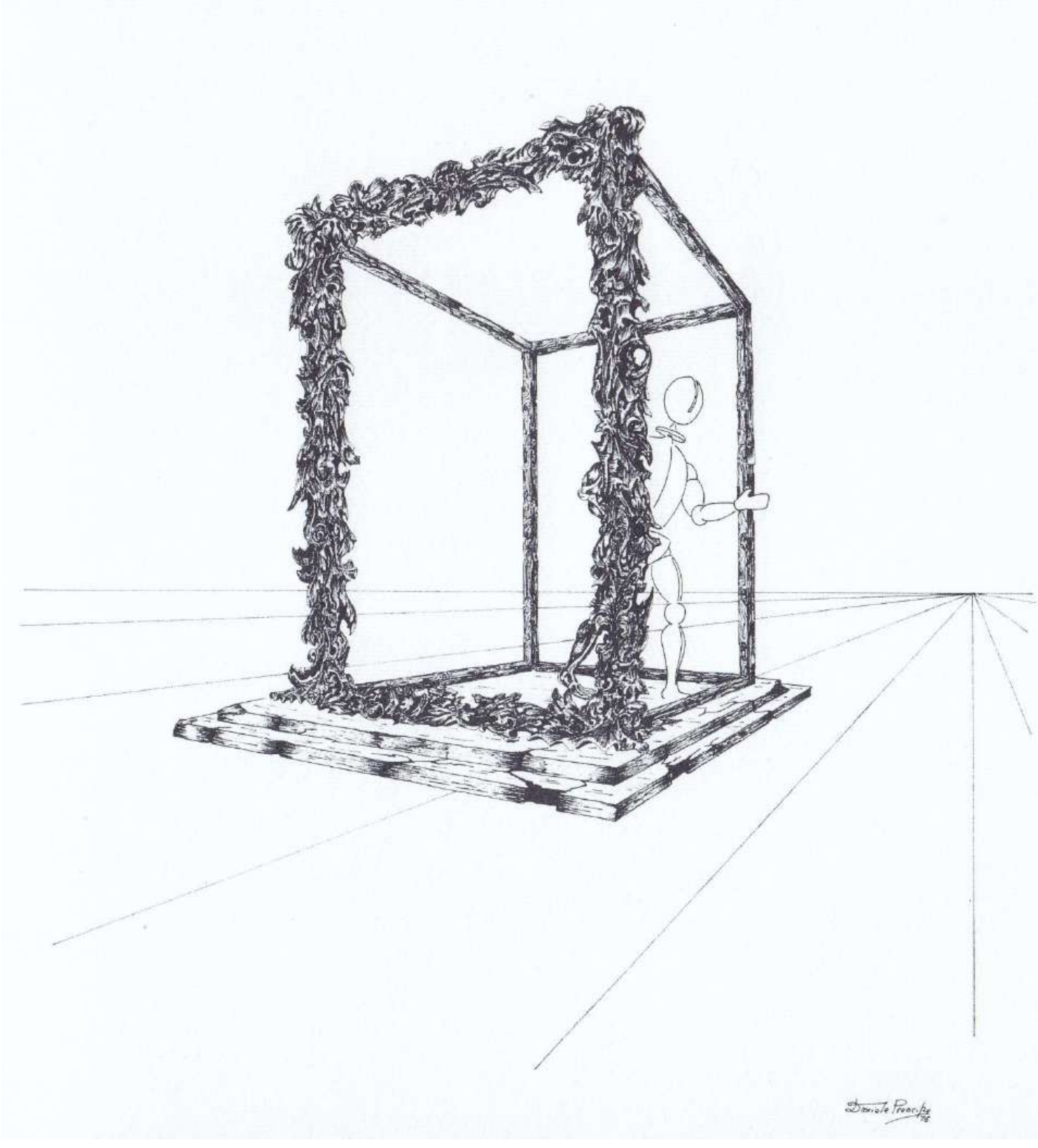

PART III: OPTIONS

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Chapter 7 – Option parameters

III.1 FUNDAMENTALS

Options are the second most traded group of derivatives. Options appeared on American markets as soon as shares trading started, and they were managed exclusively over-the-counter until 1972. In those early days, they were difficult to negotiate because of their high illiquidity: they were traded over the phone and dealers applied large spreads to investors interested in buying these derivatives. In addition, these early options could not be renegotiated and sometimes the sellers went bankrupt. As a result, growth in the trading initially remained relatively low. On April 26, 1973, after 4 years of study and planning, the Chicago Board of Trade established the Chicago Board Options Exchange and introduced standardized options on underlying stock. In the same year, and in fact, almost simultaneously, Black and Scholes provided a reasonable option pricing model. After reaching a volume of 911 contracts on the first day, the average daily volume reached 20,000 contracts the following year and, over time, many different types of options were introduced on a large number of underlyings.

An option is defined as a financial contract between two parties, the **option holder** and the **option writer**, which allows the holder to buy/sell an asset on which it is written. More specifically, there are two types of basic options: call options and put options. A **call option** gives its holder the right (but not the obligation) to buy an asset (underlying asset) on a certain date at an agreed price. The writer of a call option is obligated to sell the underlying asset to the option holder, if the latter decides to exercise it. On the other hand, a **put option** gives its holder the right (but not the obligation) to sell an asset (underlying asset) on a certain date at an agreed price. The writer of a put option is obligated to buy the underlying asset from the option holder, if the latter decides to exercise it. The agreed price is called **exercise price** or **strike price**.

A call option is said to be **in-the-money** when the price level of the underlying asset exceeds the option strike price; while a put is said to be in-the-money when the price of the underlying is lower than the exercise price.

An option is **at-the-money** when the underlying asset equals the strike price, whereas, when the underlying price is below the strike for calls and above for puts, the option is said to be **out-of-the-money**.

Another important concept is the initial purchase price of an option, which is called **premium**.

The date on which the right can be exercised is called the **exercise date**, expiration date or simply maturity.

Options that can only be exercised on the maturity date are called **European options**, while those that can be exercised at any time from the inception until expiry are called **American options**. Lastly, when entering into a **Bermuda option**, the right can be exercised not only upon expiry, but also on other contractually prefixed dates.

Similarly to forward contracts, all contractual terms are potentially customizable in options traded over-the-counter: it is sufficient for the counterparties to formally agree on the conditions. On the other hand, options traded on organized markets are standardized.

Standardization allows these derivatives to be traded on the secondary market where option holders and writers can close out their open positions by neutralizing long or short positions with a higher probability than OTC ones.

The key point in an option contract is that the holder has a right, not an obligation. This prominent feature

distinguishes options from forwards/futures, where the holder has the obligation to buy or sell the underlying asset. Only option writers are required to open and maintain a margin account to ensure the fulfillment of their contractual obligations. Option holders are not required to do so since they will only exercise the acquired right if it is convenient to do so.
The following diagrams illustrate the transactions associated with a call option. In the **initial transaction**, the call buyer pays the premium to the seller (option writer) and receives the option.

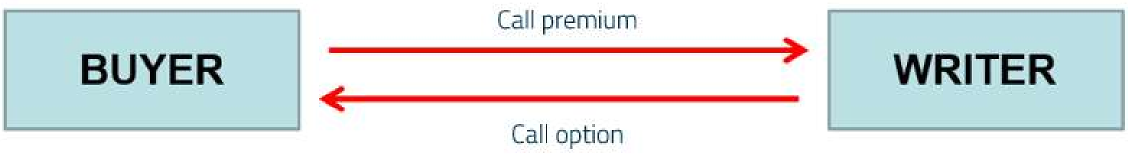

Figure III.1 Call option initial transaction

On the **exercise date**, if the call holder decides to exercise it, he pays the exercise price and returns the option to the seller. Consequently, the seller is obliged to deliver the underlying (or, if required by the contractual conditions, the cash equivalent of its value). If the buyer does not exercise, nothing happens.

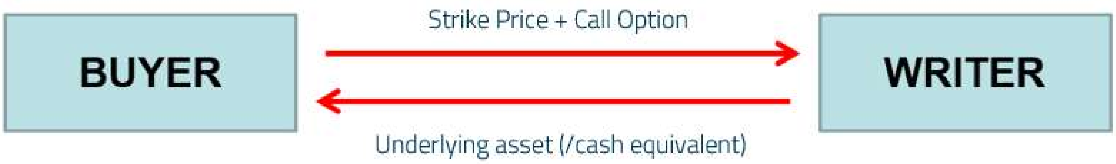

Figure III.2 Call option final transaction

Among the main advantages of options, we mention the fact that they offer asymmetrical patterns of returns that could not be achieved with a static investment in the underlying asset. Options also, allow to take a position in an underlying under more favorable restrictions compared to investing directly in the underlying. We can also state that they provide analysts with information regarding market expectations and input data for valuation models, and last but not least, options may offer tax benefits.

Options can be written on a long list of underlyings, here are the most common ones:

Equity options give the holder the right (but not the obligation) to buy (call) or sell (put) a certain number of shares at (or within, if American) an expiration date at an agreed price. The contractual terms can be adjusted in case of special events (such as stock splits or rights offerings). Traders who assume a long position in a call option do not have the same rights as those who have invested directly in the shares (such as voting rights, regular cash, special dividends).

An **Equity index option** is a call option that gives its holder the right (but not the obligation) to buy (call) or sell (put) the stock index on (or by, if American) an expiration date at an agreed price. Since an index is a measure of the price levels of a set of securities and, therefore, the underlying is synthetic, not physical, settlement must be made in cash on the exercise date.

Options on Futures are very similar to standard options, except for the fact that the underlying is a futures contract. The holder of the option acquires the right to buy (call) or the right to sell (put) a futures contract at a prefixed price by a certain exercise date. The option writer is contractually obligated to take the opposite position if the buyer decides to exercise its right. All options on futures traded in the United States have an American-like exercise right, as a result they can be exercised on any trading day, which involves a physical or cash settlement, depending on the date.

A **Foreign exchange option** is a currency option that gives the holder the right (but not the obligation) to exchange money denominated in one currency against another at an agreed exchange rate at a future date.

Interest rates options are derivatives written directly on interest rates or on financial instruments connected to them (such as bank deposits, certificates of deposit, commercial papers and T-bills). They can be classified into two categories:

- **Price-based options** are derivatives which give their holder the right to buy (/sell) a specific debt security (physical delivery) or to receive a cash payment based on the value of the underlying security (cash settlement). Many of these financial instruments are, in fact, options on a futures contract.

- **Yield-based options** are derivatives whose settlement in cash is based on the difference between the exercise price and the value of the underlying yield (or interest rate) applied to a fixed notional.

The most widespread options in the yield-based options category are **interest rate options**, known as caps, floors and collars (which are traded over-the-counter) and **yield options**, which are predominantly traded on organized markets. An **interest rate cap** can be defined as a collection of caplets, and a **caplet** is a single European call option written on a reference interest rate (e.g. Libor, Euribor, mortgage rate, commercial paper rate) which matures on a specified date. As a result, a **cap** can be represented as a portfolio of interest rate options.

The term of the cap coincides with the time horizon over which the option is effective. The **tenor** or **reset period** of the cap determines how often the reference rate is observed/adjusted and how often payments are made. The pay-off of a Libor caplet at its maturity (T) can be described algebraically by the following formula:

$$\textbf{Caplet(T)=Q x Max[Libor – K, 0] x (days/360)} \quad (\textit{Eq. III.1})$$

Where Libor is the spot rate recorded at the reset rate, K is the exercise rate and Q is the notional amount.

An interest rate cap is typically implemented to limit the reference floating rate from exceeding a maximum threshold.

For example, a typical use is the case of a loan indexed to Euribor: the counterparty who pays for the interest fixes a maximum ceiling on the future cash flows by purchasing a cap.

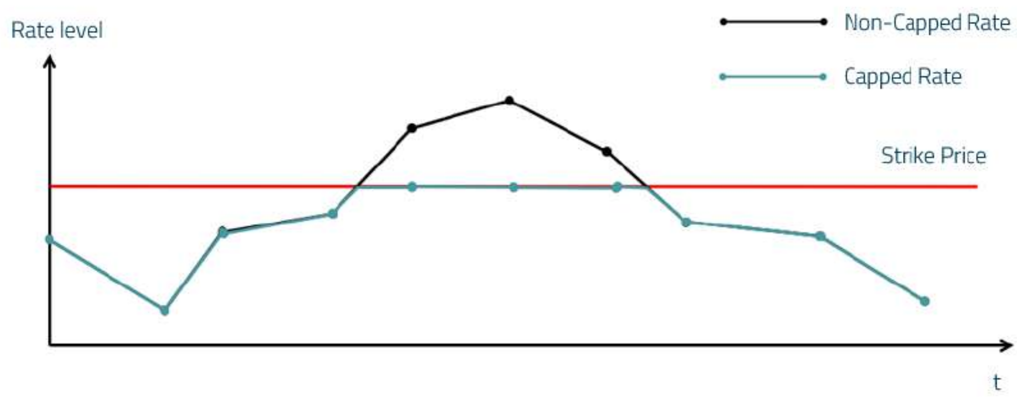

Figure III.3 Capped versus Non-Capped Rate

Let us assume that we have a caplet with a duration of one year and an exercise price (K) equal to 2% on the 12-month Euribor (reference rate). The Euribor Rate at time t is denoted by S_t.

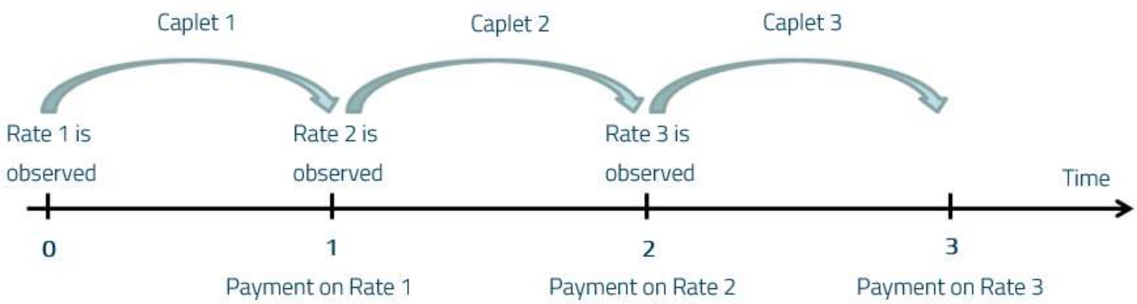

Figure III.4 Interest rate Caplets

At the beginning of the year (t=0), the Libor rate S_0 is recorded and it constitutes the reference rate. At the end of the year, the cap strike is compared to the reference rate. The pay-off is: max(S_0 – K ;0).
Assuming a notional of Q= EUR 10 million and a S_0=2.5%, the caplet has a value of:

Caplet(T=1)=EUR 10,000,000 x Max[2.5% – 2%, 0] x (360/360) = EUR 50 million

An **interest rate floor** is a collection of floorlets and a **floorlet** is a single European put option on a reference interest rate (Libor, Euribor, mortgage rate, commercial paper rate, etc…) which matures on a specified date. Thus, a floor can be managed as a portfolio of European put options written on interest rates. The pay-off of a Libor floorlet at its maturity (T) can be described by the formula:

Floorlet(T)=Q x Max [K – Libor, 0] x (days/360) (*Eq. III.2*)

where Libor is the observed spot rate on the reset date, K is the exercise rate and Q is the notional amount.

This financial instrument is typically used for the buyer to have a guaranteed minimum rate in the agreed period of time. Finally, an interest rate **collar** is a portfolio of caps and floors that limit changes in the benchmark interest rate to a specific "corridor".

Let us now focus on the valuation of option contracts and their fundamental characteristics. The applied notations in this section are the following:

K is the Strike price (or Exercise price) of the option.
C is the Call option price (American or European).
C_E is the European Call option.
C_{US} is the American Call option.
P is the Put option price (American or European).
P_E is the European Put option.
P_{US} is the American Put option.
$C(S_t,T-t,K)$ is the value at time t of a call option (European or American) written on an underlying S (for example, a share) with an expiry time of τ=T-t and strike price K.
Lastly, $P(S_t,T-t,K)$ is the value at time t of a put option (European or American) written on an underlying S (for example, a share) with an expiry time of τ=T-t and strike price K.

Before examining the characteristics of option contracts, we should consider the pay-off at maturity of "pure" (i.e. non-derivative) financial instruments, as the combination of an ordinary share (the underlying) and a default-free zero-coupon bond (used for the computation of the discount rate).

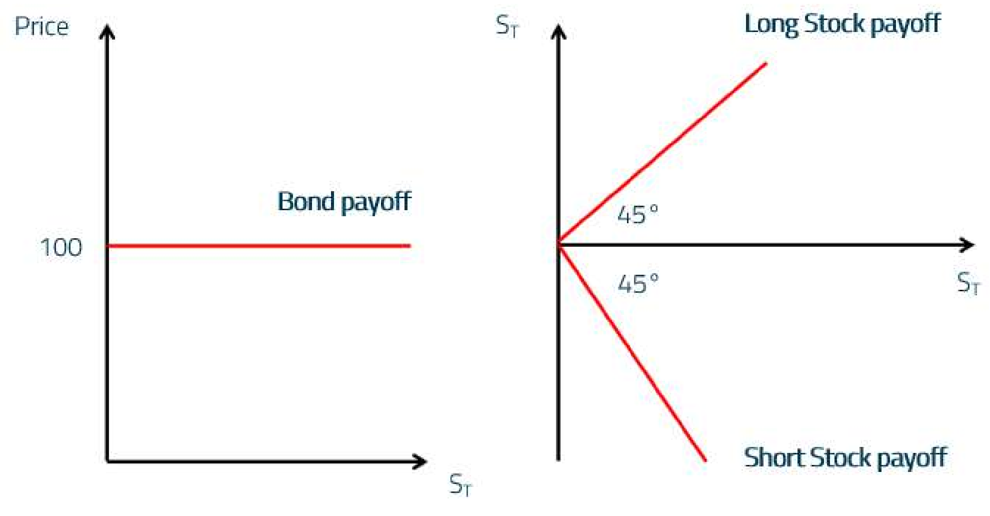

Figure III.5 Bond and Long-Short Stock payoffs

Let us calculate the value of a call option at the expiration date, i.e. at time T. We start from the description of the derivative and obtain the formula for the pay-off. We know that a call option gives its owner the right to buy the underlying at maturity at the strike price. Let us examine the two cases:

- If the share price (S_T) is higher than the strike price (K), then the option holder will exercise the call option and buy the share at price level K, while simultaneously selling the underlying at market price S_T. The profit is equal to S_T-K per each share.

- If the stock price (S_T) is lower than the strike price, then the option holder will do nothing, since he can buy the stock directly on the market at a lower price rather than exercise his option. The option will expire worthless.

Clearly, the pay-off of the option cannot be negative, in the worst case it is zero.

Expressing the concept in mathematical terms, the intrinsic value of a call option at maturity is:

$$\mathbf{C_T = max\ [S_T - K, 0]} \quad (\textit{Eq. III.3})$$

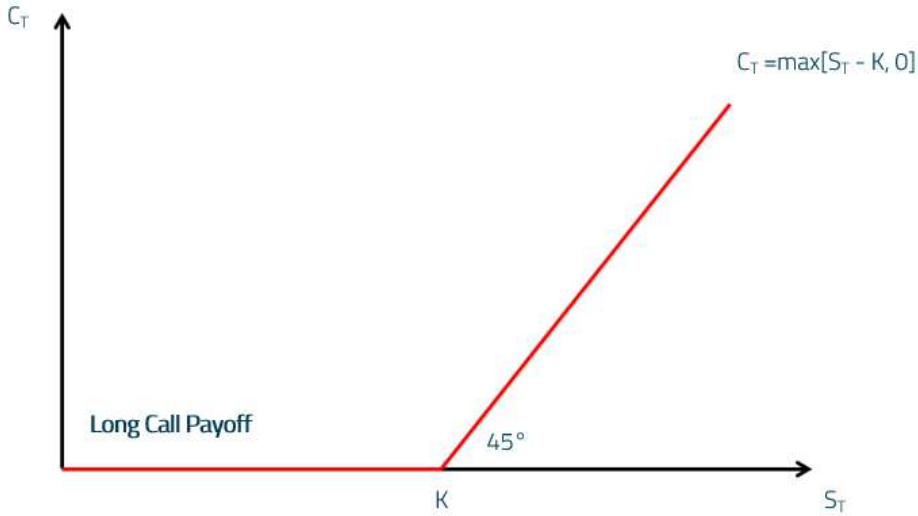

Figure III.6 Value of a Long Call option at maturity

Since the call holder pays a premium when buying the option, the net profit will be equal to the final value of the derivative (gross pay-off) minus the initial investment (option cost, premium).

The figure shows that the call holder's loss is limited to the premium paid. The gain is (theoretically) unlimited, in function of the underlying performance. The sum of the strike price plus the premium is called the **break-even point**, which represents the share price level from which the option holder starts earning.

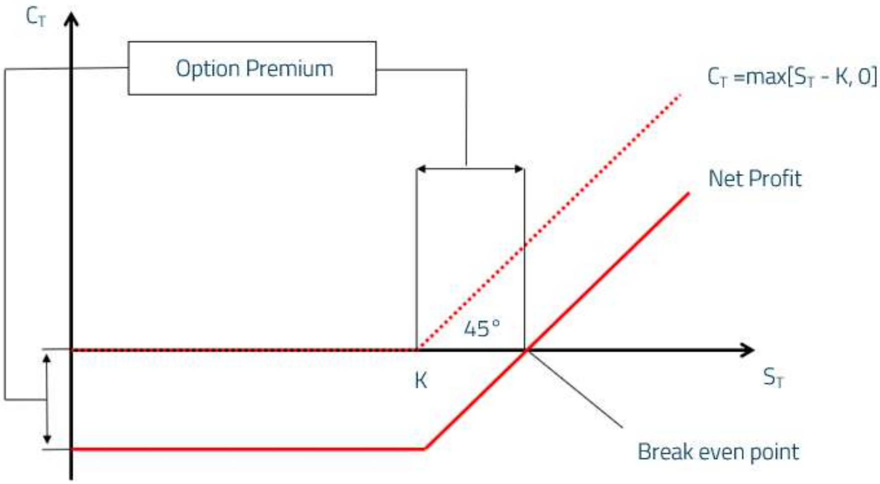

Figure III.7 Net profit for a Call-holder at maturity

As an example, let us consider a call option written on stock XY. The current price of the underlying is CHF 95. The exercise price is K = CHF 90. The contract size is 10. The premium paid for the purchase of the option is equal to 5. We can calculate the breakeven point of the investment in the derivative.

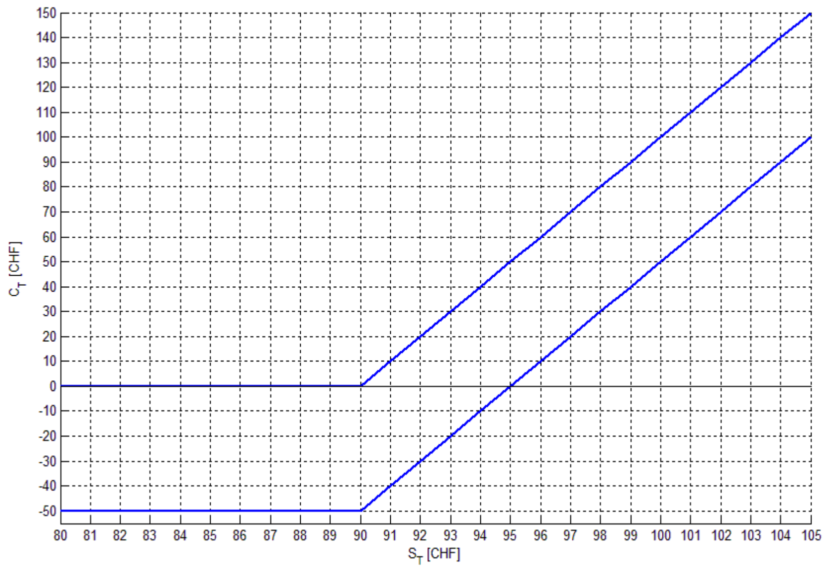

Figure III.8 Breakeven point for a Long Call option

The Break even point of the current strategy is CHF 95.

Stock Price (S_T)	Gross payoff	Net Profit	Net Payoff
80	0	-5	-50
81	0	-5	-50
82	0	-5	-50
83	0	-5	-50
84	0	-5	-50
85	0	-5	-50
86	0	-5	-50
87	0	-5	-50
88	0	-5	-50
89	0	-5	-50
90	0	-5	-50
91	1	-4	-40
92	2	-3	-30
93	3	-2	-20
94	4	-1	-10
95	**5**	**0**	**0**
96	6	1	10
97	7	2	20
98	8	3	30
99	9	4	40
100	10	5	50
101	11	6	60
102	12	7	70
103	13	8	80
104	14	9	90
105	15	10	100

Table III.1 Breakeven point for a Long Call option

Since options, like futures, are a **zero sum game**, the net profit of the seller of a call is exactly the opposite of the net profit of the call holder.

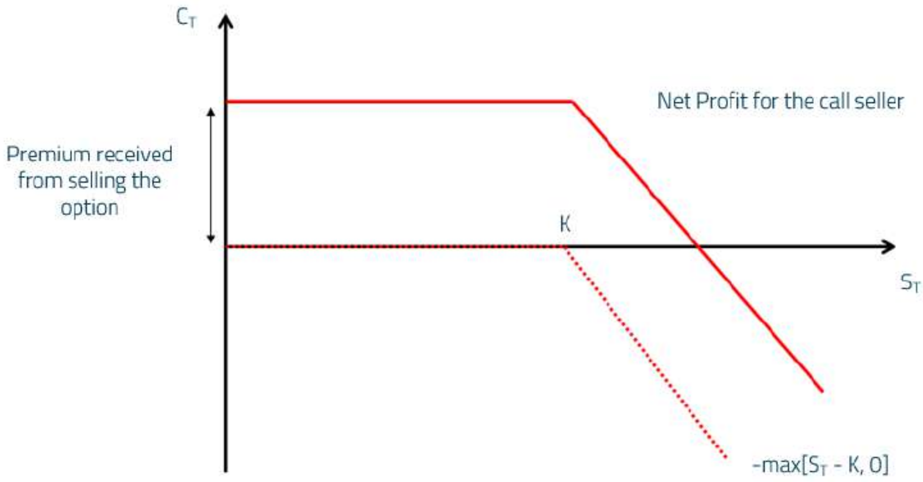

Figure III.9 Short Call payoff

For the call seller, the potential profit is limited to the premium, the potential loss is (theoretically) unlimited.

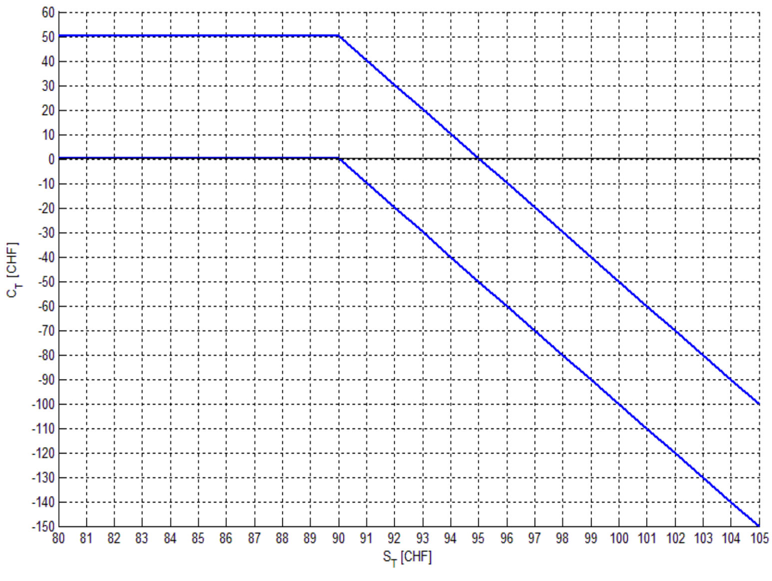

Figure III.10 Breakeven point for a Short Call option

Stock Price (S_T)	Gross payoff	Net Profit	Net Payoff
80	0	5	50
81	0	5	50
82	0	5	50
83	0	5	50
84	0	5	50
85	0	5	50
86	0	5	50
87	0	5	50
88	0	5	50
89	0	5	50
90	0	5	50
91	-1	4	40
92	-2	3	30
93	-3	2	20
94	-4	1	10
95	**-5**	**0**	**0**
96	-6	-1	-10
97	-7	-2	-20
98	-8	-3	-30
99	-9	-4	-40
100	-10	-5	-50
101	-11	-6	-60
102	-12	-7	-70
103	-13	-8	-80
104	-14	-9	-90
105	-15	-10	-100

Table III.2 Breakeven point for a Short Call option

Let us now determine the value of a put option at maturity, i.e. at time T. We start from the description of the instrument, then obtain the pay-off formula. As known, a put option gives its owner the right to sell the underlying at maturity at the strike price. We can analyze the two cases:

- If the share price (S_T) is lower than the strike price (K), then the option holder will exercise the put to sell the share at price level K and simultaneously buy the underlying at the market price S_T. The profit will be $K\text{-}S_T$ per each share.

- If the stock price (S_T) is higher than the strike price, then the option holder will do nothing, since he can sell the stock directly on the market at a higher price rather than exercise his option. The option will expire worthless.

Clearly, the pay-off of the option cannot be negative and it is zero in the worst case.

Expressing the concept in mathematical terms, the intrinsic value of a put option at maturity is the following:

$$\mathbf{P_T = max\ [K\text{-}S_T\ ,\ 0]} \quad (\textit{Eq. III.4})$$

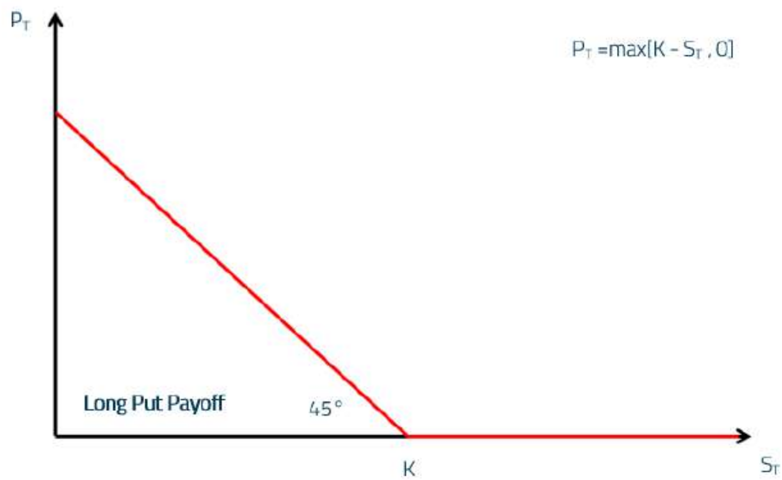

Figure III.11 Long Put payoff

Since the put holder pays a premium when buying the option, the net profit will be equal to the final value of the derivative (gross pay-off) minus the initial investment (option cost, premium).

The figure shows that the gain is limited, as is the potential loss: both quantities depend on the performance of the underlying.

As said above, the strike price minus the premium is called the break-even point, which represents the share price below which the option holder begins to gain from the current strategy.

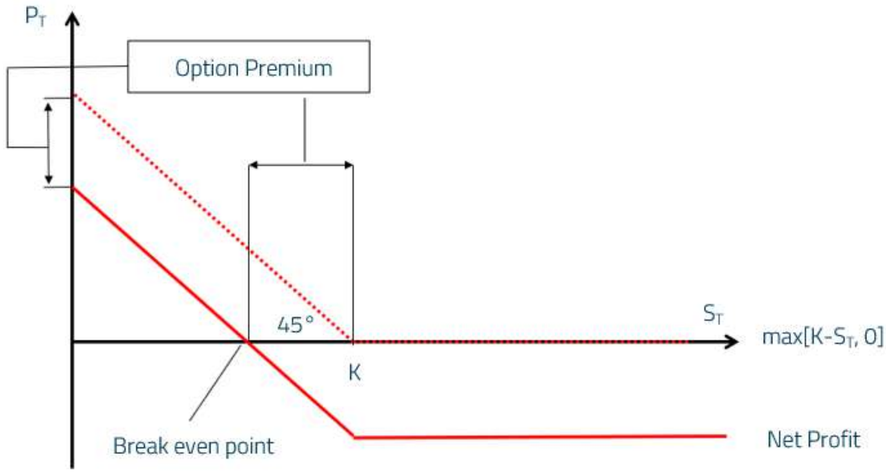

Figure III.12 Net profit for a Put-holder at maturity

Let us consider as an example a put option written on stock XY. The current price of the underlying is CHF 95. The exercise price is K = CHF 90. The contract size is 10. The premium paid for the purchase of the option is equal to 5. We can calculate the breakeven point of the investment in the derivative.

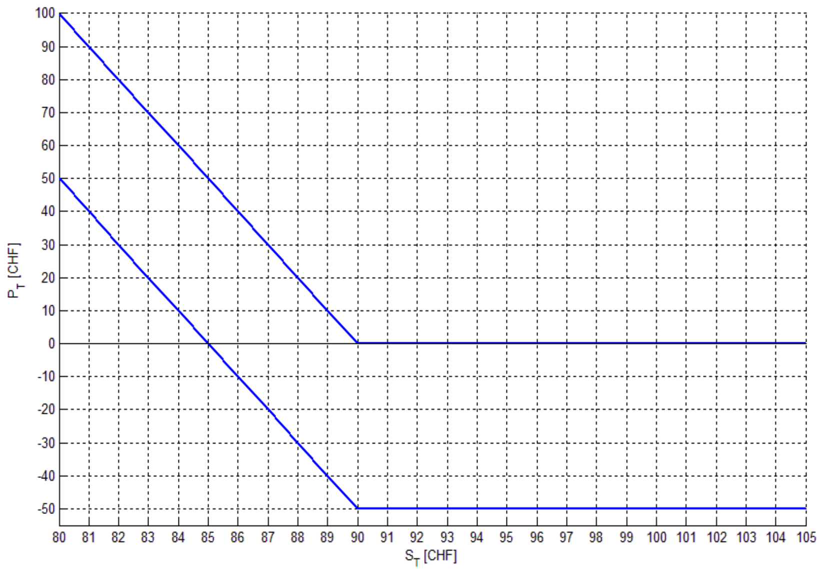

Figure III.13 Breakeven point for a Long Put option

The Break-even point of the current strategy is CHF 85.

Stock Price (S_T)	Gross payoff	Net Profit	Net Payoff
80	10	5	50
81	9	4	40
82	8	3	30
83	7	2	20
84	6	1	10
85	**5**	**0**	**0**
86	4	-1	-10
87	3	-2	-20
88	2	-3	-30
89	1	-4	-40
90	0	-5	-50
91	0	-5	-50
92	0	-5	-50
93	0	-5	-50
94	0	-5	-50
95	0	-5	-50
96	0	-5	-50
97	0	-5	-50
98	0	-5	-50
99	0	-5	-50
100	0	-5	-50
101	0	-5	-50
102	0	-5	-50
103	0	-5	-50
104	0	-5	-50
105	0	-5	-50

Table III.3 Breakeven point for a Long Put option

Since options, like futures, are a **zero sum game**, the net profit of the seller of a put is exactly the opposite of the net profit of the put holder.

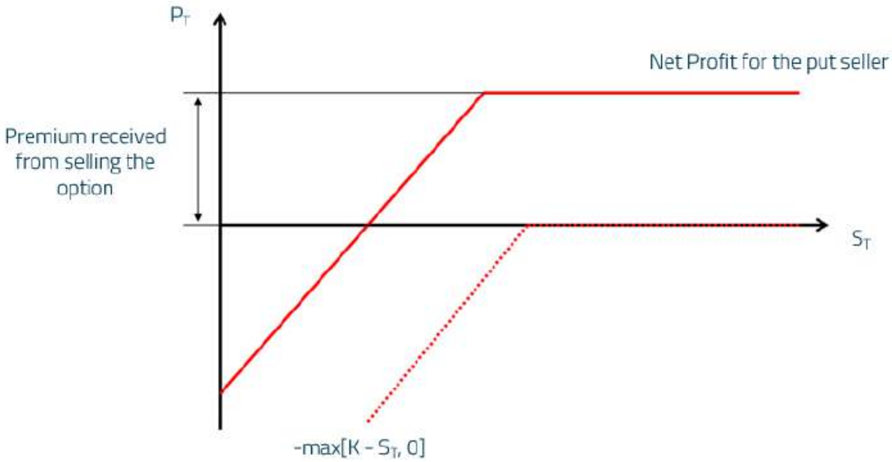

Figure III.14 Short Put payoff

For the put seller, the potential profit is limited to the premium, the potential loss is (theoretically) unlimited.

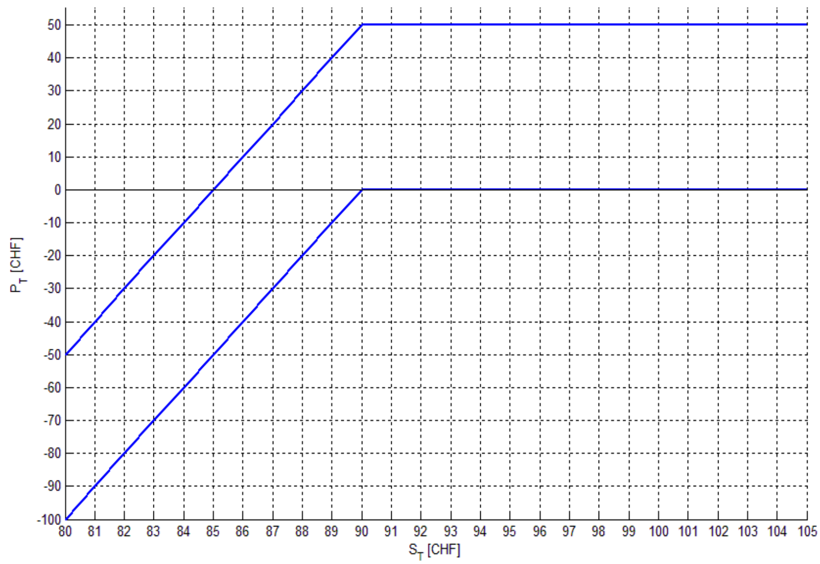

Figure III.15 Breakeven point for a Short Put option

Stock Price (S_T)	Gross payoff	Net Profit	Net Payoff
80	-10	-5	-50
81	-9	-4	-40
82	-8	-3	-30
83	-7	-2	-20
84	-6	-1	-10
85	**-5**	**0**	**0**
86	-4	1	10
87	-3	2	20
88	-2	3	30
89	-1	4	40
90	0	5	50
91	0	5	50
92	0	5	50
93	0	5	50
94	0	5	50
95	0	5	50
96	0	5	50
97	0	5	50
98	0	5	50
99	0	5	50
100	0	5	50
101	0	5	50
102	0	5	50
103	0	5	50
104	0	5	50
105	0	5	50

Table III.4 Breakeven point for a Short Put option

In this section, concepts derived from arbitrage will be employed to establish general properties inherent in the price of options, without making assumptions on the underlying asset. The implications deriving from these reasonings are useful for outlining the upper-bounds and lower-bounds within which the theoretical price of the option will necessarily remain. A market price outside of this theoretical range represents an arbitrage opportunity between the observed market price and the theoretical price, **regardless of the mathematical model** used.

For the sake of simplicity, it will be assumed below that the size of the option contract (contract size) is unitary, i.e. the holder of an option contract will have the right to buy (call) or sell (put) a single share. The first eleven properties assume that the underlying has no dividends. The following four properties relax this constraint and consider a possible payout from the underlying while the derivative is still alive. Here are the properties:

Property 1

Given that the maximum loss connected to an option is the initial investment (limited liability instrument), the value of an option cannot be negative. Therefore:

$$C_E \geq 0, C_{US} \geq 0, P_E \geq 0, P_{US} \geq 0 \text{ (Eq. III.5)}$$

If there were the possibility of a negative option price, a trader could buy it now and hold it until maturity. Such strategy would give a positive amount now and a non-negative one later. Thus, without considering the initial investment (as will be done in illustrating all the properties), an investor could earn a risk-free profit and therefore an arbitrage strategy could be implemented. If the market is efficient though, such an opportunity should not exist. This reasoning, combined with the mathematical definition of the option pay-off, ensures that Property 1 always holds.

Property 2

A call option is a limited asset security: its value cannot fall below zero. If the underlying price is zero, the call price is also zero. It follows that:

$$C_E(0, \tau, K) = C_{US}(0, \tau, K) = 0 \text{ (Eq. III.6)}$$

Property 3

The price of a call option cannot exceed the value of the asset on which it is written. Otherwise, if this were the case, an investor would prefer to buy the underlying directly rather than by exercising the option. The underlying can be considered as an American option with an infinite time to maturity and a zero strike.

$$S = C_{US}(S, \infty, 0) \geq C_{US}(S, \tau, K) \text{ (Eq. III.7)}$$

Property 4

The price of an option is equal to the pay-off at the expiration date: for a call option it is $S_T - K$ or zero; for a put option, it is $K - S_T$ or zero, as indicated below:

$$C_E(S_T, 0, K) = C_{US}(S_T, 0, K) = \max[0, S_T - K] \text{ (Eq. III.8)}$$

$$P_E(S_T, 0, K) = P_{US}(S_T, 0, K) = \max[0, K - S_T] \text{ (Eq. III.9)}$$

Property 5

The minimum value for an American call is given by zero or $S_t - K$, and the minimum value for an American put is given by zero or $K - S_t$.

In mathematical terms, we have:

$$C_{US}(S,\ \tau, K) \geq \max[0, S_t - K]\,; P_{US}(S,\ \tau, K) \geq \max[0,\ K - S_t] \text{ (Eq. III.10)}$$

The formal proof of this claim can be divided into two parts. The first part comes from Property 1 and states that the option, by nature, cannot assume a negative value. Therefore, we have:

$$\mathrm{C}_{US}(S,\ \tau, K) \geq 0 \text{ and } \mathrm{P}_{US}(S,\ \tau, K) \geq 0 \text{ (Eq. III.11)}$$

The second part assumes that S_t is greater than K. If the corresponding option value were less than $\mathrm{S}_t - \mathrm{K}$ then an arbitrage opportunity would arise and we could buy the option, exercise it immediately and sell the underlying stock. In that case, the net profit would be:

$$S_t - \{C_{US}(S,\tau,K) + K\} = S_t - K - C_{US}(S,\tau,K) \geq 0 \text{ (Eq. III.12)}$$

Consequently, we should remove arbitrage opportunities: $C_{US}(S,\tau,K) \geq S_t - K$

Similar arguments for an American put: $P_{US}(S,\tau,K) \geq K - S_t$

From properties 1, 2, 3 and 5, the mentioned boundaries can be established for an American call option.

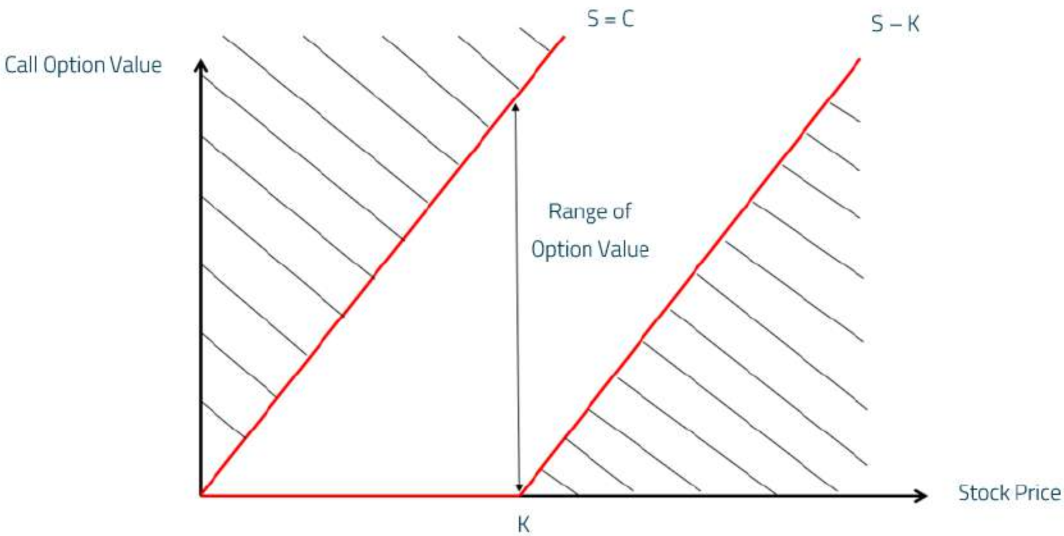

Figure III.16 American call option boundaries

Property 6

An American option cannot be worth less than a European option:

$$C_{US}(S,\tau,K) \geq C_E(S,\tau,K) \text{ (Eq. III.13)}$$

$$P_{US}(S,\tau,K) \geq P_E(S,\tau,K) \text{ (Eq. III.14)}$$

Anything that can be done with a European option can also be done with an American option. By assuming a long position in an American option, the holder has the opportunity to exercise his right not only at maturity, as happens in a European option, but at any moment of its life. Such additional freedom of choice has a positive or, in the worst case, a zero value.

Property 7

The value of a European call option will never be less than the price level of the stock it is written on, minus the present value of the option strike price.

$$C_E(S,\tau,K) \geq \max(0, S_t - Ke^{-r\tau}) \rightarrow C_{US}(S,\tau,K) \geq C_E(S,\tau,K) \geq \max(0, S_t - Ke^{-r\tau}) \text{ (Eq. III.15)}$$

In order to prove property 7, we can compare the values of the following two portfolios: the first portfolio, A consists of a long position in a European call option and a long position in a zero-coupon bond paying K at time T. The second portfolio, B, consists of a share (S).

	Value at time t	Value at time T	
		$S_T<K$	$S_T>K$
Portfolio A	$C_E(S,\tau,K) + Ke^{-r\tau}$	0+K	$(S_T-K)+K$
Portfolio B	S_t	S_T	S_T
		A>B	A=B

Table III.5 Option Property 7: proof

Since at time T, the value of portfolio A cannot be less than the value of portfolio B, the same property could be applied to a generic time t. It follows that:

$$C_E(S,\tau,K) + Ke^{-r\tau} \geq S_t \rightarrow C_E(S,\tau,K) \geq S_t - Ke^{-r\tau} \text{ (Eq. III.16)}$$

The positivity of the result is enforced using property 1. We can extend the result to an American call using property 6.

Then, from properties 6 and 7, we can establish the lower and upper bounds of the value for a European call.

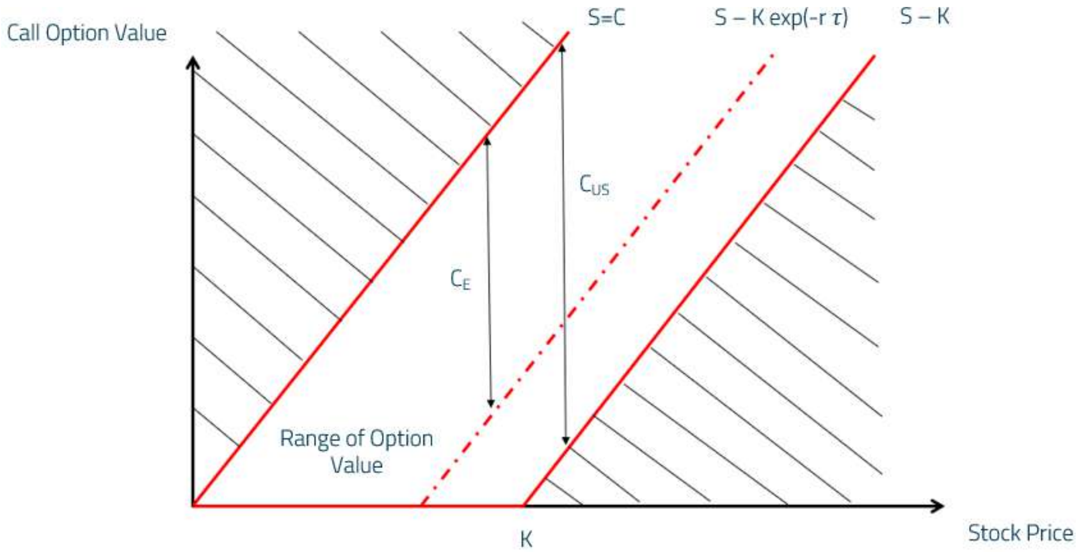

Figure III.17 European Call option boundaries

Property 8

Under the assumption that the underlying stock of the option does not pay a dividend and under the assumption of positive interest rates, it will never be convenient to exercise an American call option prematurely. In the cases mentioned, the American option will be priced like a European one.

$$C_{US}(S,\tau,K) = C_E(S,\tau,K) \text{ (Eq. III.17)}$$

From property 7, we have:

$$C_{US}(S,\tau,K) \geq C_E(S,\tau,K) \geq S_t - Ke^{-r\tau} \text{ (Eq. III.18)}$$

From property 5, we have:

$$C_{US}(S,\tau,K) \geq \max(0, S_t - K) \text{ (Eq. III.19)}$$

Since $S_t - K < S_t - Ke^{-r\tau}$, it follows that an American call written on an underlying stock that does not pay dividends is worth more alive than exercised. Thus, early exercise of a call option on a stock that does not pay dividends before its expiration is never optimal.

Property 9

Since the underlying equity is a limited liability security, its value cannot be less than zero, then the price of a put option cannot exceed its strike price. This statement is valid for both American and for European options.

$$K \geq P_{US}(S,\tau,K) \geq P_{EUR}(S,\tau,K) \text{ (Eq. III.20)}$$

Property 10

If two call options differ only in their strike prices, the call option with the lower strike price will be worth at least as much as the option with the higher strike price.

$$C_{US}(S,\tau,K_1) \leq C_{US}(S,\tau,K_2) \text{ if } K_1 \geq K_2 \text{ (Eq. III.21)}$$

$$C_E(S,\tau,K_1) \leq C_E(S,\tau,K_2) \text{ if } K_1 \geq K_2 \text{ (Eq. III.22)}$$

The contrary is true for put options:

$$P_{US}(S,\tau,K_1) \geq P_{US}(S,\tau,K_2) \text{ if } K_1 \geq K_2 \text{ (Eq. III.23)}$$

$$P_E(S,\tau,K_1) \geq P_E(S,\tau,K_2) \text{ if } K_1 \geq K_2 \text{ (Eq. III.24)}$$

From properties 1, 4, 5 and 9, we can establish the lower and upper bounds for an American put option.

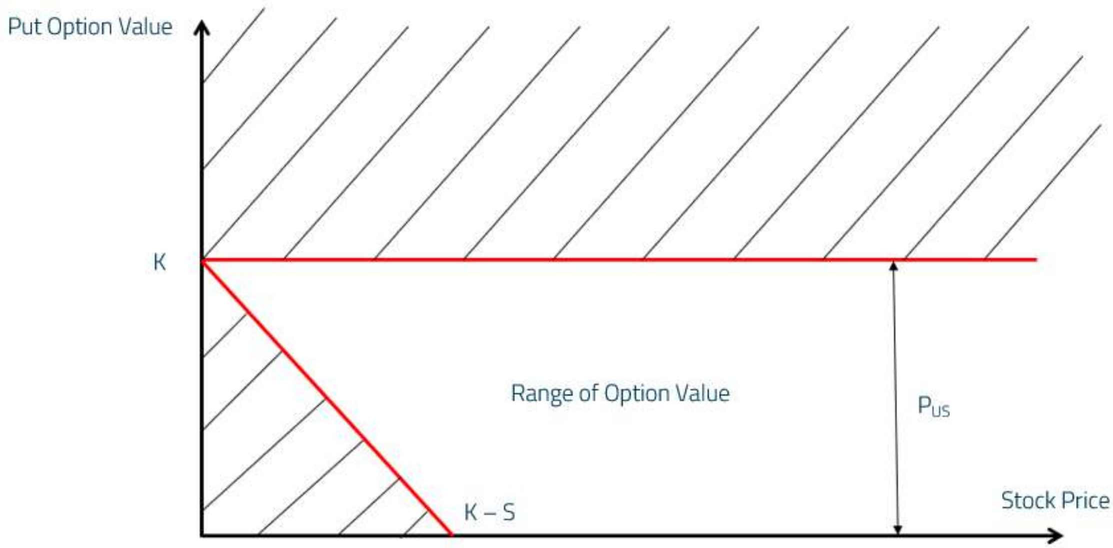

Figure III.18 American Put option boundaries

Property 11

If two American options differ only in their time to maturity, the option with the longer time will be worth at least as much as the option with the shorter one.

$$C_{US}(S,\tau_1,K) \geq C_{US}(S,\tau_2,K) \text{ if } \tau_1 \geq \tau_2 \text{ (Eq. III.25)}$$

$$P_{US}(S, \tau_1, K) \geq P_{US}(S, \tau_2, K) \text{ if } \tau_1 \geq \tau_2 \quad (Eq.\ III.26)$$

For European put options, this property does not always hold: it is not always true that a put option with a longer time to expiry has a larger value. A European put only pays at maturity: if the maturity is very far, the positive effect of the longer maturity is offset by the time value of money. Therefore, the price is affected by both factors and strictly depends on which of the two factors is dominant.
The early exercise of an American put written on an equity underlying that does not pay dividends could be valuable.

Let us now consider the general arbitrage relationships with an equity underlying that pays dividends during the life of the option.

Property 12

The value of a European call option can never be less than the share price diminished by the present value of the strike price and the present value of the dividends paid over the life of the derivative. In mathematical notation: $C_E(S, \tau, K) \geq S_t - Ke^{-r\tau} - D$.

If this relationship were not true, a trader could implement the following strategy: go short on the stock, buy the call, and buy zero-coupon bonds for an amount equal to $Ke^{-r\tau}$. Such a portfolio would give an immediate profit of:

$$-C_E(S, \tau, K) + S_t - Ke^{-r\tau} - D > 0 \quad (Eq.\ III.27)$$

D zero-coupon bonds are sold to pay dividends at the maturity to the buyer of the short position in the stock. Then the trader buys back the stock, sells (or exercises) the call, and sells K zero-coupon bonds. The pay-off of the strategy at maturity is summarized in the following table:

Value at time T	
$S_T \leq K$	$S_T > K$
We do not exercise the Call $-S_T + 0 + K = K - S_T$	We exercise the Call $-S_T + (S_T - K) + K = 0$

Table III.6 Option Property 12: proof

There can be no losses at time T and therefore the implementation of such a strategy would ensure a certain initial minimum profit. Since we have hypothesized the absence of arbitrage opportunities, we reach an absurdity. Thereby, the consistency of the relationship has been proved:

$$C_E(S, \tau, K) \geq S_t - Ke^{-r\tau} - D \quad (Eq.\ III.28)$$

Property 13

For American calls, if there is a dividend payment, it may be optimal to exercise the call shortly before its payment. If the following relationship holds just before dividend payment: $K(1-e^{-r\tau}) \geq D$ then the discounted value of the interest that can be earned by investing the strike price is greater than the discounted value of the dividends that will be paid during the life of the option. In this case, an American call option will not be exercised earlier and its value will be equal to the corresponding European option. Otherwise, it may be optimal to exercise the call and therefore its value may be greater compared to the European correspondent.

Property 14

The value of a European put option is never less than the discounted value of the strike price, plus the discounted value of the dividends that will be paid during the life of the option, minus the share price.

$$P_E(S,\tau,K) \geq -S_t + Ke^{-r\tau} + D \quad (Eq.\ III.29)$$

Property 15

Since an American put option gives its holder the same rights as a European put, plus the possibility of its early exercise, it follows that:

$$P_{US}(S,\tau,K) \geq P_E(S,\tau,K) \geq -S_t + Ke^{-r\tau} + D \quad (Eq.\ III.30)$$

From properties 14 and 15, we can establish the lower and upper bounds for a European put option.

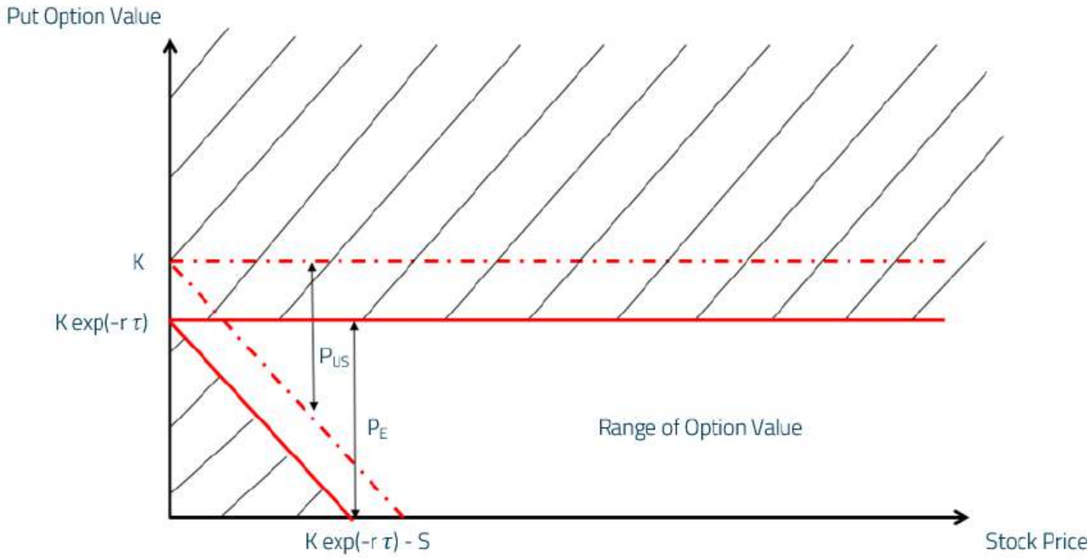

Figure III.19 European Put option boundaries

The **put-call parity** links the prices of European-style call options with puts having the same strike price and the same expiration date.

Let us now prove such relationship in the different cases, first for European calls and puts written on an underlying equity that does not pay dividends **(A)**, then for European options with an underlying that pays known dividends **(B)**. The following step is to consider the relationships between American call and put options with an underlying equity with no dividend **(C)** and known remuneration **(D)**.

A. Put-call parity: European Options written on equity without dividends

Let us consider the following portfolio:

- A long position on a stock.

- A short position on a call written on the same share, having a strike price K and a time to expiry equal to $\tau = T - t$.

- A long position on a put written on the same share, having a strike price K and a time to maturity equal to $\tau = T - t$.

- Borrowing $Ke^{-r(T-t)}$ through the sale of zero-coupon bonds with a maturity of $\tau = T - t$.

	Current Date	Expiration Date	
		$S_T < K$	$S_T > K$
Write Call	C_E	0	$K - S_T$
Buy Put	$-P_E$	$K - S_T$	0
Buy Stock	$-S$	S_T	S_T
Borrow	$Ke^{-r(T-t)}$	$-K$	$-K$
Total	$C_E - P_E - S + Ke^{-r(T-t)}$	0	0

Table III.7 Put-call parity proof

This strategy gives an amount equal to $C_E - P_E - S + Ke^{-r(T-t)}$ at the current date. But, as can be seen from the table, the future cash flows at maturity of this portfolio will be zero, regardless of the share price. Here are the different outcomes:

- If $S_T = K$ both options expire unexercised, and the gains realized from the sale of the stock exactly repay the debt.
- If $S_T < K$ the put option ends in the money, the call expires unexercised: the share is delivered at the strike price of the put and the profits exactly repay the debt.
- If $S_T > K$ the call option ends in the money, the put expires unexercised: the share is delivered when the call is exercised by the counterparty and the profits exactly repay the debt.

The reverse position (long a call, write a put, short a stock, and lend money for $Ke^{-r(T-t)}$ will also give a zero future cash flow under all circumstances. Consequently, in order to avoid arbitrage opportunities, the initial investment should be zero:

$$C_E - P_E - S + Ke^{-r(T-t)} = 0 \quad (Eq.\ III.31)$$

This equation is known as "**put-call parity**" for a European option written on an underlying that does not pay a dividend.

B. Put-call parity: European Options written on equity with a known dividend

We start by examining the effect of a cash dividend payment on the relationship found in point A. Experience shows that the share price level will decrease by an amount equal to the amount paid per share (cash-dividend per share). This effect would cause an increase in the put value and a decrease in the call value. Let us consider a stock that pays a dividend d_1 at time t_1 and two portfolios as follows:

Portfolio A includes a share and a put option written on this share with maturity T.

Portfolio B includes a call option written on the share with maturity T, K, a pure discount bond (with face value 1) that matures at time T, and d_1, a pure discount bond (with face value 1) that matures at time t_1.
The table below shows the final value of the two portfolios, assuming that the dividend is reinvested at the risk-free rate.

	Initial Value	Final Value	
		$S_T < K$	$S_T \geq K$
Portfolio A	$P_E + S$	$K + d_1 \cdot e^{r(T-t_1)}$	$S_T + d_1 \cdot e^{r(T-t_1)}$
Portfolio B	$C_E + K \cdot e^{-r(T-t)} + d_1 \cdot e^{-r(t_1-t)}$	$K + d_1 \cdot e^{r(T-t_1)}$	$S_T + d_1 \cdot e^{r(T-t_1)}$

Table III.8 Put-call parity proof with dividends

The final value of portfolios A and B is the same, regardless of the share price at maturity. Therefore, to avoid arbitrage opportunities, the initial values of the two strategies must be equal.

$$P_E + S = C_E + K \cdot e^{-r(T-t)} + d_1 \cdot e^{-r(t_1-t)} \quad (Eq.\ III.32)$$

$$C_E - P_E - S + K \cdot e^{-r(T-t)} + d_1 \cdot e^{-r(t_1-t)} = 0 \quad (Eq.\ III.33)$$

This relationship can be generalized for a generic number of dividend payments.

Now, let D be the sum of the discounted values of all cash dividends paid by the stock during the life of the option, the relation can be rewritten as follows:

$$C_E - P_E - S + D + K \cdot e^{-r\tau} = 0 \quad (Eq.\ III.34)$$

In other words, the share price is replaced by the difference between its price level and the present value of the dividends paid. It should be specified that the relations discussed in points A) and B) can only be applied for European-type options.

C. Put-call parity: American Options written on equity without dividend

As stated by Property 8, let us prove that American calls written on an underlying stock that does not pay dividends should not rationally be exercised before their maturity. Under these conditions, an American call option can be managed like a European one and therefore the considerations made in the previous section are valid. In any case, this characteristic does not apply to American put options, for which the convenience of being exercised in advance may occur during the life of the option instead. Therefore, the value of an American put will be higher than its European equivalent. Thus the existing put-call relationship becomes:

$$C_{US} - S + Ke^{-r\tau} \leq P_{US} \leq C_{US} - S + K \quad (Eq.\ III.35)$$

This statement can be proved in two steps: first, the right-hand side of the inequality is verified ($P_{US} \leq C_{US} - S + K$) and secondly we verify the left-hand side ($P_{US} \geq C_{US} - S + Ke^{-r\tau}$).

As a first step, let us suppose $P_{US} > C_{US} - S + K$; in this case, we could sell the put, buy the call, short the stock and invest K in risk-free bonds. The initial cash flow ($P_{US} - C_{US} + S - K$) is positive and therefore implies an initial profit. At maturity, if the put option is not exercised, the position can be closed according to table III.9. On the other hand, if the put were exercised earlier, the K invested in the risk-free bonds would be used to buy the put holder's stock and liquidate the short position in the stock. The profit would be equal to the price of the call plus the interest on the cash K. This strategy would lead to a risk-free profit, which is in contradiction with the hypothesis of no arbitrage opportunities. Therefore, the following is verified: $P_{US} \leq C_{US} - S + K$.

	$S_T < K$	$S_T \geq K$
Buy the Put	$-(K - S_T)$	0
Sell the Call	0	$S_T - K$
Buy the Stock	$-S_T$	$-S_T$
Proceeds from the loan	$Ke^{r\tau}$	$Ke^{r\tau}$
Result	$Ke^{r\tau} - K$	$Ke^{r\tau} - K$

Table III.9 Put-call parity proof with dividends – Step 1

The second step is proved in a similar way, considering the following trading position: we buy a put option, we sell a call, and we lend an amount of money equal to the discounted value of the strike price.

If $P_{US} < C_{US} - S + K \cdot e^{-r\tau}$ then the initial investment would give an initial profit. The call option would not be exercised prematurely, since the share does not issue a dividend, and upon expiry, the position would be closed as shown in table III.10:

	$S_T < K$	$S_T \geq K$
Sell the Put	$K - S_T$	0
Buy the Call	0	$-(S_T - K)$
Sell the Stock	S_T	S_T
Pay back the loan	$-K$	$-K$
Result	0	0

Table III.10 Put-call parity proof with dividends – Step 2

As a result, we would obtain a risk-free strategy with an initial profit. In order to avoid arbitrage opportunities, the following has to be valid:

$$P_{US} \geq C_{US} - S + K \cdot e^{-r\tau} \quad \textit{(Eq. III.36)}$$

D. Put-call parity: American Options written on equity with dividend

In the case of American options written on dividend-paying stocks, the put-call parity relationship becomes:

$$C_{US} - S + Ke^{-r\tau} \leq P_{US} \leq C_{US} - S + K + D \quad \textit{(Eq. III.37)}$$

Where D is the discounted value of the dividends. The upper bound is modified to consider the dividend payment, which has a positive effect on the put price. We omit the proof of this last relation because it can be proved in a very similar way to case C).

FURTHER READINGS

Burro G., Giribone P. G., Ligato S., Mulas M., Querci F. – "Negative interest rates effects on option pricing: back to basics?" – International Journal of Financial Engineering Vol. 4, N. 2 (2017).
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Hull J. – "Options, Futures and other derivatives" – Pearson (2014).
Klemkosky R. C., Resnick B. G. – "Put-Call Parity and Market Efficiency" – The Journal of Finance Vol. 34, Issue 5, 1141-1155 (1979).

Calls Ticker	Bid	Ask	Last	IVM	Volm	Strike	Puts Ticker	Bid	Ask	Last	IVM	Volm
20-Jan-23 (16d); CSize 10; IDiv 1.01; R 2.72; IFwd 3952.35						5	20-Jan-23 (16d); CSize 10; IDiv 1.01; R 2.72; IFwd 3952.35					
1) SX5E 1 C3900	86.40	88.10	86.50	18.25	2832	3900	77) SX5E 1 P3900	38.50	39.30	40.60	18.19	2253
2) SX5E 1 C3925	70.70	72.10	68.70	17.90	86	3925	78) SX5E 1 P3925	47.50	48.50	49.00	17.78	38
3) SX5E 1 C3950	56.60	57.60	54.40	17.50	5512	3950	79) SX5E 1 P3950	58.00	59.40	61.00	17.43	5006
4) SX5E 1 C3975	44.00	45.20	42.50	17.15	52	3975	80) SX5E 1 P3975	70.60	71.90	82.50	17.07	19
5) SX5E 1 C4000	33.60	34.40	33.30	16.81	8000	4000	101) SX5E 1 P4000	84.80	86.50	129.70y	16.74	
17-Feb-23 (44d); CSize 10; IDiv 1.97; R 2.72; IFwd 3950.77						5	17-Feb-23 (44d); CSize 10; IDiv 1.97; R 2.72; IFwd 3950.77					
6) SX5E 2 C3900	126.30	127.40	120.80	18.76	51	3900	102) SX5E 2 P3900	79.60	80.40	81.30	18.71	135
7) SX5E 2 C3925	111.10	112.10	102.80	18.40	1001	3925	103) SX5E 2 P3925	89.10	90.20	119.20y	18.36	1000
8) SX5E 2 C3950	96.60	97.70	95.70	18.08	102	3950	104) SX5E 2 P3950	100.00	100.70	106.00	18.04	17
9) SX5E 2 C3975	83.50	84.30	76.00	17.77	131	3975	105) SX5E 2 P3975	111.60	112.40	146.60y	17.71	
10) SX5E 2 C4000	71.30	72.10	70.10	17.44	374	4000	106) SX5E 2 P4000	123.90	125.40	126.50	17.40	2
17-Mar-23 (72d); CSize 10; IDiv 1.39; R 2.72; IFwd 3956.37						5	17-Mar-23 (72d); CSize 10; IDiv 1.39; R 2.72; IFwd 3956.37					
11) SX5E 3 C3900	160.00	161.20	159.80	19.18	1542	3900	107) SX5E 3 P3900	108.00	108.90	109.90	19.15	3040
12) SX5E 3 C3925	144.70	145.90	114.30y	18.86	1000	3925	108) SX5E 3 P3925	117.80	118.40	119.60	18.83	10
13) SX5E 3 C3950	130.10	131.20	123.50	18.58	1	3950	109) SX5E 3 P3950	127.90	128.80	130.00	18.55	2
14) SX5E 3 C3975	116.50	117.30	89.80y	18.27		3975	110) SX5E 3 P3975	139.00	139.90	139.30	18.24	186
15) SX5E 3 C4000	103.50	104.20	103.50	17.95	8011	4000	111) SX5E 3 P4000	150.50	151.80	151.10	17.95	41

Figure III.20 Calls/Puts written on the Euro Stoxx 50 Index (SX5E Index). Center Strike. OMON (Option Monitor). Reference Date: 4th January 2023.Source: Bloomberg®

Tenor	80	85	90	95	97.5	100	102.5	105	110	115	120
1W				24.10	20.31	18.52	18.09	20.17			
2W		34.57	27.16	21.18	19.32	17.81	16.64	16.29			
3W	37.09	30.88	24.98	20.34	18.71	17.17	15.88	15.13	16.37		
1M	34.79y	29.59y	24.92y	21.32y	19.86y	18.48y	17.23y	16.40y	16.56y	18.04y	19.07y
2M	30.98	27.10	23.72	20.98	19.65	18.34	17.19	16.12	15.37	15.73	17.02
3M	29.26	26.01	23.34	21.01	19.81	18.61	17.55	16.54	15.25	14.98	15.43
6M	26.70	24.52	22.58	20.59	19.63	18.62	17.78	16.93	15.53	14.79	14.58
9M	26.38	24.55	22.76	21.05	20.19	19.36	18.55	17.81	16.49	15.50	14.94
1Y	26.31y	24.68y	23.10y	21.58y	20.84y	20.14y	19.45y	18.78y	17.57y	16.59y	15.89y
18M	24.25y	22.99y	21.77y	20.61y	20.07y	19.65y	19.14y	18.64y	17.77y	17.06y	16.48y
2Y	24.90y	23.63y	22.43y	21.29y	20.75y	20.41y	19.91y	19.41y	18.49y	17.69y	17.04y

Figure III.21 Calls/Puts written on the Euro Stoxx 50 Index (SX5E Index). Moneyness. OMON (Option Monitor). Reference Date: 4th January 2023.Source: Bloomberg

SX5E 3949.11 +66.82
At 12:14 d O 3891.04 H 3949.57 L 3891.04 Prev 3882.29
SX5E Index 95) Actions 96) Export 97) Settings Option Monitor
Euro Stoxx 50 Pr 3949.11 66.82 1.7212% Hi 3949.57 Lo 3891.04 Volm 0 HV 17.70
Center 3949.36 Strikes 5 Exp 20-Jan-23 Exch Eurex Germany 92) Events Calendar | EVTS »
Calc Mode As of 04-Jan-2023
81) Center Strike 82) Calls/Puts 83) Calls 84) Puts 85) Term Structure 87) Moneyness

Exp	20-Jan-23				17-Feb-23				17-Mar-23				21-Apr-23				19-May-23			
Calls/Puts	Calls		Puts		Calls		Puts		Calls		Puts		Calls		Puts		Calls		Puts	
Strike	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM
3700	.92	22.47	-.07	21.97	.81	21.69	-.18	21.56	.77	21.79	-.23	21.69	.73	21.53	-.27	21.47	.67	21.08	-.31	21.02
3725	.90	21.79	-.09	21.37	.79	21.27	-.20	21.20	.75	21.47	-.25	21.37	.71	21.19	-.28	21.18	.65	20.84	-.33	20.75
3750	.88	21.20	-.11	20.87	.77	20.91	-.23	20.84	.73	21.15	-.27	21.06	.69	20.92	-.30	20.90	.64	20.54	-.34	20.48
3775	.86	20.63	-.14	20.38	.75	20.55	-.25	20.46	.71	20.86	-.29	20.74	.67	20.64	-.32	20.62	.62	20.26	-.36	20.21
3800	.83	20.14	-.17	19.92	.72	20.20	-.28	20.09	.68	20.49	-.31	20.42	.65	20.37	-.34	20.32	.60	19.99	-.38	19.94
3825	.79	19.69	-.21	19.45	.69	19.84	-.31	19.78	.66	20.20	-.34	20.11	.63	20.08	-.36	20.07	.58	19.71	-.40	19.66
3850	.74	19.23	-.25	19.04	.66	19.52	-.34	19.41	.63	19.88	-.36	19.80	.61	19.77	-.38	19.76	.56	19.44	-.42	19.39
3875	.69	18.77	-.30	18.65	.62	19.16	-.37	19.06	.61	19.56	-.39	19.49	.59	19.49	-.41	19.46	.54	19.17	-.44	19.12
3900	.64	18.42	-.36	18.23	.59	18.83	-.41	18.71	.58	19.25	-.42	19.18	.56	19.19	-.43	19.19	.52	18.91	-.46	18.86
3925	.58	18.01	-.42	17.83	.55	18.49	-.45	18.37	.55	18.98	-.45	18.88	.54	18.93	-.45	18.91	.49	18.65	-.49	18.62
3950	.51	17.63	-.49	17.46	.51	18.16	-.49	18.04	.52	18.65	-.48	18.56	.52	18.65	-.48	18.63	.47	18.38	-.51	18.35
3975	.44	17.28	-.56	17.10	.47	17.83	-.53	17.71	.49	18.33	-.51	18.27	.49	18.37	-.50	18.34	.45	18.13	-.53	18.10
4000	.37	16.95	-.63	16.75	.43	17.52	-.57	17.39	.46	18.05	-.54	17.96	.46	18.09	-.53	18.07	.42	17.86	-.56	17.86
4025	.30	16.66	-.70	16.41	.39	17.21	-.61	17.05	.43	17.75	-.57	17.68	.44	17.82	-.56	17.80	.40	17.62	-.58	17.61
4050	.24	16.40	-.76	16.12	.34	16.90	-.66	16.76	.39	17.47	-.60	17.38	.41	17.55	-.58	17.55	.38	17.39	-.60	17.38
4075	.19	16.21	-.82	15.89	.30	16.64	-.70	16.48	.36	17.20	-.64	17.09	.38	17.31	-.61	17.28	.35	17.15	-.63	17.15
4100	.14	16.08	-.86	15.80	.26	16.38	-.74	16.23	.33	16.95	-.67	16.86	.36	17.06	-.64	17.03	.33	16.93	-.65	16.93
4125	.10	16.06	-.90	15.53	.23	16.16	-.77	15.99	.30	16.70	-.70	16.61	.33	16.82	-.66	16.78	.31	16.72	-.67	16.71
4150	.08	16.03	-.93	15.40	.19	15.96	-.81	15.77	.27	16.47	-.73	16.36	.30	16.58	-.69	16.59	.28	16.50	-.70	16.48
4175	.05	16.09	-.95	15.30	.16	15.78	-.84	15.58	.24	16.26	-.76	16.13	.28	16.36	-.72	16.31	.26	16.30	-.72	16.29

Figure III.22 Calls/Puts written on the Euro Stoxx 50 Index (SX5E Index). Term Structure. OMON (Option Monitor). Reference Date: 4th January 2023.Source: Bloomberg®

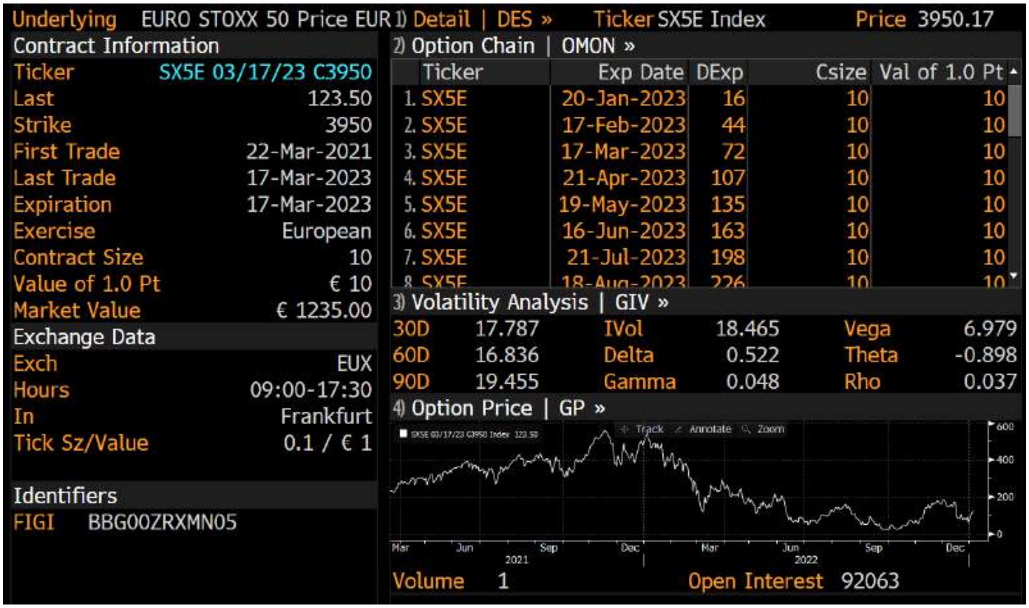


Figure III.23 SX5E 03/17/23 C3950 - Source: Bloomberg® - Reference Date: 4th January 2023

III.2 PLAIN VANILLA OPTIONS

Considering the option properties derived from the application of non-arbitrage principles we come to the definition of lower and upper bounds for European and American options prices. If we wish to compute a single value that expresses the fair value of the derivative, it is necessary to make additional assumptions though. As known, the most popular and applied formula is the Black, Scholes and Merton model (1973), which provides an analytical expression for the value of a European call and put option written on a single underlying equity. The ideas underlying the option pricing have proved revolutionary in the financial field both at a theoretical level and for the important practical implications they have had over the years. Merton and Scholes were awarded the Nobel Prize for their outstanding contributions in 1997 (Black had died two years before). We also know that the formal proof of the closed valuation formula for European options is based on the analytical solution of a partial differential equation (Black-Scholes-Merton's fundamental PDE) in continuous time and under certain assumptions. In this context, we only mention the assumptions and explain the final result. The initial assumptions made to derive the traditional Black-Scholes formula (1973) are the following:

- Markets are perfect and continuous over time: a trader can negotiate at any time, there are no transaction costs and/or taxes, there are no restrictions on short-selling.
- There are no arbitrage opportunities.
- The risk-free rate is constant over the life of the option.
- The volatility of the price for the underlying asset is constant over time.
- The changes of the underlying price level follow a lognormal distribution, which implies a normal distribution of continuously compounded returns.
- The underlying asset pays no dividends or cash flows during the life of the option.

In the following years some of these hypotheses were relaxed, obtaining models capable of describing the dynamics of the underlying more realistically and with such adjustments, the B&S pricing framework is still today the most widespread model for pricing options on different types of underlyings. In accordance with the BS framework, the formula for a European call option is:

$$C_E = S \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) \quad \textit{(Eq. III.38)}$$

$$d_1 = \frac{\ln\left(\frac{S}{K \cdot e^{-r\tau}}\right)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = \frac{\ln\left(\frac{S}{K}\right) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}};$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

Where:

S is the current underlying spot price.
τ is the Time to maturity for the call option, expressed in years.

K is the Strike price of the call.
σ is the Annualized volatility of the underlying.
r is the Annualized and continuously compounded risk-free rate.
$N(x)$ is the cumulative probability distribution function for a standard normal variable.

The pricing formula for European put options can be derived using the put-call parity:

$$P_E = K \cdot e^{-r\tau} \cdot N(-d_2) - S \cdot N(-d_1) \quad (Eq.\ III.39)$$

$N(x)$ is thus the area under the standard normal $n(x)$ up to value x.

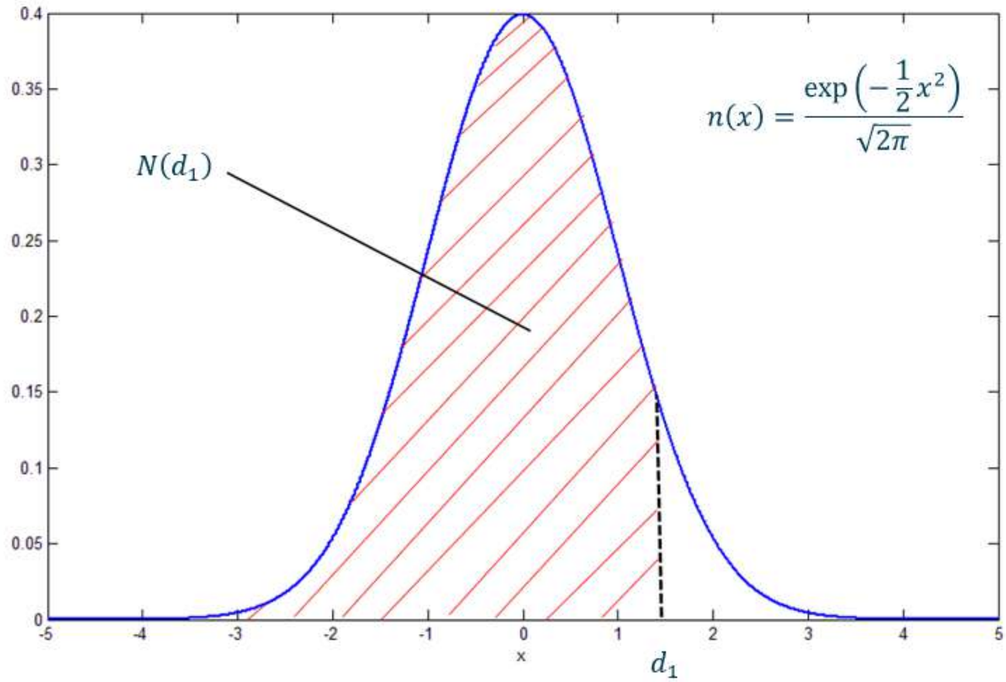


Figure III.24 Standard Normal Distribution (PDF – Probability Density Function)

The value of $N(x)$ for a given x ($x = d_1$) can be estimated using dedicated statistical functions.
$N(1.5) = 0.933$.

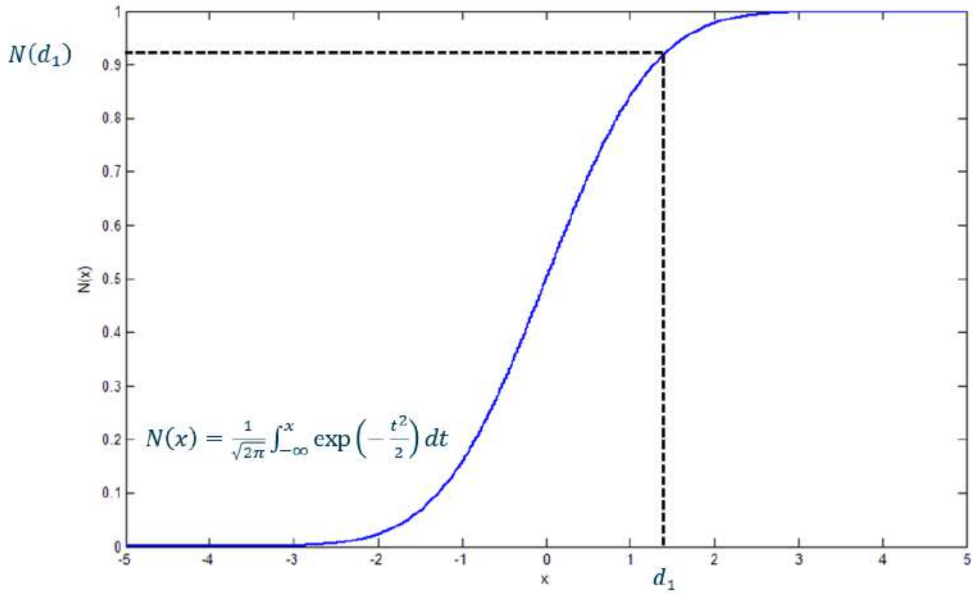


Figure III.25 Standard Normal Distribution (CDF – Cumulative Distribution Function)

Example

In accordance with the Black-Scholes pricing framework, estimate the price of an option which has the following financial characteristics:

- Stock Price, $S = \text{EUR } 100$
- Exercise price, $K = \text{EUR } 90$
- Interest Rate: $R = 0.3\%$ p.a. (simple interest rate)
- Time to Maturity: 3 months, ACT/365. $T = 90/365 = 0.247$
- Standard deviation of return: $\sigma = 30\%$ p.a.
- r (continuous compounded rate) = $\ln(1+R) = \ln(1+0.003) = 0.29955\%$

$$d_1 = \frac{\ln\left(\frac{S}{K \cdot e^{-r\tau}}\right)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = \frac{\ln\left(\frac{100}{90 \cdot e^{-0.0029955 \cdot 0.247}}\right)}{0.3 \cdot \sqrt{0.247}} + \frac{1}{2} 0.3\sqrt{0.247} = 0.786707$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = 0.786707 - 0.3 \cdot \sqrt{0.247} = 0.637738$$

$$N(d_1) = 0.784273; N(d_2) = 0.738178$$

$$C_E = S \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) =$$

$$= 100 \cdot 0.784273 - 90 \cdot e^{-0.0029955 \cdot 0.247} \cdot 0.738178 = \text{EUR } 12.04038$$

Let us examine a market case, based on the following data, as shown in the figure below:

Stock Price, $S = \text{EUR } 0.7511$; Exercise price, $K = \text{EUR } 0.7511$; Interest Rate: $r = 2.286\%$ p.a.; Time to Maturity: $T = 90$ days (ACT/360) 90/360=0.25 and Historical Volatility: $\sigma = 72.836\%$ p.a.

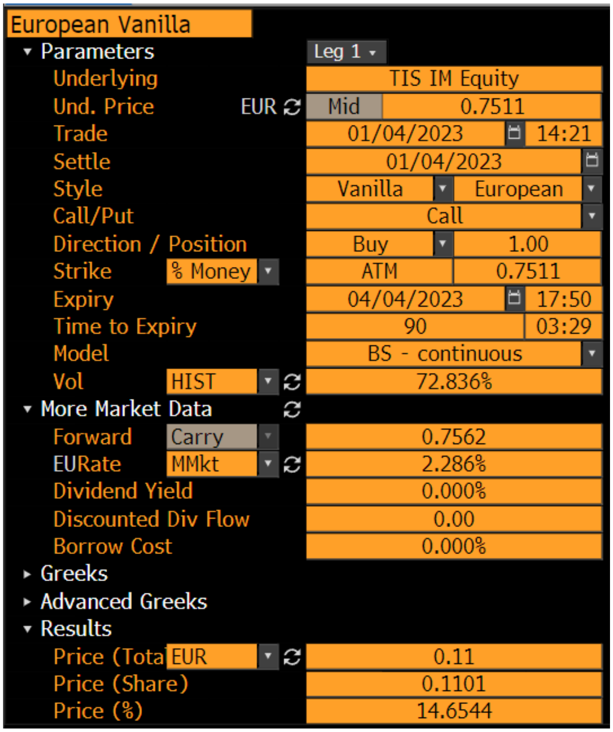


Figure III.26 European Call Option pricing written on a share without a pay-out. Source: Bloomberg®

$$d_1 = \frac{\ln\left(\frac{S}{K\cdot e^{-r\tau}}\right)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = \frac{\ln\left(\frac{0.7511}{0.7511\cdot e^{-0.02286\cdot 0.25}}\right)}{0.72836\cdot\sqrt{0.25}} + \frac{1}{2}0.72836\sqrt{0.25} = 0.197606$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = 0.197606 - 0.72836\cdot\sqrt{0.25} = -0.16657$$

$$N(d_1) = 0.578323; N(d_2) = 0.433853$$

$$C_E = S\cdot N(d_1) - K\cdot e^{-r\tau}\cdot N(d_2) = 0.7511\cdot 0.578323 - 0.7511\cdot e^{-0.02286\cdot 0.25}\cdot 0.433853 = \text{EUR } 0.11$$

$$Price[\%] = \frac{C_E}{Spot} = 14.6\%$$

The traditional Black and Scholes formula refers to an underlying equity that does not pay any dividend. We now assume that this underlying pays known dividends during the life of the option. Such remuneration will be received by the trader who invested directly in the stock, and nothing is paid by the option holder. The idea behind the adjustment of the initial formula is to reduce the spot value of the share observed on the market by the sum of the discounted values of the expected dividends that will be paid during the life of the derivative: $S^* = S - \sum_{i=1}^{I} D_i \cdot e^{-r\tau_i}$

Assuming that there will be I dividend payments, the formula becomes:

$$C_E = \left(S - \sum_{i=1}^{I} D_i \cdot e^{-r\tau_i}\right) \cdot N(d_1{}^*) - K \cdot e^{-r\tau} \cdot N(d_2{}^*) \quad (Eq.\ III.40)$$

$$P_E = K \cdot e^{-r\tau} \cdot N(-d_2{}^*) - \left(S - \sum_{i=1}^{I} D_i \cdot e^{-r\tau_i}\right) \cdot N(-d_1{}^*) \quad (Eq.\ III.41)$$

$$d_1{}^* = \frac{\ln\left[\frac{\left(S - \sum_{i=1}^{I} D_i \cdot e^{-r\tau_i}\right)}{K \cdot e^{-r\tau}}\right]}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} \qquad d_2{}^* = d_1{}^* - \sigma\sqrt{\tau}$$

t_i is the payment date of dividend i.
$\tau = T - t$ is the Time to maturity of the option.
t is the Valuation or Reference date.
$\tau_i = T - t_i$ represents the time between the ex-dividend date of the i -th dividend (in years), t_i and the maturity date T.
D_i is the dividend paid at time t_i.

Let us analyze as an example a European call option written on an underlying equity that expires in 3 months (30/360). The spot market value is EUR 61.7, the exercise price of the option is EUR 62. Assuming an annualized historical volatility of 35% and a continuously compounded risk-free rate of 2%, the fair-value of the call is determined in accordance with the BS closed-formula. Here are the data:

Stock Price, $S = \text{EUR } 61.7$
Exercise price, $K = \text{EUR } 62$
Interest Rate: $r = 2\%$ p.a.
Time to Maturity: $T = 0.25$
Historical Volatility: $\sigma = 35\%$ p.a.

Thus, the following can be calculated:

$d_1 = 0.088073$, $d_2 = -0.08693$; $N(d_1) = 0.535091$, $N(d_2) = 0.465365$ and $C_E = 4.306382$.

A financial analyst expects a dividend of € 1.5 in one month (day count 30/360). We wish to determine the decrease in the value of the call option.

$S^* = 61.7 - 1.5 \cdot \exp\left(-0.02 \cdot \frac{1}{12}\right) = \text{EUR } 60.2025$; $K =$EUR 62, $r = 2\%$ p.a. , $T = 0.25$ and $\sigma = 35\%$ p.a.

$d_1 = -0.05205$, $d_2 = -0.22705$; $N(d_1) = 0.479246$, $N(d_2) = 0.410194$ and $C_E^* = 3.546624$

A similar logic can be applied if the European option is written on an underlying that pays a known and constant dividend yield (q).

We can define the **Dividend yield** as the ratio of dividends paid by the company in the year to the share price.

$$q = \frac{Annual\ Dividends\ per\ Share}{Price\ per\ Share} \quad (\textit{Eq. III.42})$$

The key idea is that the payment of a continuous dividend at rate q causes the growth rate in the share price to be lower than it would have been in the absence of a continuous remuneration of an amount equal to q. Thus, from a financial point of view, an investor holding a stock that pays a continuous dividend yield q and that grows from S_t at time t to S_T at time T is equivalent to holding a stock that does not pay dividends and which increases from S_t at time t to $S_T \exp[+q(T-t)]$ at time T, or, equivalently, from $S_t \exp[-q(T-t)]$ at time t to S_T at time T. Taking this fact into account, a European option written on a stock with a price S that pays a continuous dividend q has the same value as the corresponding European option written on a stock with a price level equal to $S_t \exp[-q(T-t)]$ and which pays no dividends.

The traditional set of Black-Scholes framework formulas can be adjusted in this case as follows:

$$C_E = S \cdot e^{-q\tau} \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) \quad (\textit{Eq. III.43})$$

$$P_E = K \cdot e^{-r\tau} \cdot N(-d_2) - S \cdot e^{-q\tau} \cdot N(-d_1) \quad (\textit{Eq. III.44})$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r-q) \cdot \tau}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}; \ d_2 = d_1 - \sigma\sqrt{\tau}$$

Where q is the continuous dividend yield.

In market practice, this formulation is useful when the precise amount of the dividend and/or the dates on which it will be paid is not known and therefore a common approach is to estimate the parameter by looking at the historical trend of the share and applying the ratio:

$$q = Dividend/Average\ quoted\ price \quad (\textit{Eq. III.45})$$

We will now illustrate a market case based on the following data, as shown in the figure:

Stock Price, $S = \text{EUR } 2.0315$; Exercise price, $K = \text{EUR } 2.20$; Interest Rate: $r = 3.172\%$ p.a.; Historical Volatility: $\sigma = 42.774\%$ p.a.; Time to Maturity: $T = 360$ days (ACT/365) 360/365=0.9863 and Dividend Yield: $q = 3.418\%$ p.a.

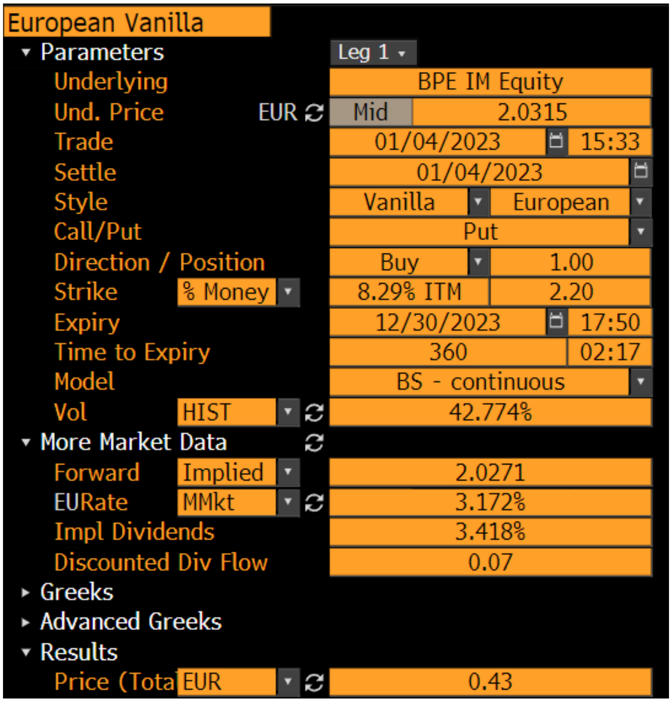


Figure III.27 European Put Option pricing written on a share with a pay-out. Source: Bloomberg®

We can proceed and calculate:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right)+(r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} = \frac{\ln\left(\frac{2.0315}{2.20}\right)+(0.03172-0.03418+0.42774^2/2)0.9863}{0.42774\sqrt{0.9863}} = 0.231511$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = 0.231511 - 0.42774 \cdot \sqrt{0.9863} = -0.19329$$

$$N(-d_1) = 0.408459, \ \ N(-d_2) = 0.576634$$

$$P_E = K \cdot e^{-r\tau} \cdot N(-d_2) - S \cdot e^{-q\tau} \cdot N(-d_1) =$$

$$= 2.20 \cdot e^{-0.03172 \cdot 0.9863} \cdot 0.576634 - 2.0315 \cdot e^{-0.03418 \cdot 0.9863} \cdot 0.408459 = \text{EUR } 0.427$$

A call option on a stock index would give its holder the right (but not the obligation) to buy the stock index (underlying asset) by a certain date and for a specified price. On the other hand, a put option on a stock index would give its holder the right (but not the obligation) to sell the index by a certain date and at the agreed price. Since an index is not physically deliverable, a **cash settlement** must be performed.

Consequently, the main difference between calls written on indices and calls written on an equity underlying is the settlement procedure:

- The holder of an in-the-money call option on a single share could receive a given number of shares (depending on the contract size) against payment of a strike price, while the holder of an in-the-money call on a stock index would receive the difference between the index value and the strike price, multiplied by the contract size.

- The holder of an in-the-money put option on a single share could deliver a given number of shares (depending on the contract size) against payment of a strike price, while the holder of an in-the-money put on a stock index would pay the difference between the strike price and the value of the index, multiplied by the contract size.

Thus, an option written on a stock index can be priced as an option written on a dividend paying stock where the index takes the role of the share. The formula for a European stock index option is given by:

$$C_E = \left(S - \sum_{j=1}^{J}\sum_{i=1}^{I} D_{j,i} \cdot e^{-r\tau_{j,i}}\right) \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) \quad \textit{(Eq. III.46)}$$

$$P_E = K \cdot e^{-r\tau} \cdot N(-d_2) - \left(S - \sum_{j=1}^{J}\sum_{i=1}^{I} D_{j,i} \cdot e^{-r\tau_{j,i}}\right) \cdot N(-d_1) \quad \textit{(Eq. III.47)}$$

$$d_1 = \frac{\ln\left[\frac{\left(S - \sum_{j=1}^{J}\sum_{i=1}^{I} D_{j,i} \cdot e^{-r\tau_{j,i}}\right)}{K \cdot e^{-r\tau}}\right]}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} \qquad d_2 = d_1 - \sigma\sqrt{\tau}$$

C_E ($/P_E$) is the price of the European call(/put) on date t.
S is the price of the index on date t.
σ is the standard deviation of the instantaneous returns of the stock index.
$D_{j,i}$ is the dividend paid at time t_i by company j .
$\tau_{j,i}$ is the time between the payment date t_i of company j and the maturity date.

Similarly to options written on dividend-paying stocks, the assumption of a continuous dividend yield is also often made for indices. Under this consideration, the pricing formulas become:

$$C_E = S \cdot e^{-q\tau} \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) \quad \textit{(Eq. III.48)}$$

$$P_E = K \cdot e^{-r\tau} \cdot N(-d_2) - S \cdot e^{-q\tau} \cdot N(-d_1) \quad \textit{(Eq. III.49)}$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r-q)\cdot\tau}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}; \;\; d_2 = d_1 - \sigma\sqrt{\tau}$$

Where q is the continuous dividend yield.

Let us now make an example, considering a European call option written on an index. The current value of the underlying is 700 and the strike price is 750. The continuously compounded risk-free rate is 1% p.a., the volatility of the equity index is 30% p.a.

The option will expire in 2 months and the expected dividend yield over that period is 0.33%. In short, the input parameters are:

$S = 700; K = 750; \tau = 0.1667; \sigma = 30\%;$

$q = 1.98\%$ (0.33% in 2 months, which means 1.98% per annum under a simple compounding regime);

$r = 1\%$.

After implementing the calculations, here are the output parameters:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right)+(r-q)\cdot\tau}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = -0.51536 \quad d_2 = d_1 - \sigma\sqrt{\tau} = -0.6379$$

$$N(d_1) = 0.30315 \; N(d_2) = 0.26179$$

$$C_E = S \cdot e^{-q\tau} \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) = 15.4903$$

We now proceed with a real market case based on the following data as shown in the below figure:

Stock Price, $S = \text{USD } 3828.04$

Exercise price, $K = \text{USD } 3828.04$

Interest Rate: $r = 4.604\%$ p.a.

Time to Maturity: $T = 90$ days (ACT/365) 90/365=0.24657

Implied Volatility: $\sigma = 22.417\%$ p.a.

Dividend Yield: $q = 1.876\%$ p.a.

$$d_1 = \frac{\ln\left(\frac{S}{K}\right)+(r-q)\cdot\tau}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = \frac{\ln\left(\frac{3828.04}{3828.04}\right)+(0.04604-0.01876)\cdot 0.24657}{0.22417\sqrt{0.24657}} + \frac{1}{2}0.22417\sqrt{0.24657} = 0.1160858$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = 0.1160858 - 0.22417\sqrt{0.24657} = 0.0047712$$

$$N(d_1) = 0.546208, \;\; N(d_2) = 0.501903$$

$$C_E = S \cdot e^{-q\tau} \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) =$$

$$= 3828.04 \cdot e^{-0.01876\cdot 0.24657} \cdot 0.546208 - 3828.04 \cdot e^{-0.04604\cdot 0.24657} \cdot 0.501903 = \text{USD } 181.64$$

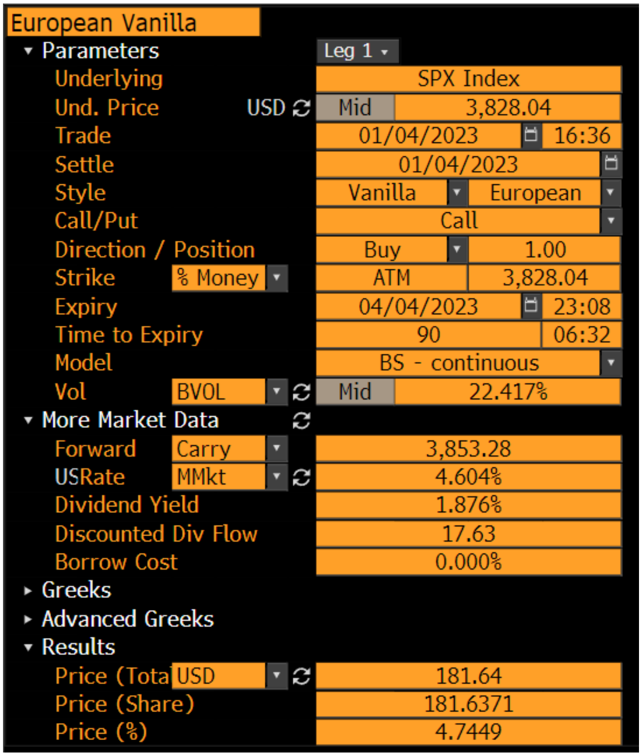


Figure III.28 European Call Option pricing written on an Equity index. Source: Bloomberg®

The traditional log-normal pricing framework can easily be extended to the valuation of options written on futures contracts. When applying the cost-of-carry model, futures contracts can be treated as assets that pay a dividend equal to the risk-free rate. Thus, an option on a futures contract can be financially modelled as an option on a stock that pays a continuous dividend yield, where the price of the futures (F) takes the place of the stock price level and the ongoing dividend yield equals the risk-free rate ($q = r$).
For European options and under the hypothesis of use of this valuation framework, the pricing formula (called Black model) becomes:

$$C_E = F \cdot e^{-r\tau} \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) = e^{-r\tau}\left[F \cdot N(d_1) - K \cdot N(d_2)\right] \quad \textit{(Eq. III.50)}$$

$$P_E = e^{-r\tau}\left[K \cdot N(-d_2) - F \cdot N(-d_1)\right] \quad \textit{(Eq. III.51)}$$

$$d_1 = \frac{\ln\left(\frac{F}{K}\right)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = \frac{\ln\left(\frac{F}{K}\right) + (\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \qquad d_2 = d_1 - \sigma\sqrt{\tau}$$

Let us consider the example of a European ATM (at-the-money) put option written on a futures contract.

Assuming the time to expiry equal to 4 months, the current price of the Futures is USD 15. The continuously compounded risk-free rate is 2% p.a. and the futures price volatility is 30% p.a.

In short, the input parameters are:

$F = \text{USD } 15$

$K = \text{USD } 15$ (given that the option is At-The-Money: $F = K$)

$r = 2\%$

$\sigma = 30\%$

After the calculations, here is the output:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = \frac{1}{2}\sigma\sqrt{\tau} = 0.086603, N(-d_1) = 0.465494$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = \frac{1}{2}\sigma\sqrt{\tau} - \sigma\sqrt{\tau} = -0.086603, N(-d_2) = 0.534506$$

$$P_E = e^{-r\tau}\left[K \cdot N(-d_2) - F \cdot N(-d_1)\right] = 1.02831$$

Let us now present a market case based on the data shown hereafter and in the figure below:

Future Price, $F = \text{USD } 838.375$

Exercise price, $K = \text{USD } 838.375$

Interest Rate: $r = 4.663\%$ p.a.

Time to Maturity: $T = 110$ days (ACT/365) 110/365=0.30137

Implied Volatility: $\sigma = 29.624\%$ p.a.

$$d_1 = \frac{\ln\left(\frac{F}{K}\right)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = \frac{\ln\left(\frac{838.375}{838.375}\right)}{0.29624\sqrt{0.30137}} + \frac{1}{2}0.29624\sqrt{0.30137} = 0.167725,$$

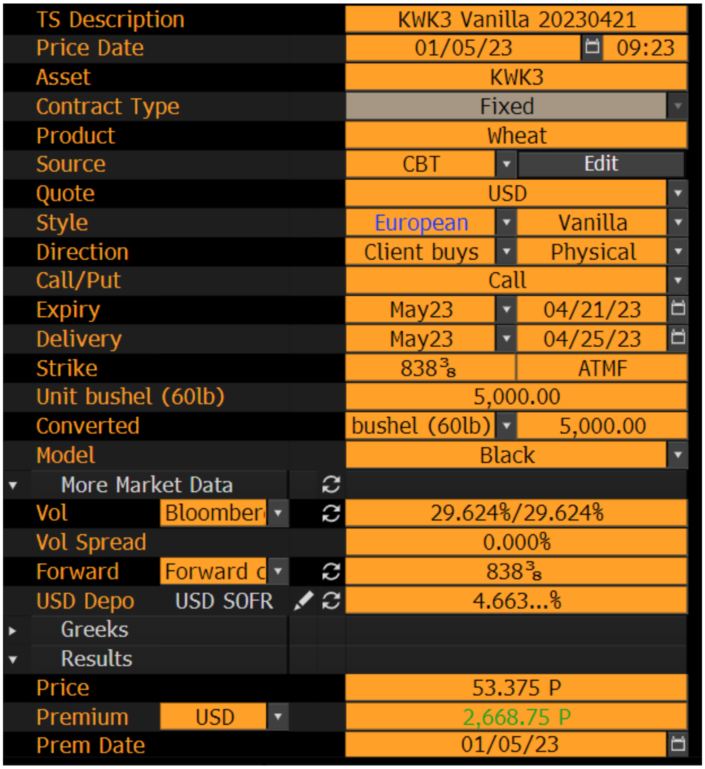
$$N(-d_1) = 0.4334$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = 0.167725 - 0.29624\sqrt{0.30137} = 0.005098,$$

$$N(-d_2) = 0.497966$$

$$P_E = e^{-r\tau}\left[K \cdot N(-d_2) - F \cdot N(-d_1)\right] = e^{-0.04663 \cdot 0.30137}\left[838.375 \cdot 0.497966 - 838.375 \cdot 0.4334\right] =$$

$$= 53.3756$$



TS Description	KWK3 Vanilla 20230421	
Price Date	01/05/23	09:23
Asset	KWK3	
Contract Type	Fixed	
Product	Wheat	
Source	CBT	Edit
Quote	USD	
Style	European	Vanilla
Direction	Client buys	Physical
Call/Put	Call	
Expiry	May23	04/21/23
Delivery	May23	04/25/23
Strike	838 3/8	ATMF
Unit bushel (60lb)	5,000.00	
Converted	bushel (60lb)	5,000.00
Model	Black	
More Market Data		
Vol Bloomber	29.624%/29.624%	
Vol Spread	0.000%	
Forward Forward c	838 3/8	
USD Depo USD SOFR	4.663...%	
Greeks		
Results		
Price	53.375 P	
Premium USD	2,668.75 P	
Prem Date	01/05/23	

Figure III.29 European Call Option pricing written on a Futures contract. Source: Bloomberg®

The Black Scholes model can be extended to the valuation of European options written on a foreign currency, since a foreign currency can be represented as an asset that pays a continuous dividend equal to the risk-free rate of the foreign currency. Therefore, a foreign currency option can be priced as an option on a stock paying a known, continuous and constant dividend where the value of the foreign currency (the exchange rate) plays the role of the share price and its current dividend yield equals the foreign currency risk-free rate.

The resulting pricing formula for European currency options, also called **Garman-Kohlhagen model**, is:

$$C_E = S \cdot e^{-r_{FOR}\tau} \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) \quad \text{(Eq. III.52)}$$

$$P_E = K \cdot e^{-r\tau} \cdot N(-d_2) - S \cdot e^{-r_{FOR}\tau} \cdot N(-d_1) \quad \text{(Eq. III.53)}$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - r_{FOR}) \cdot \tau}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = \frac{\ln\left(\frac{S}{K}\right) + (r - r_{FOR} + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\, ; d_2 = d_1 - \sigma\sqrt{\tau}$$

Where:

S is the current exchange rate.
r_{FOR} is the continuously compounded foreign risk-free rate.
σ is the annualized volatility of the exchange rate.
r is the continuously compounded domestic risk-free rate.

We now describe an example, considering a European USD-call/EUR-put option with six months to expiration. The USD/EUR exchange rate is 1.56, the strike is 1.6, the domestic risk-free interest rate in EUR is 8% per year, the foreign risk-free interest rate in USD is 6% per year and the volatility is 12% per year. Here are the calculations:

$S = 1.56, X = 1.6, T = 0.5, r = 0.06, r_{FOR} = 0.08$ and $\sigma = 0.12$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - r_{FOR} + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = \frac{\ln\left(\frac{1.56}{1.6}\right) + (0.06 - 0.08 + 0.12^2/2)0.5}{0.12\sqrt{0.5}} = -0.3738,\ N(d_1) = N(-0.3738) = 0.3543$$

$$d_2 = -0.3738 - 0.12\sqrt{0.5} = -0.4587, N(d_2) = N(-0.4587) = 0.3232$$

$$C_E = S \cdot e^{-r_{FOR}\tau} \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) = 1.56 \cdot e^{-0.08 \cdot 0.5} \cdot 0.3543 - 1.6 \cdot e^{-0.06 \cdot 0.5} \cdot 0.3232 = \\ = 0.0291$$

The option premium is thus 0.0291 USD per EUR. Alternatively, the premium can be quoted in EUR per USD 0.0291/1,562=0.012 – or as percentage of the spot, 0.0291/1.56=1.8654% of EUR (or the spot price).

Hence, if the option has a notional of EUR 100 million, the total option premium is EUR 1,865,384.62 or 1,865,384.62 x 1.56 = USD 2,910,000.00.

The most common options on interest rates are caps and floors. An interest rate **cap** consists of a series of European call options, called caplets, and each **caplet** can be priced using an analytical formula derived by the Black-Scholes framework (**Modified Black-76 Formula**). This extension may be possible using the implicit forward rate (*F*) calculated at each maturity of the caplet (*T*) or at each re-fixing of the interest rate. The fair value of the option is therefore calculated as the sum of the caplets premiums which constitute the cap. A completely similar logic can be applied to the fair value estimation of the price of a **floor**, which can be broken down into a sum of individual European put options (or **floorlets**).

$$Cap\ Value = \sum_{i=1}^{n} Caplet\ Value_i \quad (Eq.\ III.54)$$

$$Caplet\ Value = \frac{Notional \times \frac{\tau}{Basis}}{1 + FWD\frac{\tau}{Basis}} \times Black76\ Call\ Value \quad (Eq.\ III.55)$$

$$Black76\ Call\ Value = e^{-rT}[F\,N(d_1) - K\,N(d_2)] \quad (Eq.\ III.56)$$

$Floor\ Value = \sum_{i=1}^{n} Floorlet\ Value_i$ *(Eq. III.57)*

$Floorlet\ Value = \frac{Notional \times \frac{\tau}{Basis}}{1 + FWD\frac{\tau}{Basis}} \times Black76\ Put\ Value$ *(Eq. III.58)*

$Black76\ Put\ Value = e^{-rT}[K\ N(-d_2) - F\ N(-d_1)]$ *(Eq. III.59)*

With $d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln\left(\frac{F}{K}\right) - \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$

Where:

τ is the number of days included in the forward rate (tenor).
$Basis$ is the number of days in a year according to the convention applied by the market.
K is the strike price of the option.
e^{-rT} is the discount factor calculated starting from the continuously compounded zero-rate to be applied for maturity T.
$N(x)$ is the standard normal cumulative distribution computed in x.
σ is the Black volatility of the forward rate.

This model, together with all the other analytical formulas for evaluating options deriving from the Black-Scholes framework, has had a considerable diffusion in the world of quantitative finance. All the formulas we have examined have been implemented for evaluating vanilla options in all the evaluation modules of the main market platforms, such as Bloomberg and Reuters and in the main OTC derivatives management softwares.

In recent years, two relevant questions have arised: is the Black76 model still valid in a low-rates environment? And what is the main problem associated with the log-normal model if the forward rate F is negative?

We will start our dissertation with an example. We wish to calculate the value of a caplet written on a semi-annual forward rate, expiring exactly in six months and with a notional amount of EUR 100 million. We assume for our calculation that the forward rate implied by the EURIBOR 6 months term structure is 2%, the exercise price is 2%, the risk-free rate is 1% and the annualized volatility of the underlying is 25%.

$Basis$=360 days, τ=180 days, F=2%, K=2%, r=1%, σ=25%, $T = 0.5$, $Notional$= EUR 100,000,000.

$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = 0.088388$, $d_2 = d_1 - \sigma\sqrt{T} = -0.088388$

$N(d_1) = 0.535216$ e $N(d_2) = 0.464784$

$Black76\ Call\ Value = e^{-rT}[F\ N(d_1) - K\ N(d_2)] = 0.001401614$

$Caplet\ Value = \frac{Notional \times \frac{\tau}{Basis}}{1 + FWD\frac{\tau}{Basis}} \times Black76\ Call\ Value = \text{EUR } 69{,}386.82848$

Let us now review another case, where we wish to price a cap written on a bullet notional amount of USD 10,000,000. The derivative is evaluated on 30th June 2021 and it is "forward start", i.e. the protection of the rate begins from 30th September 2021 and lasts five years, consequently the maturity date is on 30th September 2026. Given the quarterly payments, the benchmark parameter has the same tenor, i.e. the 3-month USD Libor, or US0003M Index on the Bloomberg platform.

The customer wants to protect against a rise of the interest rate over 0.5%, which is then the Strike price for all the 20 caplets (X = 0.005). Considering an ACT/360 convention, the Basis is equal to 360 and the t_i are the exact number of days included in the forward rate (considering that the fixing takes place two days before the payment dates).

The maturity dates for each caplet are shown in the second column of the Table III.12.
The payment dates are the dates on which cash flows are paid out. They are useful for calculating the maturity of each caplet expressed in year fractions, T. They are summarized in the third column of the Table III.12.

The zero rates interpolated from the risk free term structure (r) denominated in USD are used for the estimation of the discount factors to be applied at each payment date T and for the forward rates computation.

The interest rates of the 3-month USD swap curve have been reported below. The interpolated zero-rates used for pricing the caplets are reported in the fourth column of Table III.12

Term	Type	Maturity	Market Rates	Zero Rates	Discount Fact.
EDU1	Futures	09/13/21	0.1396	0.1444	0.9993
3 MO	Cash Rates	09/30/21	0.1458	0.1478	0.9996
EDZ1	Futures	12/13/21	0.2038	0.1663	0.9988
EDH2	Futures	03/14/22	0.1978	0.1751	0.9983
EDM2	Futures	06/13/22	0.2663	0.1959	0.9676
EDU2	Futures	09/19/22	0.3595	0.2243	0.9967
EDZ2	Futures	12/19/22	0.5126	0.2641	0.9955
2 YR	Swap	06/30/23	0.3284	0.3279	0.9934
3 YR	Swap	06/30/24	0.5711	0.5708	0.983
4 YR	Swap	06/30/25	0.793	0.7946	0.9687
5 YR	Swap	06/30/26	0.965	0.9691	0.9526
6 YR	Swap	06/30/27	1.1054	1.1125	0.9353

Table III.11 Interest rates term structure

Caplet	Start Date	End Date	Zero Rate r[%]	Forward Rates F[%]	Black Volatility [%]	Bachelier Volatility [bps]
1	09/28/21	12/31/21	0.168	0.188	100.35	49.28
2	12/29/21	03/31/22	0.179	0.201	100.35	49.28
3	03/29/22	06/30/22	0.201	0.266	100.35	49.28
4	06/28/22	09/30/22	0.229	0.341	100.35	49.28
5	09/28/22	12/30/22	0.268	0.461	92.37	49.18
6	12/28/22	93/31/23	0.298	0.479	84.3	49.07
7	03/29/23	06/30/23	0.328	0.539	76.24	48.97
8	06/28/23	09/29/23	0.388	0.867	68.17	48.86
9	09/27/23	12/29/23	0.449	0.994	67.45	51.18
10	12/27/23	03/28/24	0.508	1.114	66.9	53.54
11	03/26/24	06/28/24	0.569	1.235	66.36	55.89
12	06/26/24	09/30/24	0.627	1.302	65.8	58.28
13	09/26/24	12/31/24	0.684	1.413	65.38	59.66
14	12/27/24	03/31/25	0.739	1.525	64.97	61.01
15	03/27/25	06/30/25	0.795	1.636	64.56	62.33
16	06/26/25	09/30/25	0.839	1.546	64.15	63.67
17	09/26/25	12/31/25	0.883	1.626	63.96	64.56
18	12/29/25	03/31/26	0.926	1.714	63.77	65.46
19	03/27/26	06/30/26	0.969	1.8	63.6	66.3
20	06/26/26	09/30/26	1.005	1.73	63.42	67.17

Table III.12 Caplets market data. Source: Bloomberg®

Analysts generally use the implied volatility quoted on the market. It is called implied because it has been derived from the listed options premiums. When the price is known, it is possible to apply a numerical inversion of the Black formula through a Newton-Raphson algorithm.

Let us observe the section of the surface closest to the relevant strike in the first row of the Implied Volatility Table below. We notice some market imperfections associated to these contributions, and the main cause for them can be found observing that the level of interest rates in 2021 was very low in America.
This negatively impacted the computation of the logarithm in auxiliary variable d_1 when the solver routine was performed by the info-provider for calculating the implied volatility.

Interpolating the 3 month USD forward volatilities along the only available strike of 1%, we can compute a proxy of the σ to be used for each caplet. The market implied volatility surface has been reported in the following figure. The interpolated Black σ used for pricing the caplets have been reported in the Table III.12 .

Volatility/Tenor	Strike	1Y	2Y	3Y	4Y	5Y	6Y
Black[%]	1%	100.35	67.99	65.78	64.14	63.41	62.46
Bachelier[bps]	0.50%	49.28	48.86	58.36	63.71	67.2	69.28

Table III.13 Implied volatilities using a normal (Bachelier) and a log-normal (Black) model

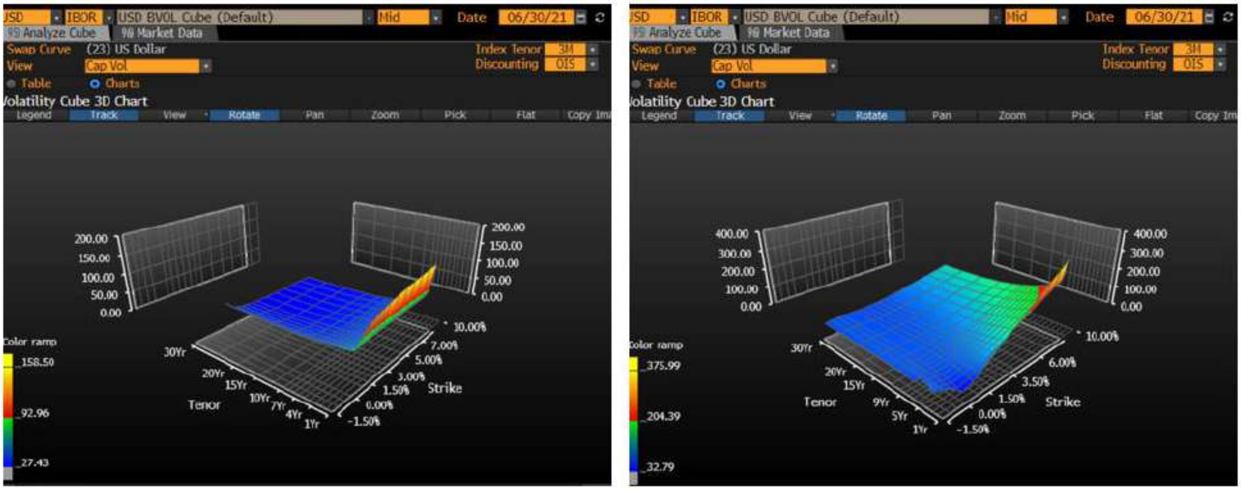


Figure III.30 Black Log-Normal Volatility [%] versus Normal Implied Volatility [bps]. Source: Bloomberg®

The gap in the Black market implied volatilities between the first and the second year, caused by the low-rates recorded on American markets, is evident. The main info providers, among them Bloomberg®, actually announced in the summer 2020, they would change the valuation model switching from the traditional well established Black 76 model to the **Bachelier** (also known as Normal) **model**. This change had already happened in the Euro Area in 2015 when the first negative rates started to appear on the European financial markets. The main concept is to model the underlying of the option using an Arithmetic instead of a Geometric Brownian motion. Such assumption allows to handle low and negative rates, as shown in the following equation, as the logarithmic term is no longer there.

$$Caplet\ Value = \frac{Notional \times \frac{\tau}{Basis}}{1+FWD\frac{\tau}{Basis}} \times Bachelier\ Call\ Value \quad (Eq.\ III.60)$$

$$Bachelier\ Call\ Value = e^{-rT}\left[(F-X)\ N(d) + \sigma\sqrt{T}n(d)\right] \quad (Eq.\ III.61)$$

$$Floorlet\ Value = \frac{Notional \times \frac{\tau}{Basis}}{1+FWD\frac{\tau}{Basis}} \times Bachelier\ Put\ Value \quad (Eq.\ III.62)$$

$$Bachelier\ Put\ Value = e^{-rT}\left[(X - F)\ N(-d) + \sigma\sqrt{T}n(d)\right]$$ *(Eq. III.63)*

With: $d = \frac{F-X}{\sigma\sqrt{T}}$ and $n(x)$ is the probability density function for a standard normal distribution.

$$n(d) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d^2}{2}\right)$$ *(Eq. III.64)*

The change of the model leads to different implied volatility quotations, which are now expressed not in percentage but in basis points (bps). Given that the evident volatility bias in the estimation of the first eight caplets (i.e. the options which mature before two years) is no longer present, we can consider applying this pricing model for having a fairer theoretical valuation for the cap. The implied forward normal volatilities term structure is reported in the second row of the "Implied Volatility Table" above, and this time there is also the proper strike in the surface, that is 0.5% allowing a better estimation of the twenty options on the short rate, US0003M Index. The Bachelier σ used for pricing are reported in the last column of the Table III.12.

At this point, we have all the market inputs for pricing all the caplets through the two interest rates pricing models. We report here the valuation of the seventh caplet, i.e., the call option written on the US003M forward that starts on 29th March 2023 and matures on the expiration date of the cap, on 30th June 2023.

Here is the valuation of the 7th caplet applying the Black model:

$Basis$=360 days, $\tau = \text{30 June 2023} - \text{29 March 2023} = 93$ days, F=0.539%, K=0.5%, r=0.328%,

$T = \frac{\text{29 March 2023} - \text{Valuation Date}}{360} = 1.769$ σ_{Black}=76.24%, $Notional$= USD 10,000,000

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.00539}{0.005}\right) + \left(\frac{0.7624^2}{2}\right)\cdot 1.769}{0.7624\sqrt{1.769}} = 0.58216$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.58216 - 0.7624\sqrt{1.769} = -0.434386$$

$$N(d_1) = 0.719769$$

$$N(d_2) = 0.332004$$

$$Black76\ Call\ Value = e^{-rT}[F\ N(d_1) - K\ N(d_2)] =$$
$$= e^{-0.00328\cdot 1.769}\cdot[0.00539\cdot\ 0.719769 - 0.005\cdot 0.332004] = 0.002207$$

$$Caplet\ Value = \frac{Notional \times \frac{\tau}{Basis}}{1 + FWD\frac{\tau}{Basis}} \times Black76\ Call\ Value = \frac{10{,}000{,}000 \times \frac{93}{360}}{1 + 0.00539\frac{93}{360}} \times 0.002207 = \text{USD } 5692.70$$

Here is the same valuation applying the Bachelier model**:**

$Basis$=360 days, $\tau = \text{30 June 2023} - \text{29 March 2023} = 93$ days, F=0.539%, K=0.5%, r=0.328%,

$$T = \frac{\text{29 March 2023} - \text{Valuation Date}}{360} = 1.769,$$

$$\sigma_{NORM} = 48.97 \text{ bps} = \frac{48.97}{10000} = 0.004897,$$

$$Notional = \text{USD } 10{,}000{,}000$$

$$d = \frac{F-X}{\sigma\sqrt{T}} = \frac{0.00539-0.005}{0.004897\sqrt{1.769}} = 0.0598785$$

$$N(d) = 0.52387378$$

$$n(d) = 0.39822773$$

$$Bachelier\ Call\ Value = e^{-rT}\left[(F-X)\,N(d) + \sigma\sqrt{T}n(d)\right] =$$

$$e^{-0.00328\cdot 1.769}\cdot\left[(0.00539\ -0.005)\cdot\ 0.52387378 + 0.004897\cdot\sqrt{1.769}\cdot 0.39822773\right] = 0.002782$$

$$Caplet\ Value = \frac{Notional\times\frac{\tau}{Basis}}{1+FWD\frac{\tau}{Basis}}\times Bachelier\ Call\ Value = \frac{10{,}000{,}000\times\frac{93}{360}}{1+0.00539\frac{93}{360}}\times 0.002782 = \text{USD } 7{,}176.47$$

Figure III.31 SWPM module > Products > Options > Cap. Source: Bloomberg®

Applying iteratively the two pricing formulas, we obtain the theoretical price for each caplet. The sum of these 20 values then gives the cap price. In accordance with the Black log-normal model the cap is worth USD 344,052, while using the Bachelier Normal model its price is USD 360,294.

We compare these results using the Bloomberg® module and obtain a very close result. The spread over the benchmark parameter can easily be taken into consideration adding this quantity to the implied forward rates.

Considering the traditional formula proposed by Black-Scholes for European options:

$$C_E = S \cdot N(d_1) - K \cdot e^{-rT} \cdot N(d_2)$$

$$P_E = K \cdot e^{-rT} \cdot N(-d_2) - S \cdot N(-d_1)$$

$$d_1 = \frac{\ln\left(\frac{S}{K \cdot e^{-rT}}\right)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{T} = \frac{\ln\left(\frac{S}{K}\right) + (r + \sigma^2/2)T}{\sigma\sqrt{T}};\ d_2 = d_1 - \sigma\sqrt{T}$$

A sensitivity analysis of the price can be implemented to assess how the price changes in function of the variation of one of the parameters.

The input data are the following: S=50, K=55, T=0.5, r=5%, σ=30%.

In accordance with the BS framework, the price of a European call option is $C_E = 2.7935$ and the theoretical fair value of a put option is $P_E = 6.4356$.

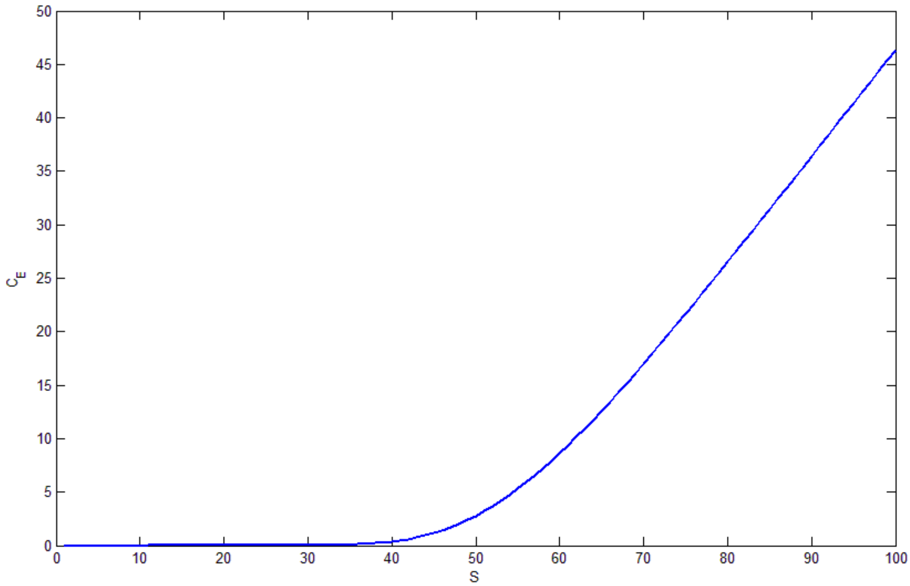


Figure III.32 European Call Price change as the spot varies. Input: S=1:100, K=55, T=0.5, r=5%, σ=30%

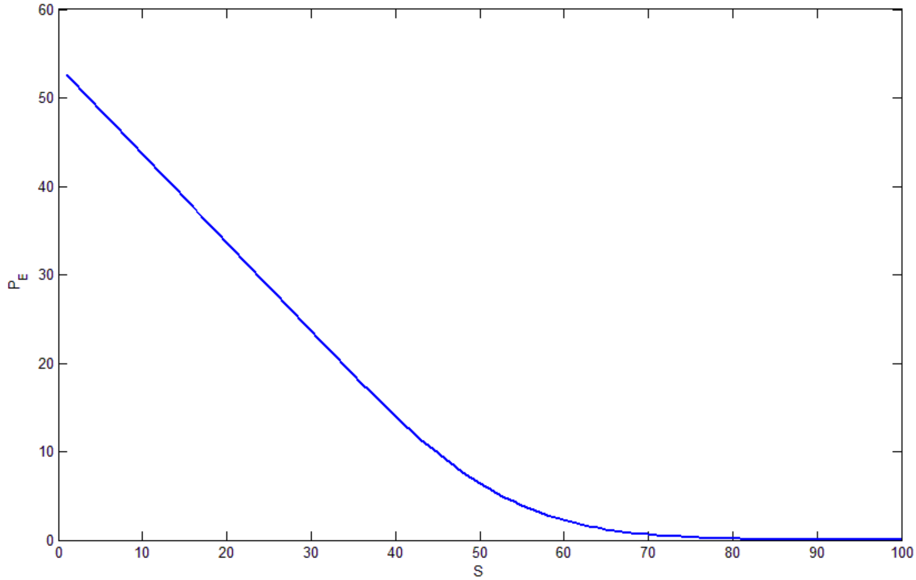


Figure III.33 European Put Price change as the spot varies. Input: S=1:100, K=55, T=0.5, r=5%, σ=30%

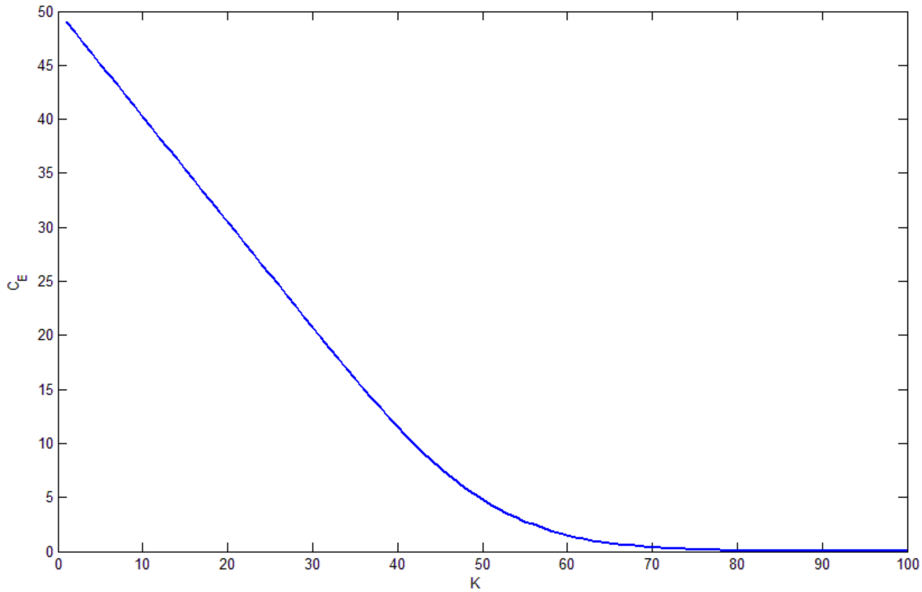


Figure III.34 European Call Price change as the strike varies. Input: S=50, K=1:100, T=0.5, r=5%, σ=30%

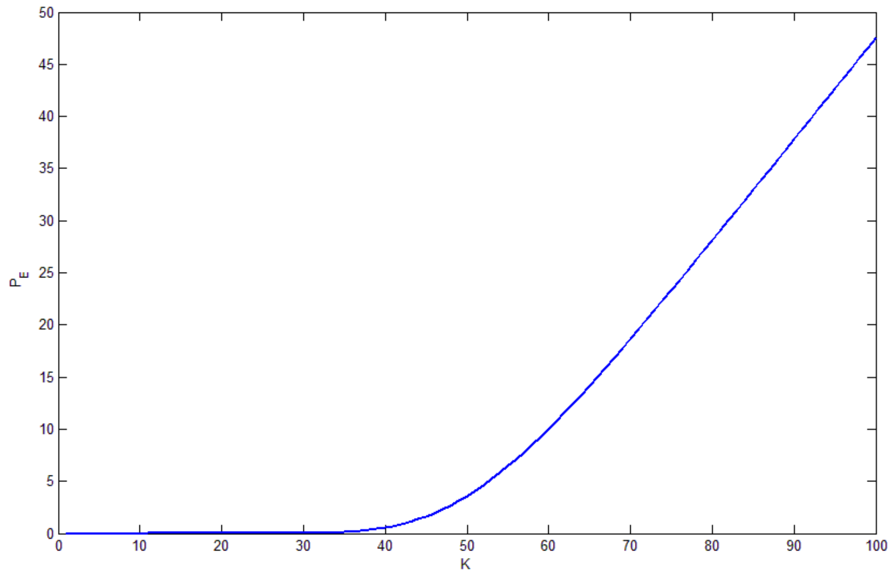


Figure III.35 European Put Price change as the strike varies. Input: S=50, K=1:100, T=0.5, r=5%, σ=30%

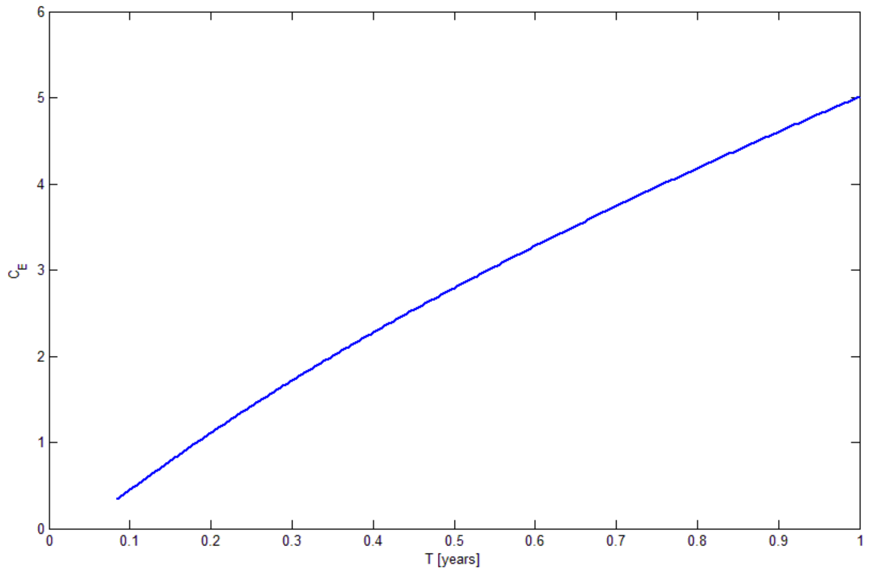


Figure III.36 European Call Price change as the Time to Maturity varies. Input: S=50, K=55, T =1 month to 1 year, r=5%, σ=30%

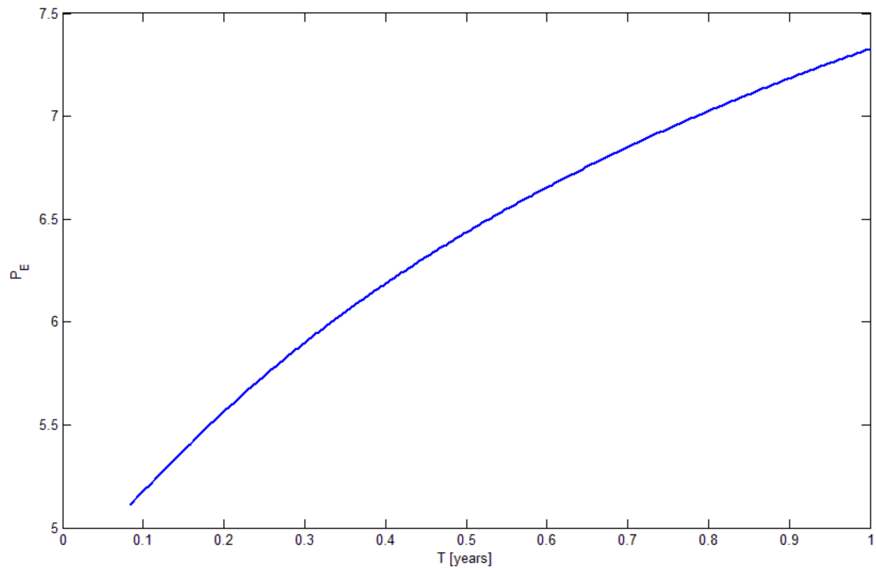


Figure III.37 European Put Price change as the Time to Maturity varies. Input: S=50, K=55, T =1 month to 1 year, r=5%, σ=30%

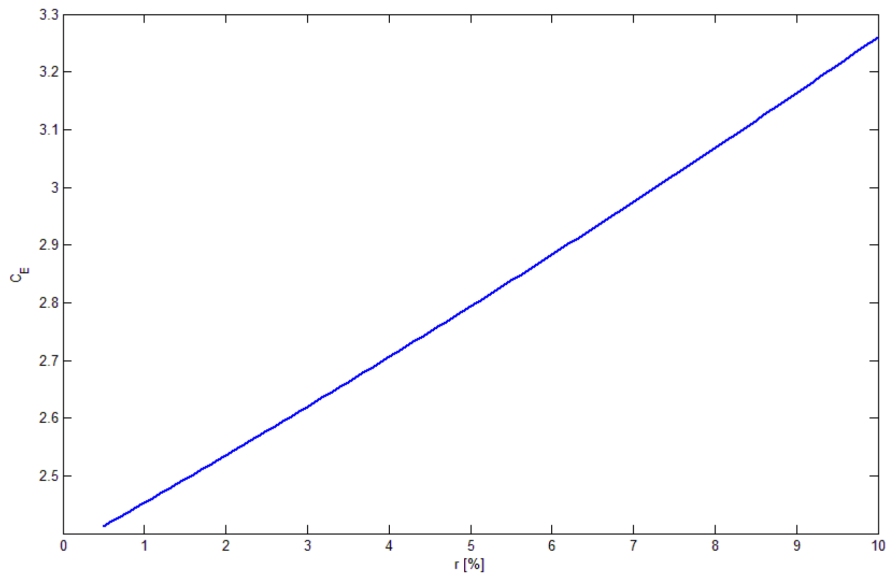


Figure III.38 European Call Price change as the risk-free rate varies. Model Input: S=50, K=55, T=0.5, r=0.5%:10%, σ=30%

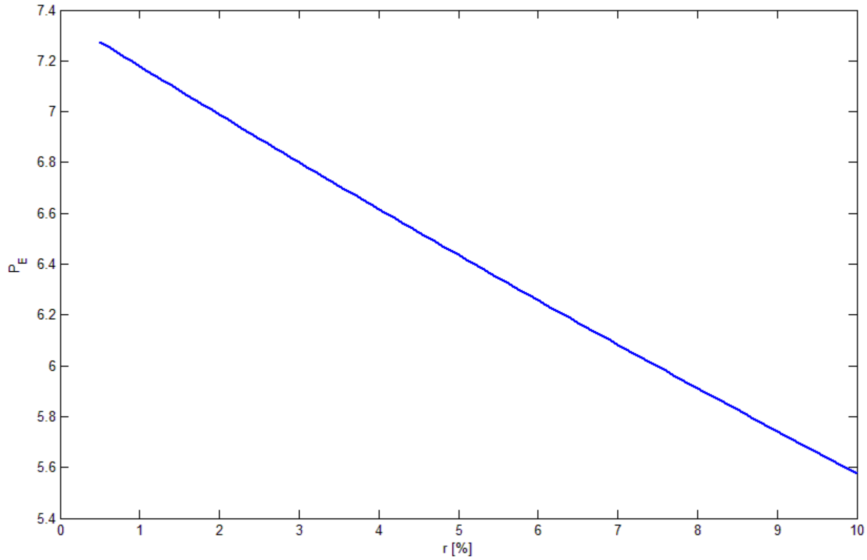


Figure III.39 European Put Price change as the risk-free rate varies. Model Input: S=50, K=55, T=0.5, r=0.5%:10%, σ=30%

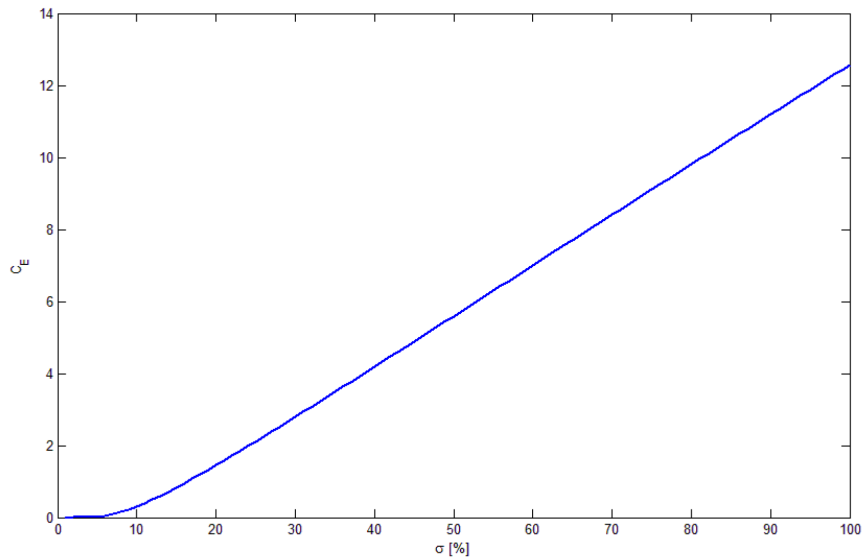


Figure III.40 European Call Price change as the volatility varies. Model Input: S=50, K=55, T=0.5, r=5%, σ=1%:100%

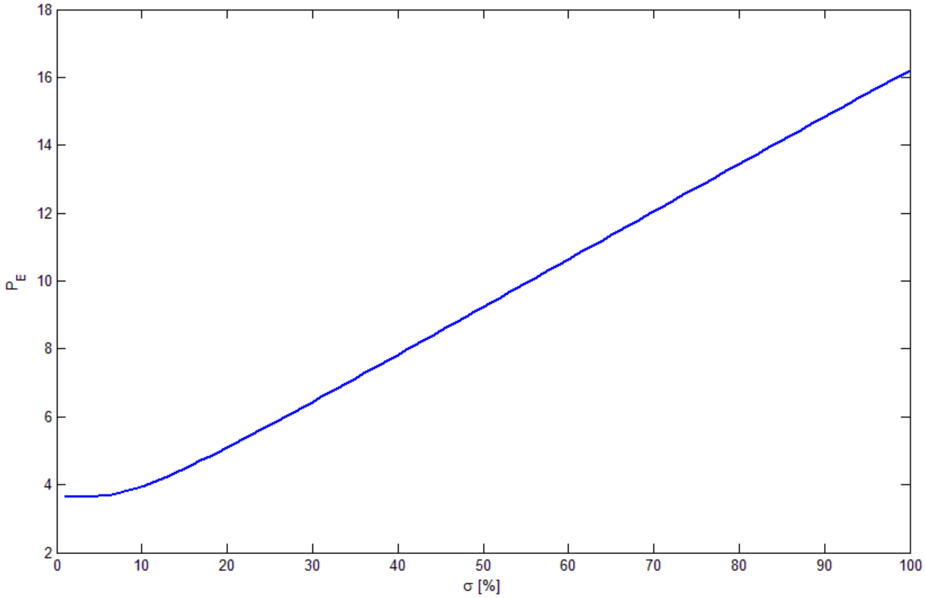


Figure III.41 European Put Price change as the volatility varies. Model Input: S=50, K=55, T=0.5, r=5%, σ=1%:100%

From the what-if analyses conducted in accordance with the Black-Scholes-Merton pricing framework, the following relationships between the variables can be deduced (ceteris paribus):

- As the price level of the underlying increases, the value of a European call increases, while the value of a put decreases.

- As the strike price increases, the value of a European call decreases, while the value of a put increases.

- As the volatility of the underlying on which a European option is written increases, its value increases.

- As the risk-free rate increases, the value of a European call increases, while the value of a put decreases.

The sensitivity analysis of the options premium with respect to a significant pricing parameter (for example: S, K, r, T, σ) can be managed in a more complete way by calculating the partial derivatives of the price with respect to one of the inputs.

These quantities representing the sensitivities are called "**Greeks**" in the financial jargon and here is a brief description of them:

Delta measures the sensitivity of the option price to a small change in the price level of the underlying:

$$\Delta = \frac{Change\ in\ value\ of\ option}{Change\ in\ value\ of\ underlying\ asset} \quad \textit{(Eq. III.65)}$$

Gamma measures the sensitivity of the Delta to a small change in the price level of the underlying:

$$\Gamma = \frac{Change\ in\ Delta}{Change\ in\ value\ of\ underlying\ asset} \quad \textit{(Eq. III.66)}$$

Theta measures the sensitivity of the option price with respect to the passage of time (i.e. it quantifies the temporal decay in the value of the option):

$$\Theta = \frac{Change\ in\ value\ of\ option}{Change\ in\ time} \quad \textit{(Eq. III.67)}$$

Rho measures the sensitivity of the option price to a small change in the interest rate:

$$\rho = \frac{Change\ in\ value\ of\ option}{Change\ in\ interest\ rates} \quad \textit{(Eq. III.68)}$$

Vega measures the sensitivity of the option price to a small change in the volatility of the underlying:

$$\vartheta = \frac{Change\ in\ value\ of\ option}{Change\ in\ volatility} \quad \textit{(Eq. III.69)}$$

Within the Black-Scholes pricing framework and for standard European options, the Greeks can be analytically calculated in a closed form by solving the partial derivatives of the price with respect to the reference risk parameter. Expressing the concept in mathematical terms, we obtain:

$$\Delta = \frac{\partial Price}{\partial S}, \Gamma = \frac{\partial^2 Price}{\partial S^2} = \frac{\partial \Delta}{\partial S}, \ \Theta = \frac{\partial Price}{\partial \tau}, \ \rho = \frac{\partial Price}{\partial r}, \vartheta = \frac{\partial Price}{\partial \sigma}$$

$$\Delta_{CALL} = \frac{\partial C_E}{\partial S} = N(d_1) > 0 \quad \textit{(Eq. III.70)}$$

$$\Delta_{\mathrm{PUT}} = \frac{\partial P_E}{\partial S} = N(d_1) - 1 < 0 \quad \textit{(Eq. III.71)}$$

The Delta of a call option is a positive number while the delta of a put option is a negative value, since, as already seen in the what-if analysis, the value of the put moves in the opposite direction compared to the price level of the underlying. Then the value of the delta depends on whether the option is in, at or out of the money:

- For in-the-money options, the delta approaches +1 for calls and -1 for puts.

- For at-the-money options, the delta approaches +0.5 for calls and -0.5 for puts.

- For out-of-the-money options, the delta approaches zero.

In the case of the base scenario (S=50, K=55, T=0.5, r=5%, σ=30%), we can estimate a $\Delta_{CALL} = 0.4108$ and a $\Delta_{PUT} = -0.5892$.

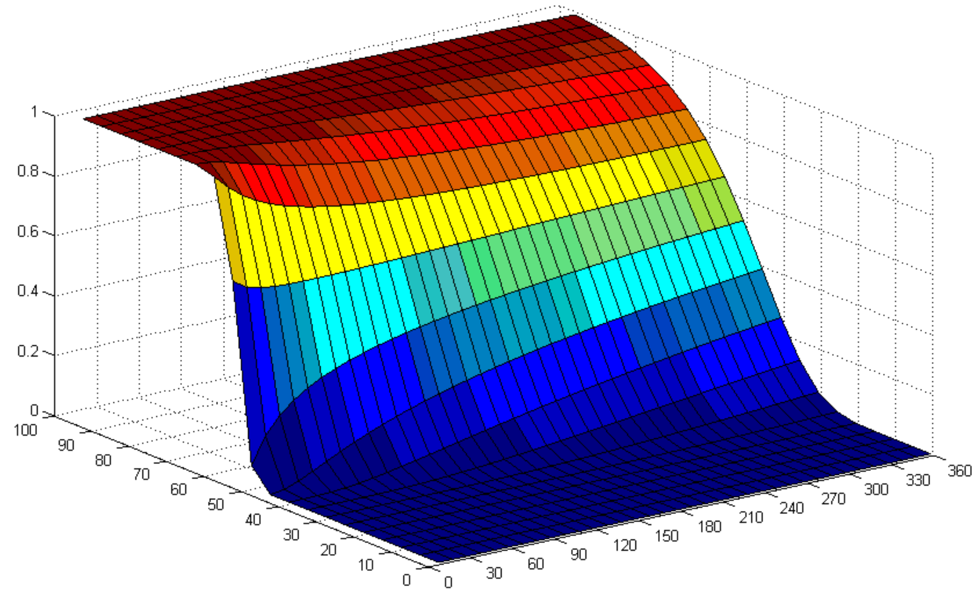


Figure III.42 Spot Delta Call Surface. Parameters: S=1:100, K=55, T=from 10 to 360 days, r=5%, σ=30%

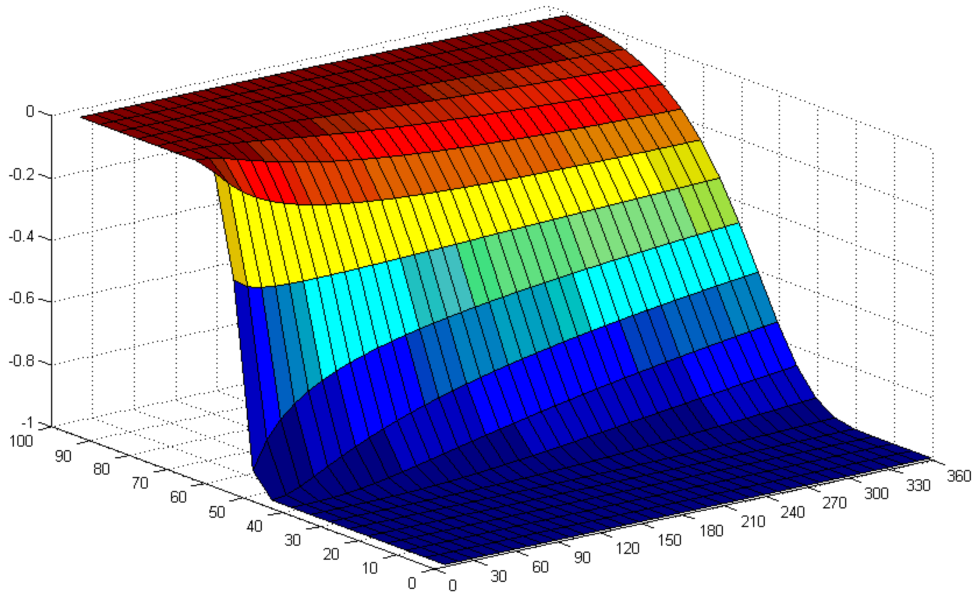


Figure III.43 Spot Delta Put Surface. Parameters: S=1:100, K=55, T=from 10 to 360 days, r=5%, σ=30%

As regards the Gamma, the following holds:

$$\Gamma_{CALL} = \Gamma_{PUT} = \frac{\partial \Delta_E^{CALL}}{\partial S} = \frac{\partial^2 C_E}{\partial S^2} = \frac{n(d_1)}{S\sigma\sqrt{\tau}} > 0 \quad \textit{(Eq. III.72)}$$

where:

$n(d_1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{d_1^2}{2}}$ is the standard normal probability distribution.

$\tau = T - t$ is the time to maturity, expressed in fractions of a year.

S is the spot price of the underlying on which the option is written.

σ is the annualized volatility of the underlying.

If the Gamma is small, the option's Delta will vary slowly; if the Gamma is large, the Delta will be very sensitive to changes in the underlying. Gamma is a fundamental measurement that traders usually observe to correctly rebalance a "**Delta neutral**" portfolio, i.e. a portfolio that does not have risk exposures to changes in the price levels of the underlying.

In the case of the base scenario (S=50, K=55, T=0.5, r=5%, σ=30%), the Gamma is $\Gamma_{CALL} = \Gamma_{PUT} = 0.0367$.

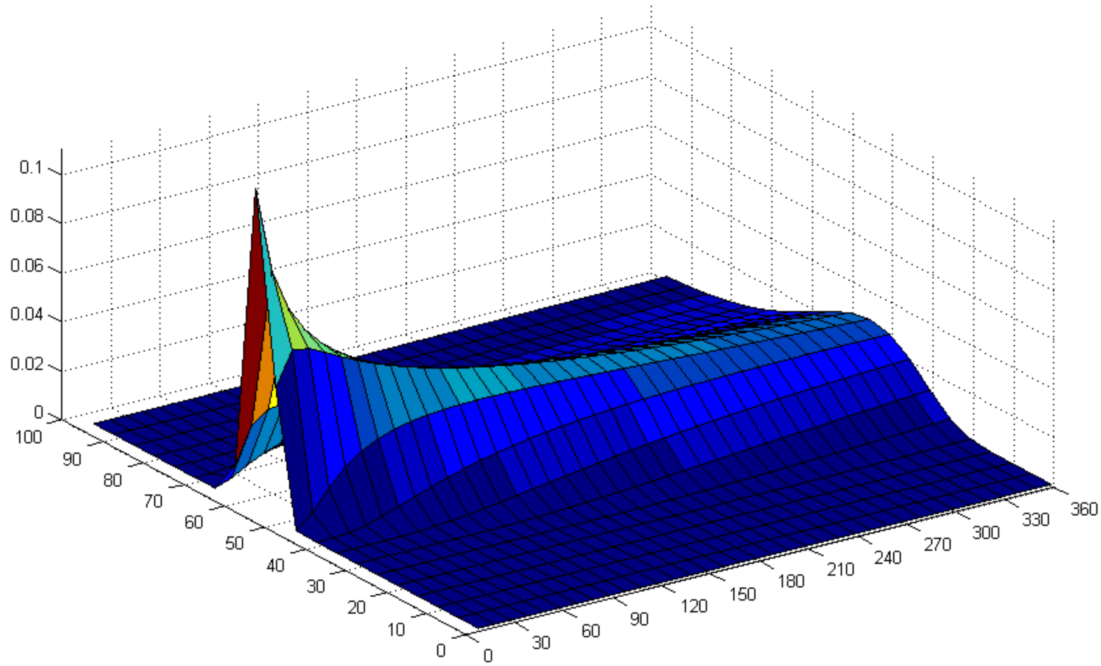


Figure III.44 Gamma Call/Put Surface. Parameters: S=1:100, K=55, T=from 10 to 360 days, r=5%, σ=30%

Here are the relations regarding Theta:

$\Theta_{CALL} = \frac{\partial C_E}{\partial t} = -\frac{S \cdot \sigma}{2 \cdot \sqrt{\tau}} \cdot n(d_1) - Ke^{-r\tau} \cdot r \cdot N(d_2)$ *(Eq. III.73)*

$\Theta_{PUT} = \frac{\partial P_E}{\partial t} = -\frac{S \cdot \sigma}{2 \cdot \sqrt{\tau}} \cdot n(d_1) - Ke^{-r\tau} \cdot r \cdot [N(-d_2)]$ *(Eq. III.74)*

$n(d_1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{d_1^2}{2}}$ is the standard normal probability distribution.

$N(d_2)$ is the standard normal cumulative probability distribution.

$\tau = T - t$ is the time to maturity, expressed in year fractions.

It is a good and common practice to divide the Theta by 360 (or 365, depending on the adopted day basis convention) with the aim of expressing the time decay of the option price for one day.

In the case of the base scenario presented above (S=50, K=55, T=0.5, r=5%, σ=30%), we obtain:

$\Theta_{CALL} = -\frac{5.0127}{360} = -0.0139$; $\Theta_{PUT} = -\frac{2.3306}{360} = -0.0065$

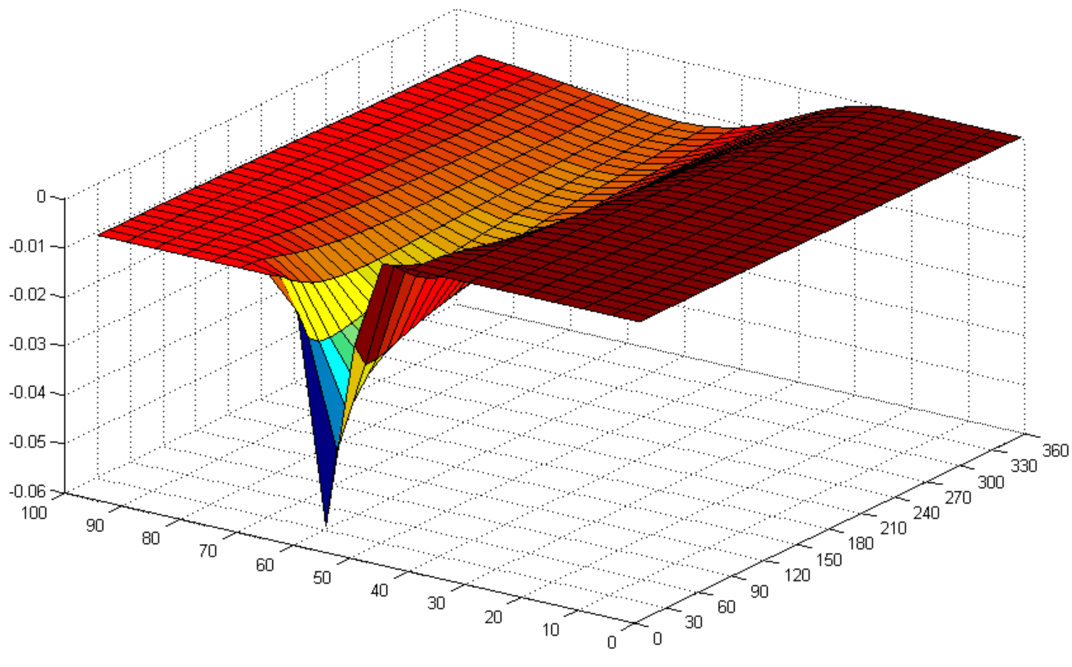


Figure III.45 Theta Call Surface (one-day time decay). Parameters: S=1:100, K=55, T=from 10 to 360 days, r=5%, σ=30%

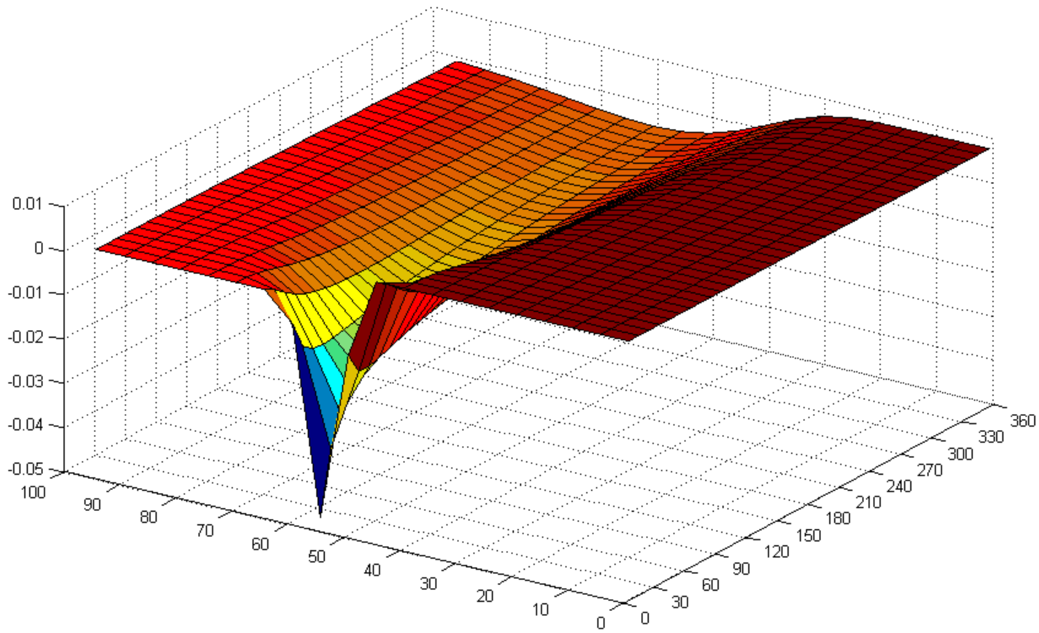


Figure III.46 Theta Put Surface (one-day time decay). Parameters: S=1:100, K=55, T=from 10 to 360 days, r=5%, σ=30%

We now present the relations valid for Rho:

$$\rho_{CALL} = \frac{\partial C_E}{\partial r} = \tau \cdot K \cdot e^{-r\tau} \cdot N(d_2) > 0 \text{ (Eq. III.75)}$$

$$\rho_{PUT} = \frac{\partial P_E}{\partial r} = \tau \cdot K \cdot e^{-r\tau} \cdot [N(d_2) - 1] < 0 \text{ (Eq. III.76)}$$

In the case of the base scenario (S=50, K=55, T=0.5, r=5%, σ=30%), we obtain: $\rho_{CALL} = +8.8743$ and $\rho_{PUT} = -17.9467$.

Lastly, the relations that hold for Vega are the following:

$$\vartheta_{CALL} = \frac{\partial C_E}{\partial \sigma} = S \cdot \sqrt{\tau} \cdot n(d_1) > 0 \text{ (Eq. III.77)}$$

$$\vartheta_{PUT} = \frac{\partial P_E}{\partial \sigma} = S \cdot \sqrt{\tau} \cdot n(d_1) = \vartheta_{CALL} > 0 \text{ (Eq. III.78)}$$

The price of an option (call/put) is an increasing function as the volatility of the underlying asset increases.

In the case of the base scenario (S=50, K=55, T=0.5, r=5%, σ=30%), we have $\vartheta_{CALL} = \vartheta_{PUT} = +13.7510$.

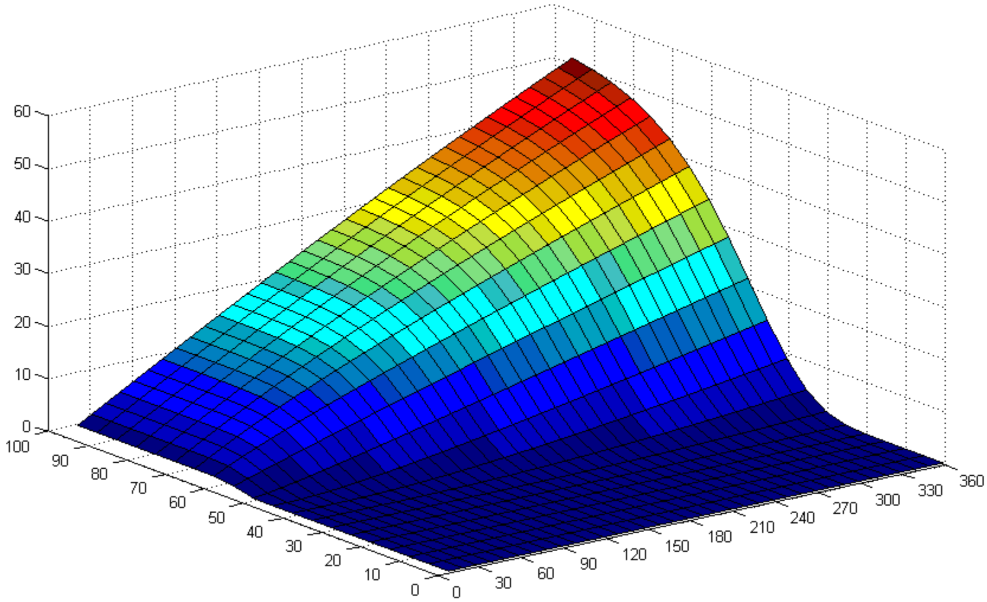


Figure III.47 Rho Call Surface. Parameters: S=1:100, K=55, T=from 10 to 360 days, r=5%, σ=30%

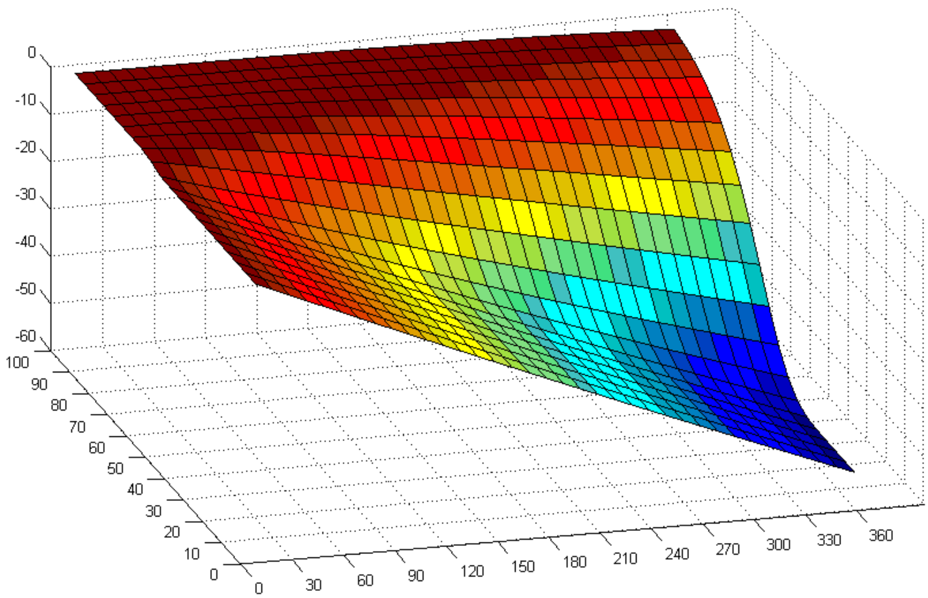


Figure III.48 Rho Put Surface. Parameters: S=1:100, K=55, T=from 10 to 360 days, r=5%, σ=30%

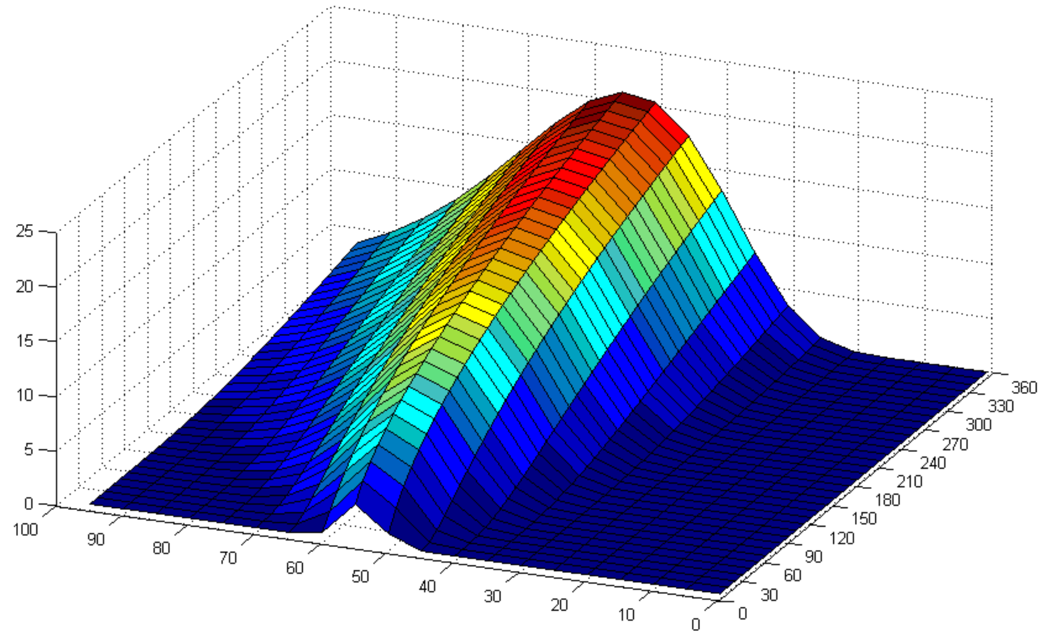


Figure III.49 Vega Call/Put Surface. Parameters: S=1:100, K=55, T=from 10 to 360 days, r=5%, σ=30%

The previous formulas can be "generalized" introducing the **cost-of-carry** parameter, b. Depending on the value assumed by this parameter, the following formula (known in the literature as the Generalized BSM formula) can be used for pricing European plain-vanilla call and put options written on different kinds of underlyings:

$$C_E = S \cdot e^{(b-r)\tau} \cdot N(d_1) - K \cdot e^{-r\tau} \cdot N(d_2) \text{ (Eq. III.79)}$$

$$P_E = K \cdot e^{-r\tau} \cdot N(-d_2) - S \cdot e^{(b-r)\tau} \cdot N(-d_1) \text{ (Eq. III.80)}$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (b + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

In this context:

- If $b = r$, the GBSM formula returns the classic Black-Scholes 73 formula suitable for pricing options written on shares that do not pay a dividend.

- If $b = r - q$, the GBSM formula returns the Black-Scholes formula used for pricing options on shares/indexes characterized by a continuous dividend yield q.

- If $b = 0$, the GBSM formula returns the formula suitable for the pricing of options on futures.

- If $b = r - r_{FOR}$, the GBSM formula can be traced back to the Garman-Kohlhagen framework.

The table below represents a summary of the main Greeks described above.

Greeks	Call option	Put Option
Delta	$e^{(b-r)\tau} \cdot N(d_1) > 0$	$e^{(b-r)\tau} \cdot [N(d_1) - 1] < 0$
Gamma	$\frac{n(d_1) \cdot e^{(b-r)\tau}}{S\sigma\sqrt{\tau}} > 0$	
Theta	$-\frac{S \cdot e^{(b-r)\tau} \cdot \sigma \cdot n(d_1)}{2 \cdot \sqrt{\tau}} - Ke^{-r\tau} \cdot r \cdot N(d_2) +$ $-(b-r) \cdot S \cdot e^{(b-r)\tau} N(d_1)$	$-\frac{S \cdot e^{(b-r)\tau} \cdot \sigma \cdot n(d_1)}{2 \cdot \sqrt{\tau}} + Ke^{-r\tau} \cdot r \cdot N(-d_2) +$ $+(b-r) \cdot S \cdot e^{(b-r)\tau} N(-d_1)$
Rho	$\tau \cdot K \cdot e^{-r\tau} \cdot N(d_2) > 0$	$-\tau \cdot K \cdot e^{-r\tau} \cdot N(-d_2) < 0$
Vega	$S \cdot e^{(b-r)\tau} n(d_1)\sqrt{\tau} > 0$	

Table III.14 The main generalized Black-Scholes Greeks

The problem of pricing vanilla options characterized by early exercise features or derivatives with highly non-linear payoffs, in jargon called "**exotic options**", cannot be solved using an exact closed-form formula. For such cases, we have to implement numerical methodologies such as Stochastic Trees, Finite Difference Method (FDM) applied to Partial Differential Equation (PDE) or Monte Carlo simulations that provide a solution to the problem using approximation schemes. On the other hand, if the Greeks of the financial derivatives are provided/computed, an approximated valuation of the instrument can be implemented, since Greeks provide a good approximation of the option price, given a **small change** in the risk parameter.

Using the first-order derivatives of the option price with respect to the risk parameter considered (for example: Delta, Vega, Rho and Theta), the following formula can be used to obtain a linear approximation:

Approx. New Price = Old Price + First Order Greek x (parameter shock) *(Eq. III.81)*

In order to incorporate the second order term (called convexity) we can use the more accurate formula:

Approx. New Price = Old Price + First Order Greek x (parameter shock) +

+ 0.5 x Second Order Greek x (parameter shock)² *(Eq. III.82)*

Obviously, the greater the shock applied to the reference parameter, the less the approximation will be precise. Let us now examine an example of an approximation of the price of a European call as the underlying on which it is written varies. The Greeks to be estimated are therefore Delta (for a linear approximation of the first order) and Gamma (for a convex approximation of the second order).

For this Delta-Gamma approximation, let us consider the base scenario presented above for a European call: S=50, K=55, T=0.5, r=5%, σ=30%. The key parameters for estimating the approximated price as the underlying varies are:

$$\Delta_{CALL} = 0.4108, \Gamma_{CALL} = 0.0367, C_E = 2.7935$$

Starting from the approximation provided by the partial derivatives, we want to estimate a reasonable price of the call option assuming a change in the reference risk parameter of + EUR 1:

Approx. New Price = Old Price + First Order Greek x (parameter shock)

Approx. New Price = 2.7935 + 0.4108 x (51-50) = EUR 3.2043

The new exact price of the European call option calculated from the BS formula would be EUR 3.2227.

Using the first-order approximation provided by the Delta, an estimation error of less than 2 cents would be committed. In order to make an even more precise estimate, convexity, i.e. the second order partial derivative, can be added to the linear approximation.

Approx. New Price = Old Price + First Order Greek x (parameter shock) +

+ 0.5 x Second Order Greek x (parameter shock)²

Approx. New Price = 2.7935 + 0.4108 x (51-50) + 0.5 x 0.0367 x (51-50)² = EUR 3.2227

Implementing the second-order approximation provided by the Delta-Gamma, an estimation error would be committed to the fourth decimal digit. As the bump applied to the price level of the underlying increases, the error in approximation of the option premium increases. The following graphs compare the exact value of the derivative, estimated starting from the BS closed formula, with the one approximated by Delta and Delta-Gamma, as the magnitude of the shock applied to S varies. The baseline scenario has been reported in bold, and the values in red do not represent a feasible approximated price for the call option.

S	BS formula [A]	Shock	Delta Approx. [B]	First order error [A-B]	Delta-Gamma Approx [C]	Second order error [A-C]
40	0.372246642	-10	-1.314453107	ND	0.520546893	-0.148300251
41	0.483828236	-9	-0.903653107	ND	0.582696893	-0.098868657
42	0.618783998	-8	-0.492853107	ND	0.681546893	-0.062762894
43	0.779601982	-7	-0.082053107	ND	0.817096893	-0.03749491
44	0.968614938	-6	0.328746893	0.639868045	0.989346893	-0.020731955
45	1.187943734	-5	0.739546893	0.448396842	1.198296893	-0.010353158
46	1.439449309	-4	1.150346893	0.289102416	1.443946893	-0.004497584
47	1.724694737	-3	1.561146893	0.163547844	1.726296893	-0.001602156
48	2.044918348	-2	1.971946893	0.072971455	2.045346893	-0.000428545
49	2.401018105	-1	2.382746893	0.018271212	2.401096893	-7.8788E-05
50	**2.793546893**	**0**	**2.793546893**	**0**	**2.793546893**	**0**
51	3.222717852	1	3.204346893	0.018370959	3.222696893	2.09594E-05
52	3.688418547	2	3.615146893	0.073271654	3.688546893	-0.000128346
53	4.190232498	3	4.025946893	0.164285605	4.191096893	-0.000864395
54	4.727466506	4	4.436746893	0.290719613	4.730346893	-0.002880387
55	5.299182146	5	4.847546893	0.451635253	5.306296893	-0.007114747
56	5.904229897	6	5.258346893	0.645883004	5.918946893	-0.014716996
57	6.541284498	7	5.669146893	0.872137606	6.568296893	-0.027012394
58	7.208880275	8	6.079946893	1.128933382	7.254346893	-0.045466618
59	7.905445404	9	6.490746893	1.414698511	7.977096893	-0.071651489
60	8.629334285	10	6.901546893	1.727787392	8.736546893	-0.107212608

Table III.15 Delta vs Delta-Gamma price approximation

In Figure III.50 below, the black price curve is the one related to the exact valuation of the European call recalculated using the BS formula; while the red line is its approximation using the Delta.

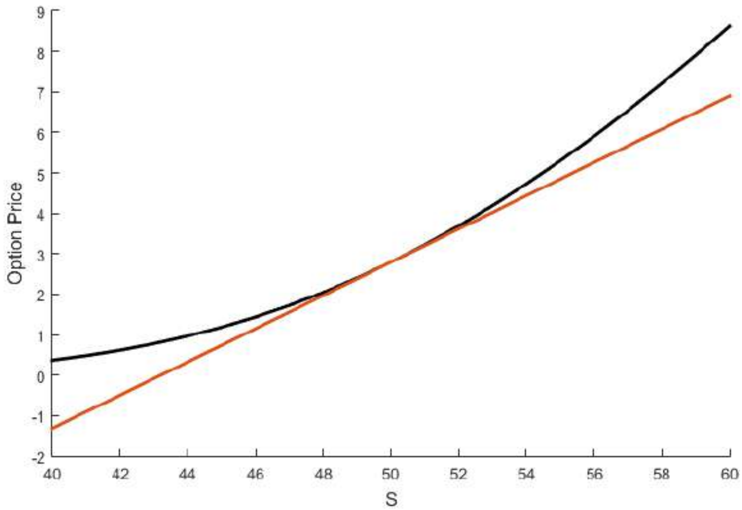


Figure III.50 Approximation of the price using Greeks: first-order approximation

In Figure III.51 below, the black price curve has been computed using the exact European call option BS formula; while the green one constitutes its approximation carried out using the Delta-Gamma Greeks.

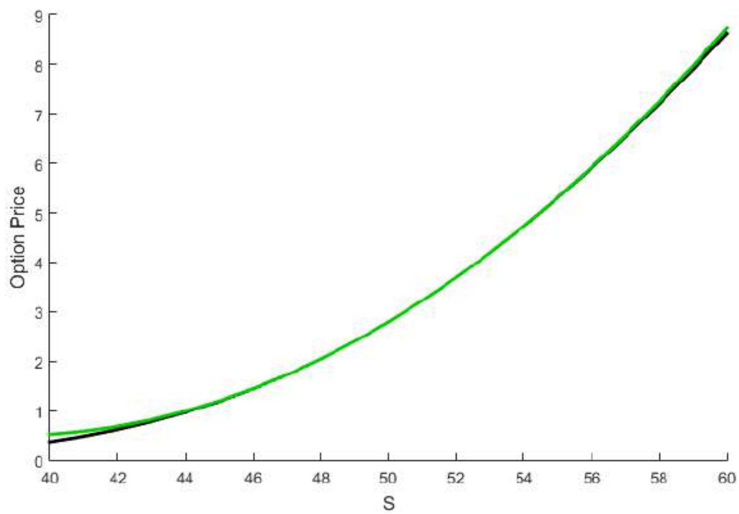


Figure III.51 Approximation of the price using Greeks: second-order approximation

In Figure III.52 below, the red line represents the Delta approximation, the green line shows the Delta-Gamma approximation and the black line shows the BS formula.

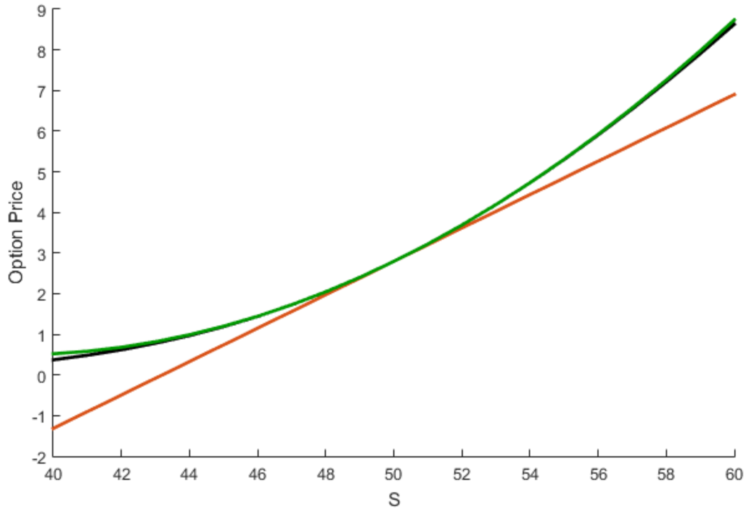


Figure III.52 Approximation of the price using Greeks: comparison

In analogy with what was discussed for the Delta and Gamma, the other Greeks can also be used to obtain linear approximations of the price, as the reference risk parameter varies. This will be shown through examples for the three remaining Greeks.

Example for Theta: taking the base scenario as a reference, we calculate the approximation of the fair value of the option using Theta for a decrease in the time to maturity of 5 days.

Approx. New Price = Old Price + First Order Greek x (parameter shock)

$\Theta_{CALL} = -0.0139$ $C_E = 2.7935$, T varies from 0.5 to 0.48611 years (parameter shock is 5/360)

Approximated New Price = 2.7935 – 0.0139 x (0.01318889) = 2.793317 vs Exact New Price = 2.7236

Example for Vega: we want to calculate the variation in the price of the call option, if volatility increases by 1%.

Approximated New Price = 2.7935 + 13.751 x (0.01) = 2.93101 vs Exact New Price = 2.9313

Example for Rho: lastly, we calculate the variation in the price of the call option, if the interest rate decreases by 30bps.

Approximated New Price = 2.7935 + 8.8743 x (0.003) = 2.8201 vs Exact Price = 2.8203.

FURTHER READINGS

Black F., Scholes M. – "The Pricing of Options and Corporate Liabilities" – Journal of Political Economy Vol. 81, N. 3 (1973) .

Giribone P. G. – "Derivazione analitica delle formule chiuse per il calcolo delle greche nelle opzioni plain vanilla" – Seminario tenuto ad Ingegneria Gestionale (2018).

Giribone P. G. – "Derivazione analitica della formula di Black & Scholes" – Seminario tenuto ad Ingegneria Gestionale (2014).

Haug E. G. – "The Complete Guide to Option Pricing Formulas" – McGraw Hill (2007).

Hull J. – "Options, Futures and other derivatives" – Pearson (2014).

III.3 OPTION STRATEGIES

We have analyzed the profit profiles originating with a single European option, now we consider the patterns that result when an option is traded together with other financial instruments. In particular, we examine the properties of portfolios having the following positions:

- An option and a zero-coupon bond.
- An option and the asset underlying the option.
- Two or more options written on the same asset.

Options are often used to create principal-protected notes for the retail market sector, which is primarily composed of very prudent investors. A typical feature of these financial products is that the financial return earned by an investor depends on the performance of a risky asset (shares, indices, exchange rates) but the initial amount invested is not at risk. A **principal-protected note** can be synthetized as a European call option plus a zero-coupon bond.

Let us examine an example, assuming that the continuously compounded 3-year interest rate is 5%. This means that $1000\,\exp(-0.05 \cdot 3) = \text{EUR}\,860.708$ will grow to EUR 1,000 in three years. The difference between EUR 1,000 and EUR 860.708 is EUR 139.292.

Now let us suppose a customer has a portfolio of shares with a value of EUR 1,000 that provides a return of 1.5% per annum. Let us then suppose that a European at-the-money call option (strike equal to EUR 1,000) on this stock portfolio can be bought for less than EUR 139.292. A bank can offer the customer the opportunity to invest EUR 1,000 in the following strategy:

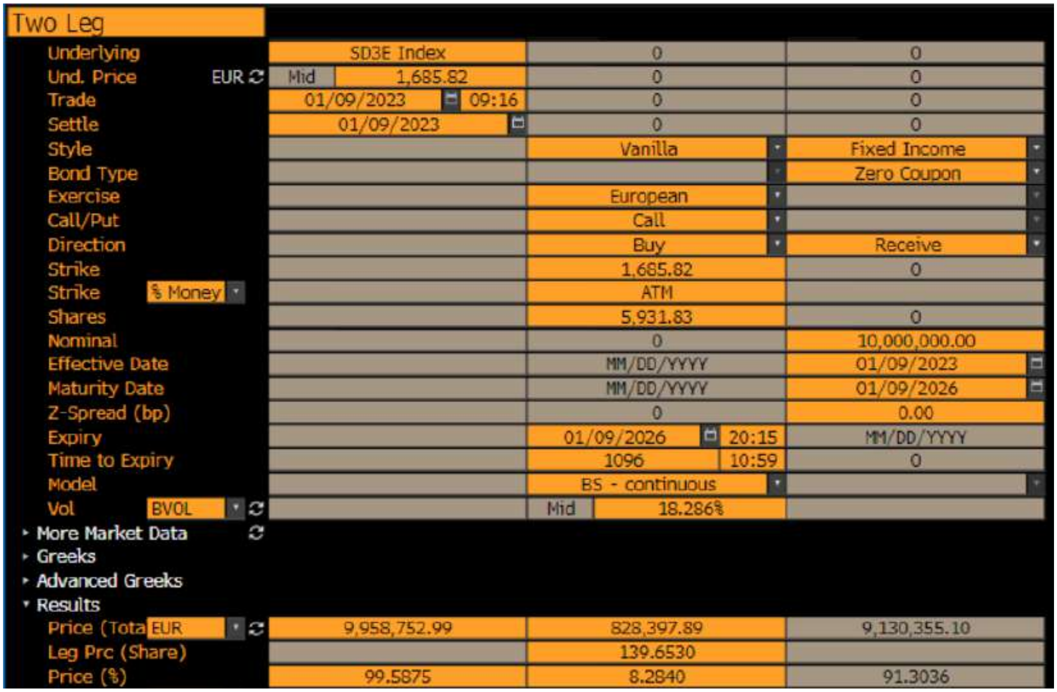
- A zero coupon bond with a principal of EUR 1,000.
- An at-the-money call option written on the stock portfolio.

If the value of the portfolio increases, the investor receives the interest accrued from the zero coupon and the yield given by the option. If the call option has no value, the investor still receives the initial investment amount (EUR 1,000).

Let us now present a real market case on principal protected notes. In this case, the note starts on 9th January 2023 and expires after three years, at par. No coupons are paid during its life. The advantage for the holder is to link the final pay-off of the structured note to the performance of the index.
In the worst case, the holder does not receive any remuneration from the equity market and receives the initial amount of his investment (ATM Long call). The main data are shown in the below table:

Long Call Price: 8.2840%
Zero Coupon Bond: 91.3036%

Total structured notes: 99.5875%



Two Leg			
Underlying	SD3E Index	0	0
Und. Price EUR	Mid 1,685.82	0	0
Trade	01/09/2023 09:16	0	0
Settle	01/09/2023	0	0
Style		Vanilla	Fixed Income
Bond Type			Zero Coupon
Exercise		European	
Call/Put		Call	
Direction		Buy	Receive
Strike		1,685.82	0
Strike % Money		ATM	
Shares		5,931.83	0
Nominal		0	10,000,000.00
Effective Date		MM/DD/YYYY	01/09/2023
Maturity Date		MM/DD/YYYY	01/09/2026
Z-Spread (bp)		0	0.00
Expiry		01/09/2026 20:15	MM/DD/YYYY
Time to Expiry		1096 10:59	0
Model		BS - continuous	
Vol BVOL		Mid 18.286%	
▸ More Market Data			
▸ Greeks			
▸ Advanced Greeks			
▾ Results			
Price (Tota EUR	9,958,752.99	828,397.89	9,130,355.10
Leg Prc (Share)		139.6530	
Price (%)	99.5875	8.2840	91.3036

Figure III.53 Principal-protected notes. Source: Bloomberg®

It is assumed that the underlying asset of the option is a stock, Although very similar trading strategies can also be developed for different underlyings.
The profit deriving from the strategy will be calculated as the final pay-off minus the initial costs, without considering the discount factor.

There are in fact different trading strategies that can be formed by investing in an option written on the underlying and the underlying itself. Some of the most popular profit patterns include:

(a) Long position in a stock + Short position in a call.
(b) Short position in a stock + Long position in a call.
(c) Long position in a put + Long position in a stock.
(d) Short position in a put + Short position in a stock.

In the following figures, the dotted line shows the relationship between the profit and the share price level, for the single security included in the portfolio, while the solid line shows the relationship between the profit and the share price for the entire portfolio.

Let us analyze the four cases:

(a) Long position in a stock + Short position in a call

The portfolio consists of a long position in the stock and a short position in a European call option written on such stock. This strategy is known in the literature as **writing a covered call**. The long position on the stock hedges the investor from the characteristic pay-off of the short position in the call, in case of a sharp rise in the level of the share price at maturity.

We illustrate this case with an example:

Call Strike Price = 50

Call Premium = 1

Initial value of the stock S(0) = 48

S(T)	long stock	short call	initial premium	final pay-off
35	-13	0	1	-12
36	-12	0	1	-11
37	-11	0	1	-10
38	-10	0	1	-9
39	-9	0	1	-8
40	-8	0	1	-7
41	-7	0	1	-6
42	-6	0	1	-5
43	-5	0	1	-4
44	-4	0	1	-3
45	-3	0	1	-2
46	-2	0	1	-1
47	-1	0	1	0
48	0	0	1	1
49	1	0	1	2
50	2	0	1	3
51	3	-1	1	3
52	4	-2	1	3
53	5	-3	1	3
54	6	-4	1	3

Table III.16 Covered call strategy

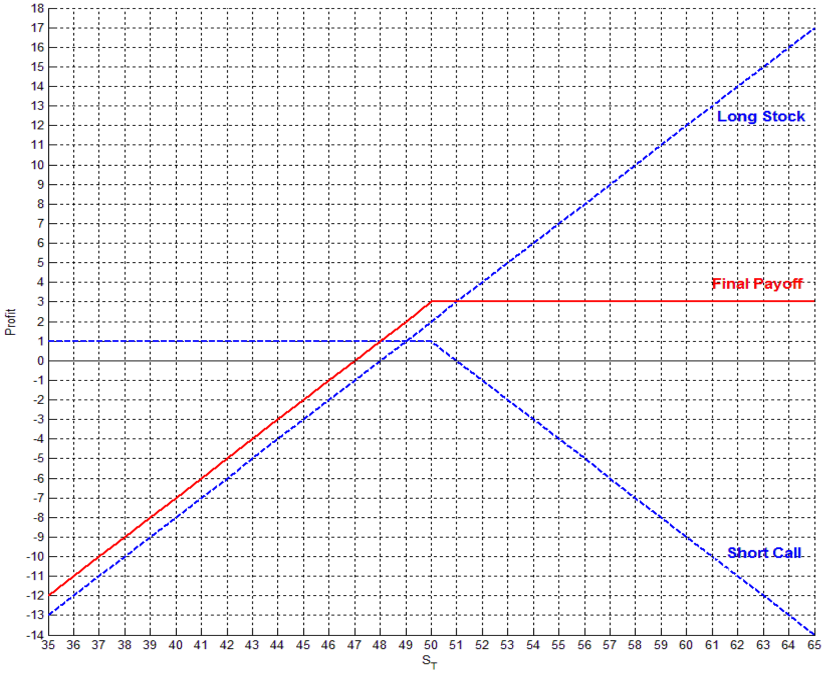


Figure III.54 Writing a Covered Call strategy

Figure III.55 Writing a Covered Call strategy. Source: Bloomberg®

Figure III.56 Writing a Covered Call strategy: Terminal pay-off. Source: Bloomberg®

(b) Short position in a stock + Long position in a call

The portfolio consists of a short position in the stock and a long position in a European call option written on such stock. This strategy is known in the literature as the reverse of writing a covered call. The long position in the call hedges the investor from the characteristic pay-off of the short position on the stock, in case of a sharp rise in the level of the stock price at maturity.

Let us make an example of this strategy:

Call Strike Price = 50; Initial investment for the Call = 1 and Initial value of the stock S(0) = 48.

S(T)	short stock	long call	initial premium	final pay-off
43	5	0	-1	4
44	4	0	-1	3
45	3	0	-1	2
46	2	0	-1	1
47	1	0	-1	0
48	0	0	-1	-1
49	-1	0	-1	-2
50	-2	0	-1	-3
51	-3	1	-1	-3
52	-4	2	-1	-3
53	-5	3	-1	-3

Table III.17 Reverse of writing a Covered call

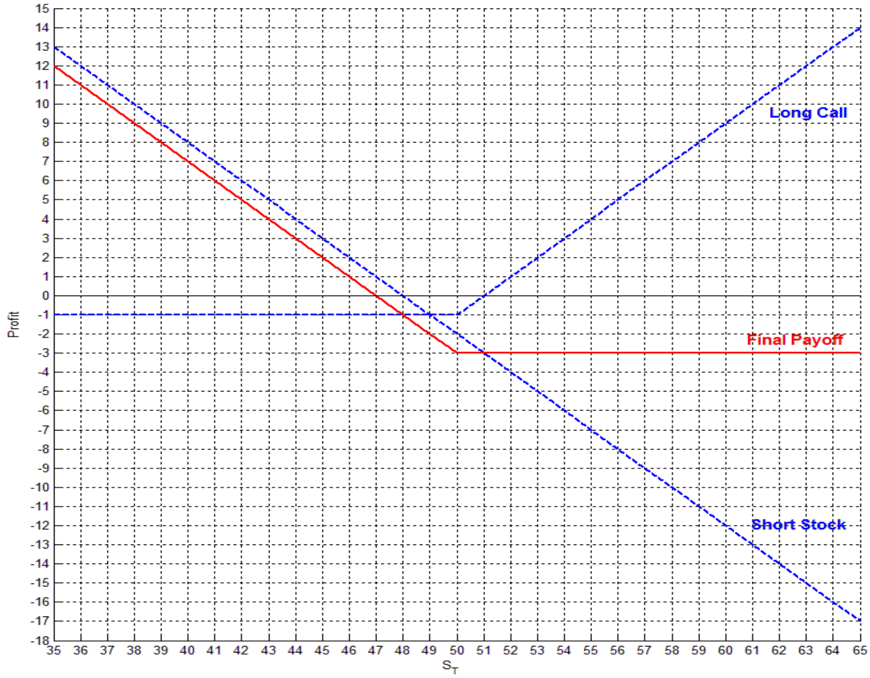
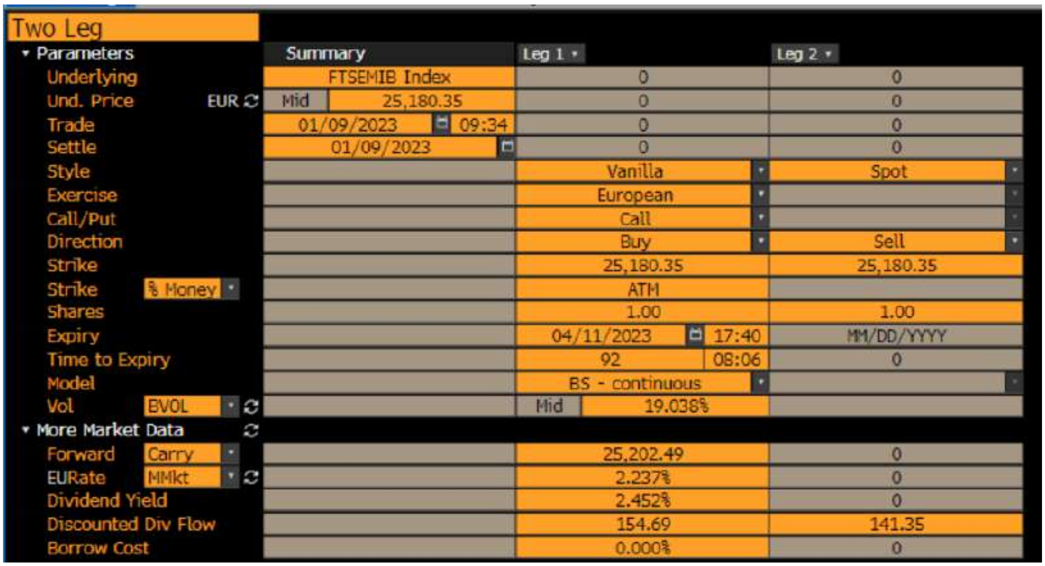


Figure III.57 Reverse of writing a Covered Call strategy



Two Leg	Summary	Leg 1	Leg 2
Parameters			
Underlying	FTSEMIB Index	0	0
Und. Price EUR	Mid 25,180.35	0	0
Trade	01/09/2023 09:34	0	0
Settle	01/09/2023	0	0
Style		Vanilla	Spot
Exercise		European	
Call/Put		Call	
Direction		Buy	Sell
Strike		25,180.35	25,180.35
Strike % Money		ATM	
Shares		1.00	1.00
Expiry		04/11/2023 17:40	MM/DD/YYYY
Time to Expiry		92 08:06	0
Model		BS - continuous	
Vol BVOL		Mid 19.038%	
More Market Data			
Forward Carry		25,202.49	0
EURate MMkt		2.237%	0
Dividend Yield		2.452%	0
Discounted Div Flow		154.69	141.35
Borrow Cost		0.000%	0

Figure III.58 Reverse of writing a Covered Call strategy. Source: Bloomberg®

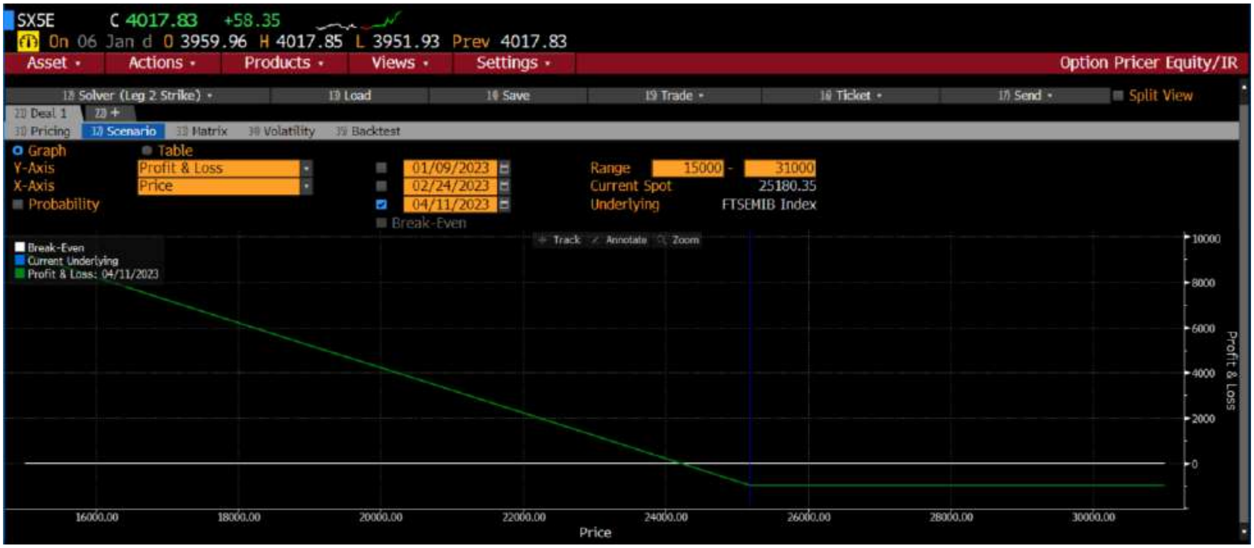


Figure III.59 Reverse of Writing a Covered Call strategy: Terminal pay-off. Source: Bloomberg®

(c) Long position in a stock + Long position in a put

The portfolio consists of a long position in the stock and a long position in a European put option written on such stock. This strategy is known in literature with the name of protective put. The long position in the put hedges the investor from the characteristic pay-off of the long position on the stock, in the event of a sharp decline in the level of the share price at maturity.

Let us present an example:

Put Strike Price = 50; Initial investment for the Put = 4; Initial value of the stock S(0) = 48.

S(T)	long stock	long put	initial premium	final pay-off
48	0	2	-4	-2
49	1	1	-4	-2
50	2	0	-4	-2
51	3	0	-4	-1
52	4	0	-4	0
53	5	0	-4	1
54	6	0	-4	2
55	7	0	-4	3
56	8	0	-4	4
57	9	0	-4	5

Table III.18 Protective Put strategy

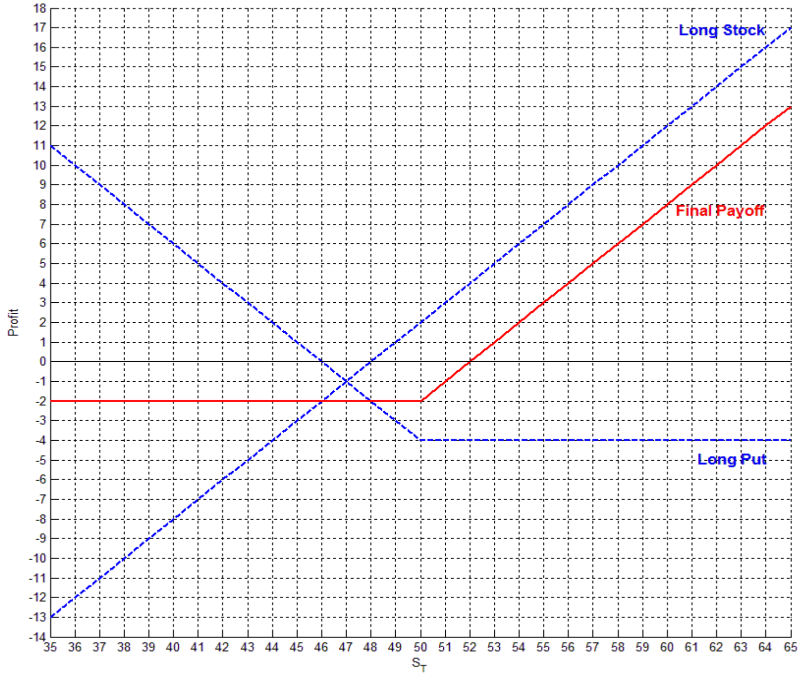


Figure III.60 Protective put strategy

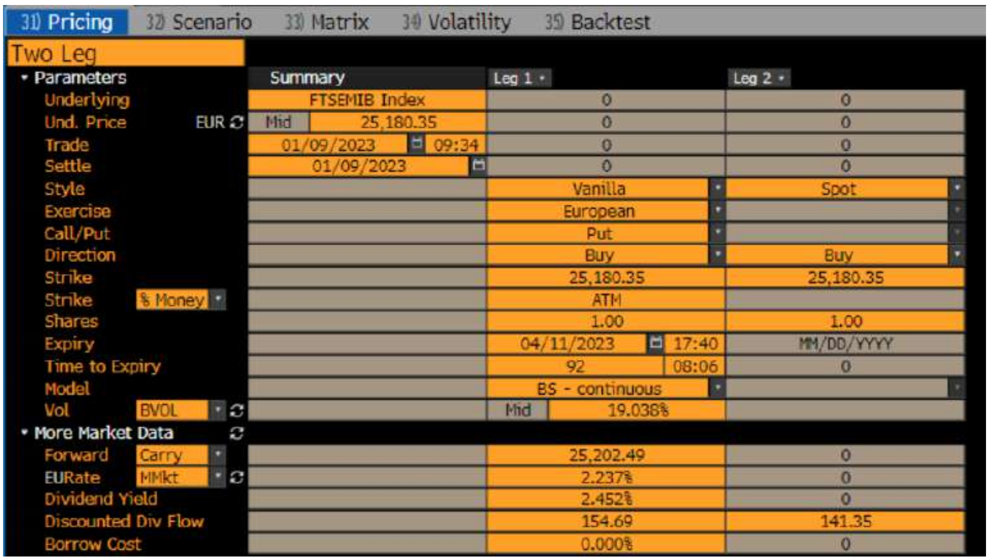


Figure III.61 Protective Put strategy. Source: Bloomberg®

Figure III.62 Protective Put strategy: Terminal pay-off. Source: Bloomberg®

(d) Short position in a stock + Short position in a put

The portfolio consists of a short position in the stock and a short position in a European put option written on the stock itself. This strategy is known in literature with the name of reverse of protective put. The short position in the stock hedges the investor from the characteristic pay-off of the short position on the put, in case of a sharp fall in the level of the share price at maturity.

Let us examine an example of this strategy:

Put Strike Price = 50; Initial Put Premium = 4 and Initial value of the stock S(0) = 48.

S(T)	short stock	short put	initial premium	final pay-off
48	0	-2	4	2
49	-1	-1	4	2
50	-2	0	4	2
51	-3	0	4	1
52	-4	0	4	0
53	-5	0	4	-1
54	-6	0	4	-2
55	-7	0	4	-3
56	-8	0	4	-4
57	-9	0	4	-5

Table III.19 Reverse of Protective Put

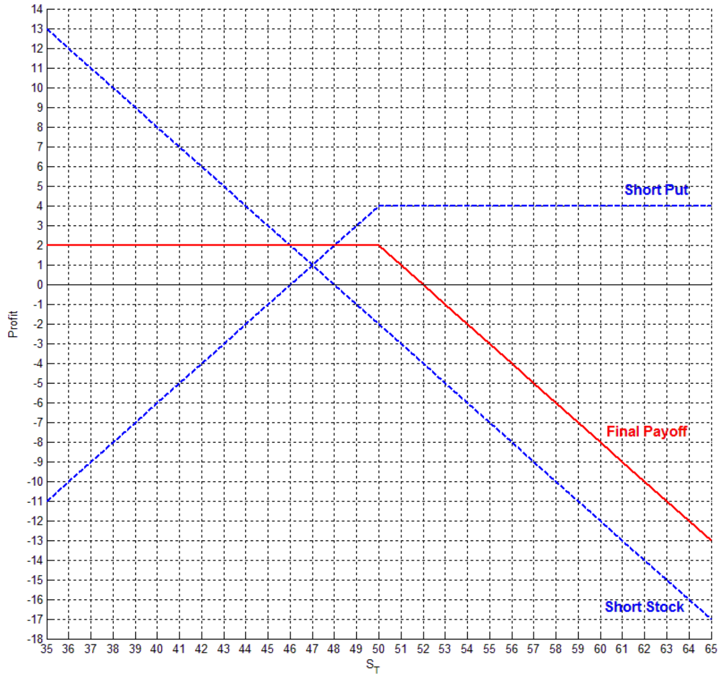


Figure III.63 Reverse of Protective Put strategy

31) Pricing | 32) Scenario | 33) Matrix | 34) Volatility | 35) Backtest

Two Leg

Parameters	Summary	Leg 1	Leg 2
Underlying	FTSEMIB Index	0	0
Und. Price EUR	Mid 25,180.35	0	0
Trade	01/09/2023 09:34	0	0
Settle	01/09/2023	0	0
Style		Vanilla	Spot
Exercise		European	
Call/Put		Put	
Direction		Sell	Sell
Strike		25,180.35	25,180.35
Strike % Money		ATM	
Shares		1.00	1.00
Expiry		04/11/2023 17:40	MM/DD/YYYY
Time to Expiry		92 08:06	0
Model		BS - continuous	
Vol BVOL		Mid 19.038%	
More Market Data			
Forward Carry		25,202.49	0
EURate MMkt		2.237%	0
Dividend Yield		2.452%	0
Discounted Div Flow		154.69	141.35
Borrow Cost		0.000%	0

Figure III.64 Reverse of a Protective Put strategy. Source: Bloomberg®

Figure III.65 Reverse of a Protective Put strategy: Terminal pay-off. Source: Bloomberg®

The profit patterns generated by the combinations between an option and the corresponding underlying generate the following pay-off:

(a) a Long position in a stock + a Short position in a call generates a short put.
(b) a Short position in a stock + a Long position in a call generates a long put.
(c) a Long position in a put + a Long position in a stock generates a long call.
(d) a Short position in a put + a Short position in a stock generates a short call.

The **Put-call parity** can help to generalize these statements:

$$p + S_0 = c + K \cdot \exp(-r \cdot T) + D \quad \textit{(Eq. III.83)}$$

where:
p is the price of a European put.
S_0 is the price level of the asset.
K is the same strike price for both the call and the put.
D is the discounted value of dividends paid during the life of the options.
r is the risk-free interest rate.
T is the time to maturity, equal for the call and for the put.
c is the price of a European call.

The **Put-Call parity** shows that a long position in a European put option, combined with a long position in the stock equals a long position in a European call option, plus a certain amount of money $K \cdot \exp(-r \cdot T) + D$.

This explains the profit pattern in the case of a protective put strategy, or case **(c)** above.
On the other hand, the strategy payoff graph **(d)** is the inverse of the case just discussed and leads to having a short position in a European call.

For the other two cases, the put-call parity relationship can be rearranged as follows:

$$S_0 - c = K \cdot \exp(-r \cdot T) + D - p \text{ (Eq. III.84)}$$

The equation shows that a long position in a stock combined with a short position in a European call option equals a short position in a European put, plus a certain amount of cash $\exp(-r \cdot T) + D$ – case **(a)**.

Strategy payoff graph **(b)** is the inverse of case **(a)** and leads to having a long position in a European put option.

We now introduce another strategy, called **spread trading strategy,** which requires investing in two or more options of the **same type**. Among them, one of the most popular types is the **bull spread**. This strategy can be created by buying a European call option written on a stock with a given strike (K_1) and selling a European call option on the same underlying, but with a higher strike price (K_2). Both options must have the same maturity (T). Another way in which a bull spread can be synthetized is by using two European put options: a long position is assumed on the put with a lower strike price (K_1) and a short position on the European put with a higher strike price (K_2). In this case, again, both options must obviously have the same maturity date (T). This strategy requires an initial investment if the bull spread strategy is implemented using call options. In fact, the price of a call decreases as the strike price increases: the value of the option sold will therefore always be lower than the one purchased. On the other hand, if the strategy is implemented by buying and selling puts, there is a positive up-front cash flow (assuming we disregard any margin requirements).

In fact, the price of a put option increases as the strike price increases: the value of the sold option will be characterized by a higher value than the one purchased. Such an options strategy limits upside and downside risk and this is ideal for an investor who has bought a call option with a strike price of K_1 and is willing to sacrifice part of the potential future profit by selling a call with a strike price of K_2 , $K_2 > K_1$ in order to spend less for the hedging.

The payoff created by a bull spread strategy using call options is shown in Table III.20 below:

Stock Price Range	Payoff Long Call	Payoff Short Call	Total Payoff
$S_T \leq K_1$	0	0	0
$K_1 \leq S_T < K_2$	$S_T - K_1$	0	$S_T - K_1$
$S_T \geq K_2$	$S_T - K_1$	$-(S_T - K_2)$	$K_2 - K_1$

Table III.20 Bull spread strategy using calls

The payoff created by a bull spread strategy using put options is shown below in Table III.21:

Stock Price Range	Payoff Long Put	Payoff Short Put	Total Payoff
$S_T \leq K_1$	$K_1 - S_T$	$-(K_2 - S_T)$	$K_1 - K_2$
$K_1 \leq S_T < K_2$	0	$-(K_2 - S_T)$	$S_T - K_2$
$S_T \geq K_2$	0	0	0

Table III.21 Bull spread strategy using puts

If the bull spread strategy is synthesized via calls, an initial investment is necessary for the holder: $premium_2 - premium_1 = \Delta_{premium} < 0$.

Stock Price Range	Payoff Long Call	Payoff Short Call	Total Payoff
$S_T \leq K_1$	$-premium_1$	$+premium_2$	$\Delta^-_{premium}$
$K_1 \leq S_T < K_2$	$S_T - K_1 - premium_1$	$+premium_2$	$S_T - K_1 + \Delta^-_{premium}$
$S_T \geq K_2$	$S_T - K_1 - premium_1$	$-(S_T - K_2) + premium_2$	$K_2 - K_1 + \Delta^-_{premium}$

Table III.22 Bull spread strategy using calls with premiums

On the other hand, if the bull spread is obtained through puts, an upfront premium is received:

Stock Price Range	Payoff Long Put	Payoff Short Put	Total Payoff
$S_T \leq K_1$	$K_1 - S_T - premium_1$	$-(K_2 - S_T) + premium_2$	$K_1 - K_2 + \Delta^+_{premium}$
$K_1 \leq S_T < K_2$	$-premium_1$	$-(K_2 - S_T) + premium_2$	$S_T - K_2 + \Delta^+_{premium}$
$S_T \geq K_2$	$-premium_1$	$+premium_2$	$\Delta^+_{premium}$

Table III.23 Bull spread strategy using puts with premiums

Here is an example of a Bull Spread strategy implemented using call options:

Long position in a call, $K_1 = 50$

Short position in a call, $K_2 = 55$

Investment for the Call with Strike $K_1 = 3$

Premium for the Call with Strike $K_2 = 1$

Stock Price Range	Total Payoff
$S_T \leq 50$	$1 - 3 = -2$
$50 \leq S_T < 55$	$S_T - 50 - 2 = S_T - 52$
$S_T \geq 55$	$55 - 50 - 2 = 3$

S(T)	Long call (1)	Initial investment	Short call (2)	Initial premium	Final pay-off
46	0	-3	0	1	-2
47	0	-3	0	1	-2
48	0	-3	0	1	-2
49	0	-3	0	1	-2
50	0	-3	0	1	-2
51	1	-3	0	1	-1
52	2	-3	0	1	0
53	3	-3	0	1	1
54	4	-3	0	1	2
55	5	-3	0	1	3
56	6	-3	-1	1	3

Table III.24 Bull spread strategy using Calls

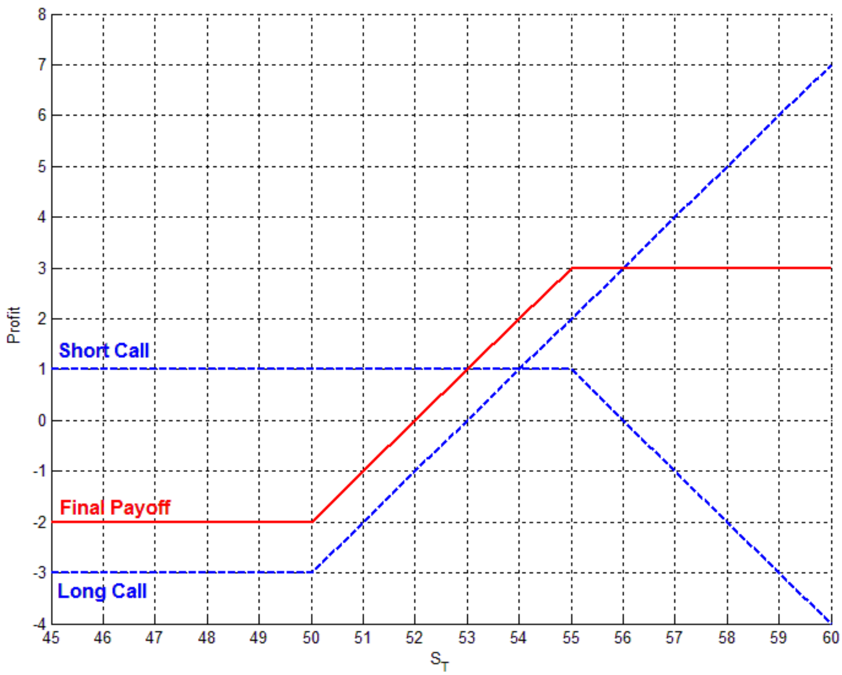


Figure III.66 Bull spread strategy using Calls

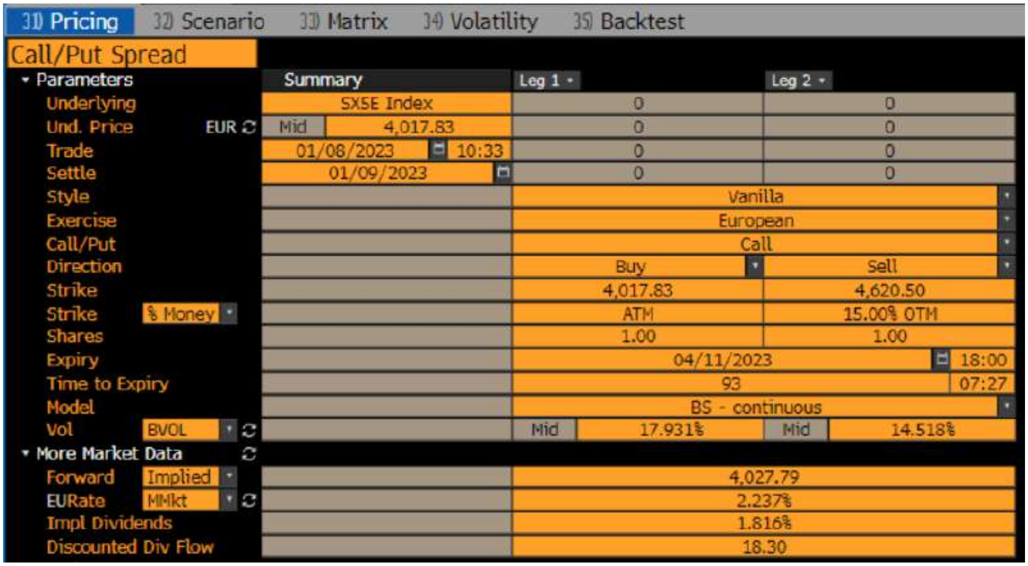


Figure III.67 Bull spread strategy using Calls. Source: Bloomberg®

Figure III.68 Bull spread strategy using Calls: Terminal pay-off. Source: Bloomberg®

Here is an example of a Bull Spread strategy implemented using put options:

Long position in a put, $K_1 = 50$

Short position in a put, $K_2 = 55$

Investment for the Put with Strike, $K_1 = 1$

Premium for the Put with Strike, $K_2 = 3$

Stock Price Range	Total Payoff
$S_T \leq 50$	$50 - 55 + 2 = 3$
$50 \leq S_T < 55$	$S_T - 55 + 2 = S_T - 53$
$S_T \geq 55$	$3 - 1 = +2$

S(T)	Long put (1)	Initial investment	Short put (2)	Initial premium	Final pay-off
48	2	-1	-7	3	-3
49	1	-1	-6	3	-3
50	0	-1	-5	3	-3
51	0	-1	-4	3	-2
52	0	-1	-3	3	-1
53	0	-1	-2	3	0
54	0	-1	-1	3	1
55	0	-1	0	3	2
56	0	-1	0	3	2

Table III.25 Bull spread strategy using Puts

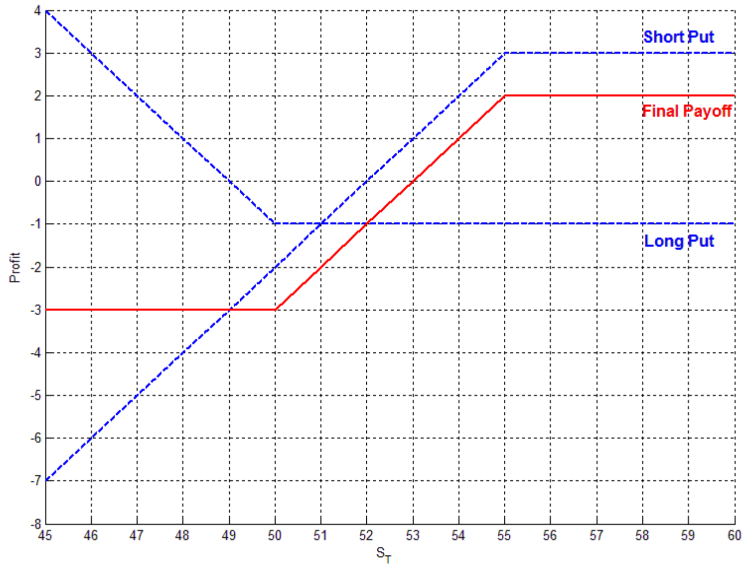


Figure III.69 Bull spread strategy using Puts

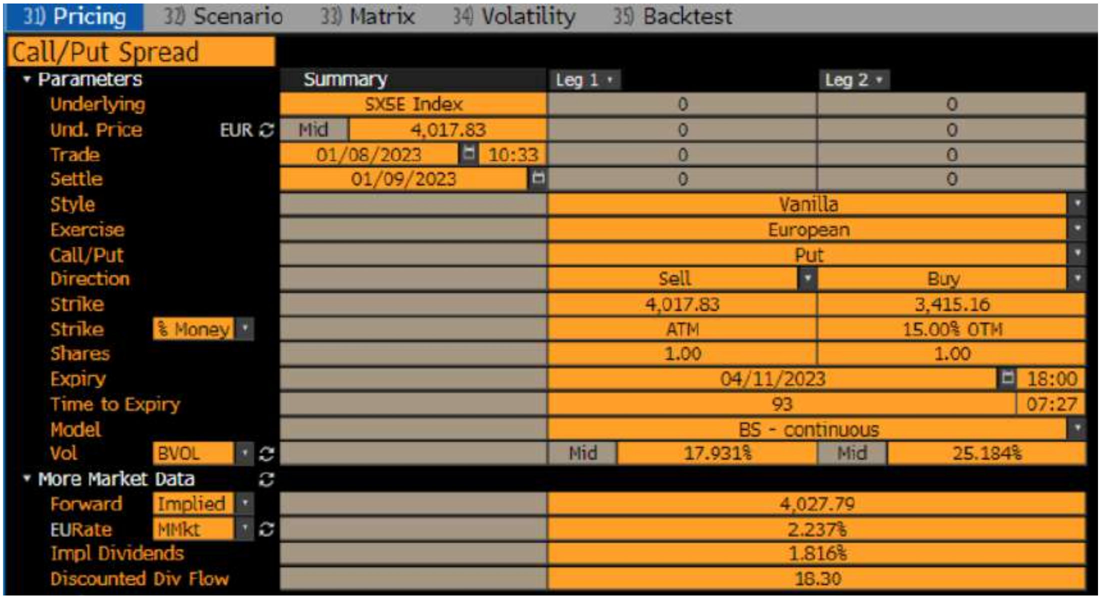


Figure III.70 Bull spread strategy using Puts. Source: Bloomberg®

Figure III.71 Bull spread strategy using Puts: Terminal pay-off. Source: Bloomberg®

We have analyzed the bull spread strategy, and it is clearly implemented by an investor having the expectation that the price level of the stock will rise.
Conversely, when the investor believes that the stock price will fall, then he will enters a **bear spread** strategy. Such a strategy can be created by selling a put option with a strike K_1 and buying a put with a different strike price K_2, in particular $K_1 < K_2$.
Since the price of the sold put is lower than the price of the purchased put, a bear spread strategy is characterized by an initial cash outflow.
Essentially, the investor bought a put option with a specified strike price and chose to sacrifice part of the potential profit by selling a put option with a lower strike price.
In return for the potential sacrificed profit, the investor receives the premium of the derivative. Like bull spreads, bear spreads also limit both upside and downsize risk. Another way to implement a bear spread strategy is by using call options.
In this case, the trader will take a long position in a call option with a higher strike price (K_2) and a short position in a call with a lower strike price (K_1). Thanks to this strategy, if we disregard margin requirements, the investor receives an initial cash inflow.

The payoff created by a bear spread strategy using put options is shown in Table III.26:

Stock Price Range	Payoff Long Put	Payoff Short Put	Total Payoff
$S_T \geq K_2$	0	0	0
$K_1 < S_T < K_2$	$K_2 - S_T$	0	$K_2 - S_T$
$S_T \leq K_1$	$K_2 - S_T$	$-(K_1 - S_T)$	$K_2 - K_1$

Table III.26 Bear spread strategy using Puts

The payoff created by a bear spread strategy using call options is shown below in Table III.27:

Stock Price Range	Payoff Long Call	Payoff Short Call	Total Payoff
$S_T \geq K_2$	$S_T - K_2$	$-(S_T - K_1)$	$K_1 - K_2$
$K_1 < S_T < K_2$	0	$-(S_T - K_1)$	$K_1 - S_T$
$S_T \leq K_1$	0	0	0

Table III.27 Bear spread strategy using Calls

In case the spread strategy is synthesized via put options, an initial investment is necessary for the holder: $premium_1 - premium_2 = \Delta_{premium} < 0$.

Stock Price Range	Payoff Long Put	Payoff Short Put	Total Payoff
$S_T \geq K_2$	$-premium_2$	$+premium_1$	$\Delta^-_{premium}$
$K_1 < S_T < K_2$	$K_2 - S_T - premium_2$	$+premium_1$	$K_2 - S_T + \Delta^-_{premium}$
$S_T \leq K_1$	$K_2 - S_T - premium_2$	$-(K_1 - S_T) + premium_1$	$K_2 - K_1 + \Delta^-_{premium}$

Table III.28 Bear spread strategy using Puts with premiums

On the other hand, if the bull spread is obtained through call options, an upfront premium is received: $premium_1 - premium_2 = \Delta_{premium} > 0$

Stock Price Range	Payoff Long Call	Payoff Short Call	Total Payoff
$S_T \geq K_2$	$S_T - K_2 - premium_2$	$-(S_T - K_1) + premium_1$	$K_1 - K_2 + \Delta^+_{premium}$
$K_1 < S_T < K_2$	$-premium_2$	$-(S_T - K_1) + premium_1$	$K_1 - S_T + \Delta^+_{premium}$
$S_T \leq K_1$	$-premium_2$	$+premium_1$	$\Delta^+_{premium}$

Table III.29 Bear spread strategy using Calls with premiums

Let us make an example of a Bear Spread strategy using Put options:

Long position in a put, $K_2 = 55$

Short position in a put, $K_1 = 50$

Investment for the put with Strike $K_2 = 3$

Premium for the put with Strike $K_1 = 1$

Stock Price Range	Total Payoff
$S_T \geq 55$	$1 - 3 = -2$
$50 < S_T < 55$	$55 - S_T - 2 = 53 - S_T$
$S_T \leq 50$	$55 - 50 - 2 = +3$

S(T)	Long put (2)	Initial investment	Short put (1)	Initial premium	Final pay-off
47	8	-3	-3	1	3
48	7	-3	-2	1	3
49	6	-3	-1	1	3
50	5	-3	0	1	3
51	4	-3	0	1	2
52	3	-3	0	1	1
53	2	-3	0	1	0
54	1	-3	0	1	-1
55	0	-3	0	1	-2
56	0	-3	0	1	-2
57	0	-3	0	1	-2

Table III.30 Bear spread strategy using Puts

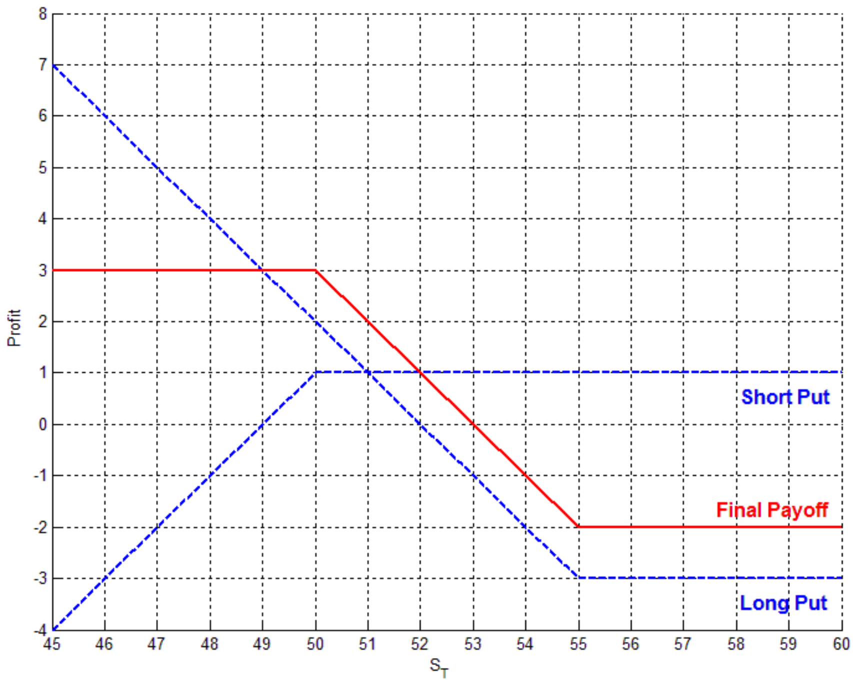


Figure III.72 Bear spread strategy using Puts

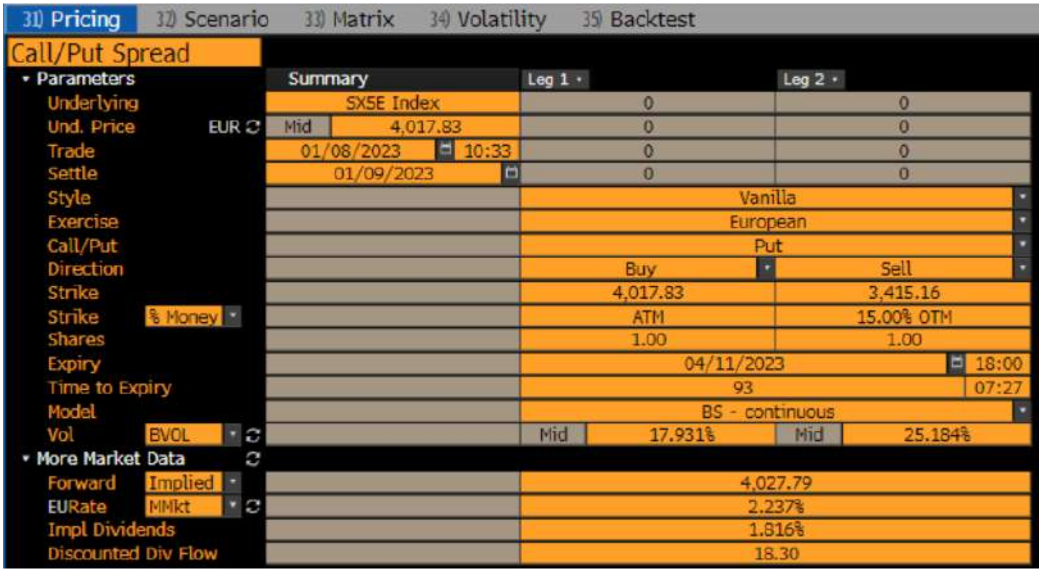


Figure III.73 Bear spread strategy using Puts. Source: Bloomberg®

Figure III.74 Bear spread strategy using Puts: Terminal pay-off. Source: Bloomberg®

Here is an example of a Bear Spread strategy using Call options:

Long position in a call, $K_2 = 55$

Short position in a call, $K_1 = 50$

Investment for the call with Strike $K_2 = 1$

Premium for the call with Strike $K_1 = 3$

Stock Price Range	Total Payoff
$S_T \geq 55$	$50 - 55 + 2 = -3$
$50 < S_T < 55$	$50 - S_T + 2 = 52 - S_T$
$S_T \leq 50$	$3 - 1 = +2$

S(T)	Long Call (2)	Initial investment	Short Call (1)	Initial premium	Final pay-off
49	0	-1	0	3	2
50	0	-1	0	3	2
51	0	-1	-1	3	1
52	0	-1	-2	3	0
53	0	-1	-3	3	-1
54	0	-1	-4	3	-2
55	0	-1	-5	3	-3
56	1	-1	-6	3	-3

Table III.31 Bear spread strategy using Calls

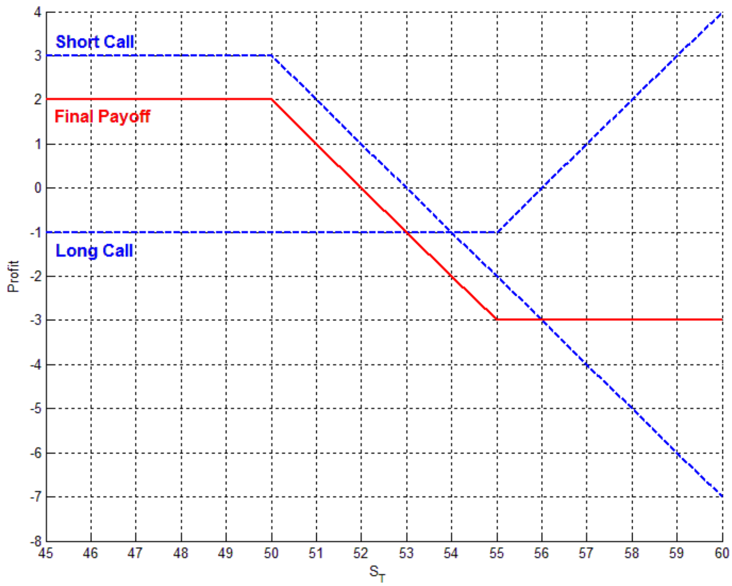


Figure III.75 Bear spread strategy using Calls

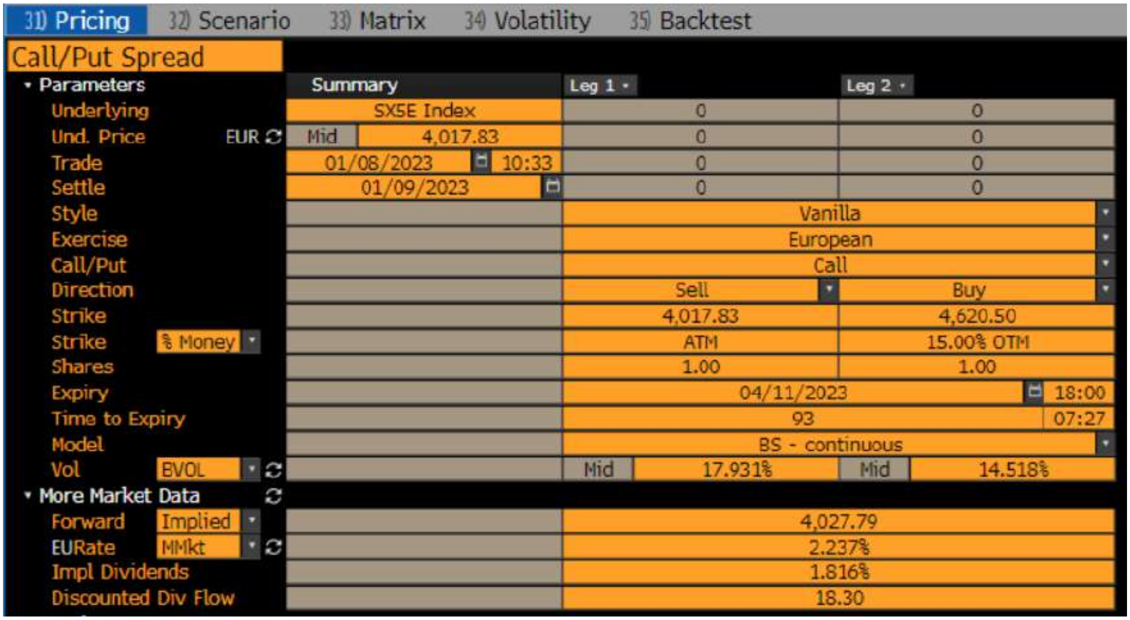


Figure III.76 Bear spread strategy using Calls. Source: Bloomberg®

Figure III.77 Bear spread strategy using Puts: Terminal pay-off. Source: Bloomberg®

Let us now introduce another strategy, called a **box spread:** it is a combination of a bull call spread with strike prices K_1 and K_2 and a bear put spread with the same strike prices.

The profit profile of such a strategy is fixed and is equal to: $K_2 - K_1$. The value of a box spread is therefore constituted by the value of the discounted deterministic pay-off: $(K_2 - K_1) \cdot \exp(-r \cdot T)$.

As per our typical reasoning, we observe that, if it were traded for different values, arbitrage opportunities would arise. Specifically:

- If the market price of the box spread were too low, it would be worth buying it. This would mean buying a call option with strike price K_1, buying a put with strike price K_2, selling a call with strike price K_2 and selling a put with strike price K_1.

- On the other hand, if the market price of the box spread were too high, it would be worth selling it. This would involve buying a call option with strike price K_2, buying a put with strike price K_1, selling a call with strike price K_1 and selling a put with strike price K_2.

It is worth to highlight that this arbitrage must be implemented using exclusively European-type options.

The payoff generated by a box spread strategy is shown below in Table III.32:

Stock Price Range	Payoff bull call spread	Payoff bear put spread	Total Payoff
$S_T \geq K_2$	$K_2 - K_1$	0	$K_2 - K_1$
$K_1 < S_T < K_2$	$S_T - K_1$	$K_2 - S_T$	$K_2 - K_1$
$S_T \leq K_1$	0	$K_2 - K_1$	$K_2 - K_1$

Table III.32 Box strategy

Considering the initial rewards/investments to implement the strategy, the payoff becomes:

Stock Price Range	Payoff bull call spread	Payoff bear put spread	Total Payoff
$S_T \geq K_2$	$K_2 - K_1 + \Delta_{premium}^-$	$\Delta_{premium}^+$	$K_2 - K_1 + \Delta_{premium}$
$K_1 < S_T < K_2$	$S_T - K_1 + \Delta_{premium}^-$	$K_2 - S_T + \Delta_{premium}^+$	$K_2 - K_1 + \Delta_{premium}$
$S_T \leq K_1$	$\Delta_{premium}^-$	$K_2 - K_1 + \Delta_{premium}^+$	$K_2 - K_1 + \Delta_{premium}$

Table III.33 Box strategy with premiums

We present an example of a Box strategy:

Long position in a call, $K_1 = 50, outflow = 3$
Short position in a call, $K_2 = 55, inflow = 1$
Long position in a put, $K_2 = 55, outflow = 3$
Short position in a put, $K_1 = 50, inflow = 1$

Stock Price Range	Total Payoff
$S_T \geq 55$	$55 - 50 - 4 = +1$
$50 < S_T < 55$	$55 - 50 - 4 = +1$
$S_T \leq 50$	$55 - 50 - 4 = +1$

S(T)	Long Call (1)	Initial Investment	Short Call (2)	Initial Premium	Bull call spread	Long put (2)	Initial Investment	Short Put (1)	Initial Premium	Bear put spread	Final payoff
46	0	-3	0	1	-2	9	-3	-4	1	3	1
47	0	-3	0	1	-2	8	-3	-3	1	3	1
48	0	-3	0	1	-2	7	-3	-2	1	3	1
49	0	-3	0	1	-2	6	-3	-1	1	3	1
50	0	-3	0	1	-2	5	-3	0	1	3	1
51	1	-3	0	1	-1	4	-3	0	1	2	1
52	2	-3	0	1	0	3	-3	0	1	1	1
53	3	-3	0	1	1	2	-3	0	1	0	1
54	4	-3	0	1	2	1	-3	0	1	-1	1
55	5	-3	0	1	3	0	-3	0	1	-2	1
56	6	-3	-1	1	3	0	-3	0	1	-2	1
57	7	-3	-2	1	3	0	-3	0	1	-2	1

Table III.34 Box spread strategy

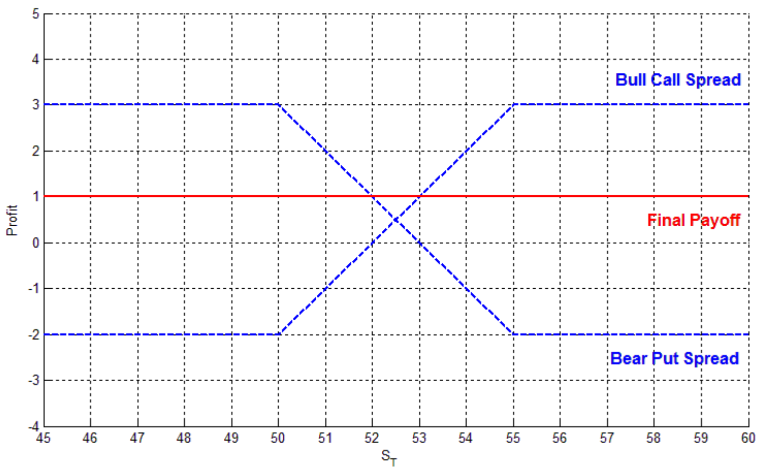


Figure III.78 Box spread strategy (Bull Call spread + Bear Put Spread)

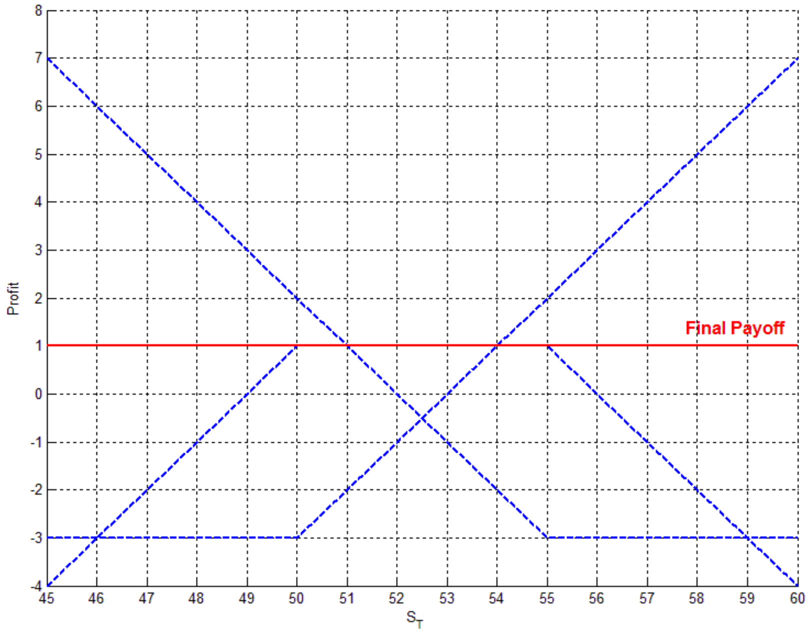
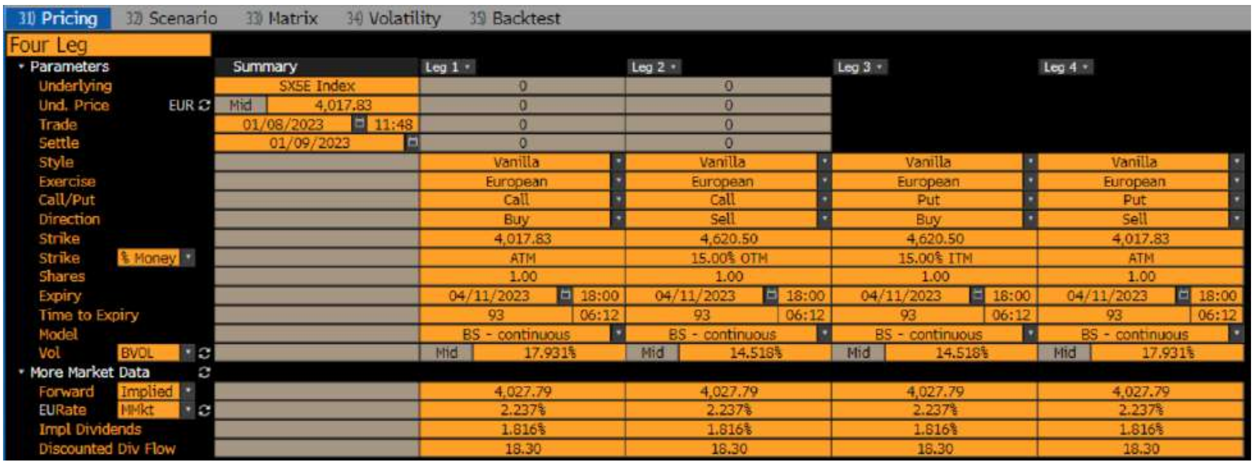


Figure III.79 Box spread strategy



31) Pricing | 32) Scenario | 33) Matrix | 34) Volatility | 35) Backtest

Four Leg

	Summary	Leg 1	Leg 2	Leg 3	Leg 4
▪ Parameters					
Underlying	SX5E Index	0	0		
Und. Price EUR	Mid 4,017.83	0	0		
Trade	01/08/2023 11:48	0	0		
Settle	01/09/2023	0	0		
Style		Vanilla	Vanilla	Vanilla	Vanilla
Exercise		European	European	European	European
Call/Put		Call	Call	Put	Put
Direction		Buy	Sell	Buy	Sell
Strike		4,017.83	4,620.50	4,620.50	4,017.83
Strike % Money		ATM	15.00% OTM	15.00% ITM	ATM
Shares		1.00	1.00	1.00	1.00
Expiry		04/11/2023 18:00	04/11/2023 18:00	04/11/2023 18:00	04/11/2023 18:00
Time to Expiry		93 06:12	93 06:12	93 06:12	93 06:12
Model		BS - continuous	BS - continuous	BS - continuous	BS - continuous
Vol BVOL		Mid 17.931%	Mid 14.518%	Mid 14.518%	Mid 17.931%
▪ More Market Data					
Forward Implied		4,027.79	4,027.79	4,027.79	4,027.79
EURate MMkt		2.237%	2.237%	2.237%	2.237%
Impl Dividends		1.816%	1.816%	1.816%	1.816%
Discounted Div Flow		18.30	18.30	18.30	18.30

Figure III.80 Box spread strategy. Source: Bloomberg®

Figure III.81 Box spread strategy. Terminal Payoff. Source: Bloomberg®

We now introduce the **butterfly spread,** which is a strategy that can be implemented by taking a position on options with three different strike prices. In other words:

- Buying a call option with a relatively low strike price K_1.
- Buying a call option with a relatively high strike price K_3.
- Selling two call options with a strike price K_2 such that $K_1 < K_2 < K_3$.

Typically, K_2 is close to the current level of the stock price. The generated payoff is shown in Table III.35 below:

Stock Price Range	Payoff long call (1)	Payoff long call (3)	Payoff from short calls (2)	Final Payoff
$S_T \leq K_1$	0	0	0	0
$K_1 < S_T \leq K_2$	$S_T - K_1$	0	0	$S_T - K_1$
$K_2 < S_T \leq K_3$	$S_T - K_1$	0	$-2(S_T - K_2)$	$2K_2 - K_1 - S_T$
$S_T > K_3$	$S_T - K_1$	$S_T - K_3$	$-2(S_T - K_2)$	$2K_2 - K_1 - K_3$

Table III.35 Butterfly spread using calls

In the particular case where $K_2 = \frac{K_1+K_3}{2}$, the profit profile of the strategy simplifies: for $K_2 < S_T < K_3$ we have $K_3 - S_T$ and for $S_T > K_3$ we have 0.

Such a strategy is suitable for investors who believe that the future value of the asset will not change very much from the current spot price level.

The specific strategy requires a small initial investment. Thus, considering the initial rewards/investments to create the strategy, we obtain:

$$\Delta_{premium}^{-} = 2 \cdot premium_2 - premium_1 - premium_3$$

Stock Range	Payoff long call (1)	Payoff long call (3)	Payoff from short calls	Final Payoff
$S_T \leq K_1$	$-prem_1$	$-prem_3$	$+2 \cdot prem_2$	Δ_{prem}^{-}
$K_1 < S_T \leq K_2$	$S_T - K_1 - prem_1$	$-prem_3$	$+2 \cdot prem_2$	$S_T - K_1 + \Delta_{prem}^{-}$
$K_2 < S_T \leq K_3$	$S_T - K_1 - prem_1$	$-prem_3$	$-2(S_T - K_2) + 2prem_2$	$2K_2 - K_1 - S_T + \Delta_{prem}^{-}$
$S_T > K_3$	$S_T - K_1 - prem_1$	$S_T - K_3 - prem_3$	$-2(S_T - K_2) + 2prem_2$	$2K_2 - K_1 - K_3 + \Delta_{prem}^{-}$

Table III.36 Butterfly strategy using calls with premiums

In case $K_2 = \frac{K_1+K_3}{2}$, the profit pattern for the butterfly spread simplifies:

Stock Range	Payoff long call (1)	Payoff long call (3)	Payoff from short calls	Final Payoff
$S_T \leq K_1$	$-prem_1$	$-prem_3$	$+2 \cdot prem_2$	$\Delta_{premium}^{-}$
$K_1 < S_T \leq K_2$	$S_T - K_1 - prem_1$	$-prem_3$	$+2 \cdot prem_2$	$S_T - K_1 + \Delta_{premium}^{-}$
$K_2 < S_T \leq K_3$	$S_T - K_1 - prem_1$	$-prem_3$	$-2(S_T - K_2) + 2 \cdot prem_2$	$K_3 - S_T + \Delta_{premium}^{-}$
$S_T > K_3$	$S_T - K_1 - prem_1$	$S_T - K_3 - prem_3$	$-2(S_T - K_2) + 2 \cdot prem_2$	$\Delta_{premium}^{-}$

Table III.37 Symmetric Butterfly strategy using calls with premiums

Here is an example of a Butterfly strategy using call options:

Long position in a call, $K_1 = 50, outflow = 10$

Short position in a call $K_2 = 55, inflow = 7$

Short position in a call $K_2 = 55, inflow = 7$

Long position in a call, $K_3 = 60, outflow = 5$

Stock Price Range	Total Payoff
$S_T \leq 50$	$-10 + 7 + 7 - 5 = -1$
$50 < S_T \leq 55$	$S_T - 50 - 1 = S_T - 51$
$55 < S_T \leq 60$	$60 - S_T - 1 = 59 - S_T$
$S_T > 60$	$-10 + 7 + 7 - 5 = -1$

S(T)	Long Call (1)	Initial Investment	Long Call (3)	Initial Investment	Short Call x 2 (2)	Initial Premium	Final Pay-off
49	0	-10	0	-5	0	14	-1
50	0	-10	0	-5	0	14	-1
51	1	-10	0	-5	0	14	0
52	2	-10	0	-5	0	14	1
53	3	-10	0	-5	0	14	2
54	4	-10	0	-5	0	14	3
55	5	-10	0	-5	0	14	4
56	6	-10	0	-5	-2	14	3
57	7	-10	0	-5	-4	14	2
58	8	-10	0	-5	-6	14	1
59	9	-10	0	-5	-8	14	0
60	10	-10	0	-5	-10	14	-1
61	11	-10	1	-5	-12	14	-1

Table III.38 Butterfly strategy using calls

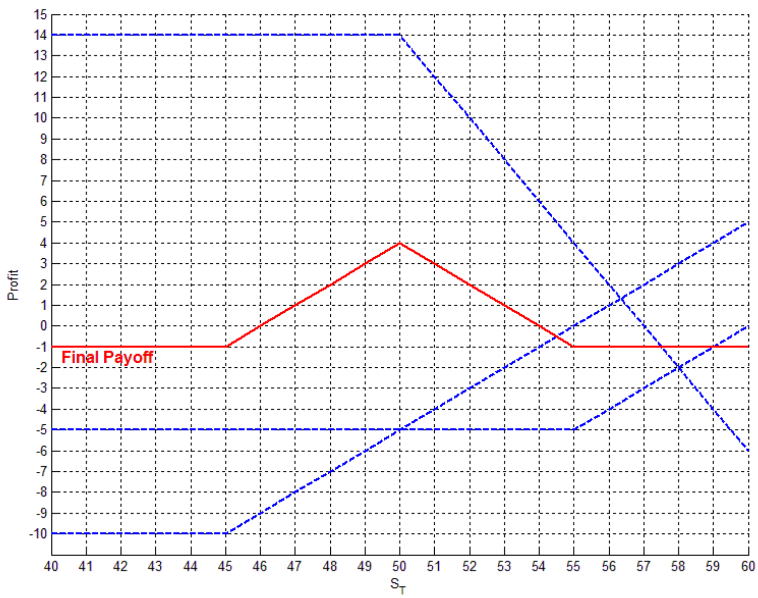
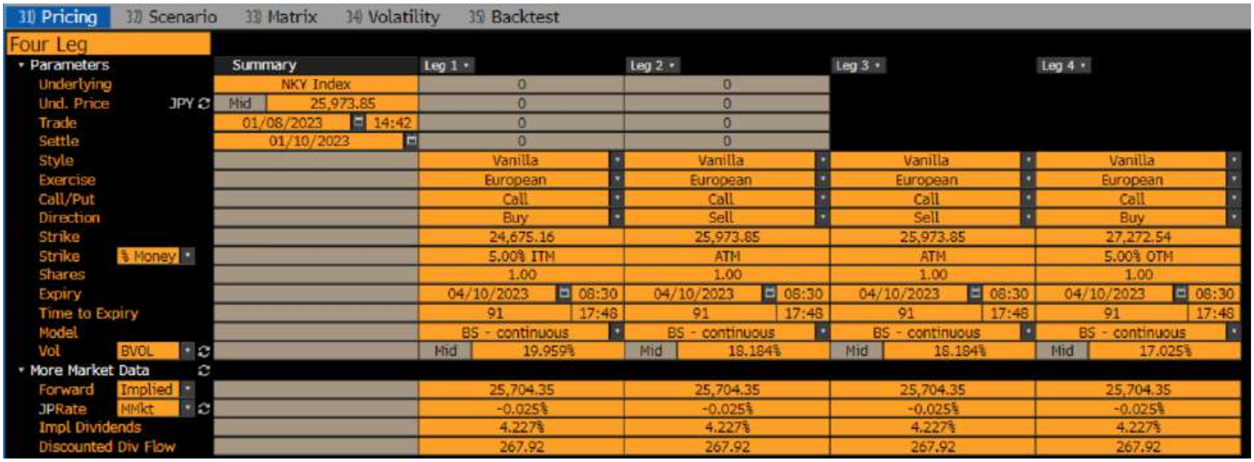


Figure III.82 Butterfly strategy using calls



31) Pricing | 32) Scenario | 33) Matrix | 34) Volatility | 35) Backtest

Four Leg

Parameters	Summary	Leg 1	Leg 2	Leg 3	Leg 4
Underlying	NKY Index	0	0		
Und. Price JPY	Mid 25,973.85	0	0		
Trade	01/08/2023 14:42	0	0		
Settle	01/10/2023	0	0		
Style		Vanilla	Vanilla	Vanilla	Vanilla
Exercise		European	European	European	European
Call/Put		Call	Call	Call	Call
Direction		Buy	Sell	Sell	Buy
Strike		24,675.16	25,973.85	25,973.85	27,272.54
Strike % Money		5.00% ITM	ATM	ATM	5.00% OTM
Shares		1.00	1.00	1.00	1.00
Expiry		04/10/2023 08:30	04/10/2023 08:30	04/10/2023 08:30	04/10/2023 08:30
Time to Expiry		91 17:48	91 17:48	91 17:48	91 17:48
Model		BS - continuous	BS - continuous	BS - continuous	BS - continuous
Vol BVOL		Mid 19.959%	Mid 18.184%	Mid 18.184%	Mid 17.025%
More Market Data					
Forward Implied		25,704.35	25,704.35	25,704.35	25,704.35
JPRate MMkt		-0.025%	-0.025%	-0.025%	-0.025%
Impl Dividends		4.227%	4.227%	4.227%	4.227%
Discounted Div Flow		267.92	267.92	267.92	267.92

Figure III.83 Butterfly strategy using calls. Source: Bloomberg®

Figure III.84 Butterfly strategy using calls. Terminal Payoff. Source: Bloomberg®

In a similar way, a butterfly spread strategy can also be created using put options:

- the investor buys a put option with a low strike price.
- the investor buys a put with a high strike price.
- the investor sells two put options with an intermediate strike price.

Stock Price Range	Payoff long put (1)	Payoff long put (3)	Payoff short puts (2)	Final Payoff
$S_T \geq K_3$	0	0	0	0
$K_2 < S_T \leq K_3$	0	$K_3 - S_T$	0	$K_3 - S_T$
$K_1 < S_T \leq K_2$	0	$K_3 - S_T$	$-2(K_2 - S_T)$	$S_T - 2K_2 + K_3$
$S_T < K_1$	$K_1 - S_T$	$K_3 - S_T$	$-2(K_2 - S_T)$	$K_1 + K_3 - 2K_2$

Table III.39 Butterfly strategy using puts

In the particular case where $K_2 = \frac{K_1+K_3}{2}$, the profit profile of the strategy simplifies: for $K_1 < S_T < K_2$ we have $S_T - K_1$ and for $S_T < K_1$ we have 0. The Put-call parity can then be used to demonstrate that the strategy cost of a butterfly spread strategy implemented using calls is equivalent to that using put options.

We now present an example of a Butterfly strategy implemented using put options:

Long position in a put, $K_1 = 50, outflow = 5$

Short position in a put $K_2 = 55, inflow = 7$

Short position in a put $K_2 = 55, inflow = 7$

Long position in a put, $K_3 = 60, outflow = 10$.

Stock Price Range	Total Payoff
$S_T < 50$	$-10 + 7 + 7 - 5 = -1$
$50 < S_T \leq 55$	$S_T - 50 - 1 = S_T - 51$
$55 < S_T \leq 60$	$60 - S_T - 1 = 59 - S_T$
$S_T \geq 60$	$-10 + 7 + 7 - 5 = -1$

S(T)	Long Put (1)	Initial Investment	Long Put (3)	Initial Investment	Short Put x 2 (2)	Initial Premium	Final Pay-off
49	1	-5	11	-10	-12	14	-1
50	0	-5	10	-10	-10	14	-1
51	0	-5	9	-10	-8	14	0
52	0	-5	8	-10	-6	14	1
53	0	-5	7	-10	-4	14	2
54	0	-5	6	-10	-2	14	3
55	0	-5	5	-10	0	14	4
56	0	-5	4	-10	0	14	3
57	0	-5	3	-10	0	14	2
58	0	-5	2	-10	0	14	1
59	0	-5	1	-10	0	14	0
60	0	-5	0	-10	0	14	-1
61	0	-5	0	-10	0	14	-1

Table III.40 Butterfly strategy using puts

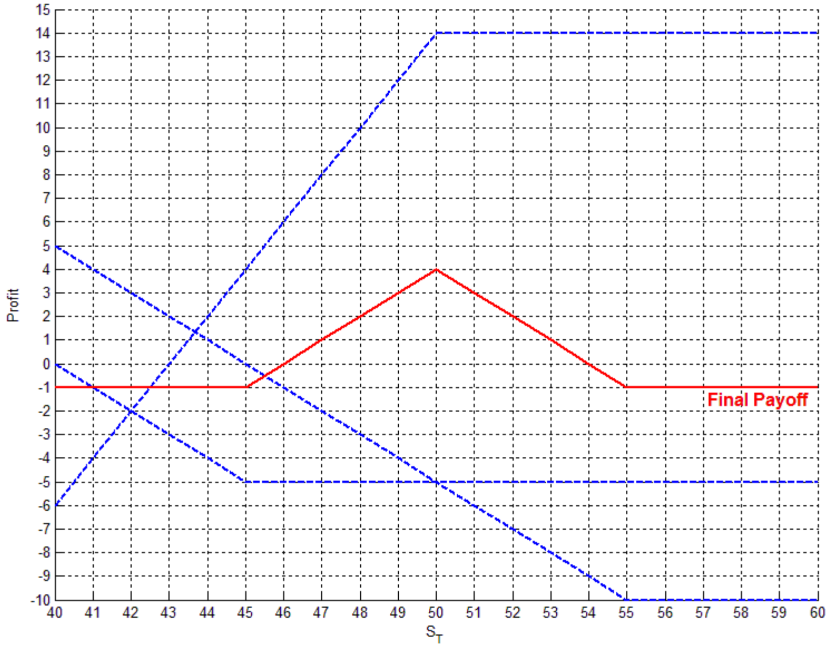


Figure III.85 Butterfly strategy using puts

31) Pricing | 32) Scenario | 33) Matrix | 34) Volatility | 35) Backtest

Four Leg

Parameters	Summary	Leg 1	Leg 2	Leg 3	Leg 4
Underlying	NKY Index	0	0		
Und. Price JPY	Mid 25,973.85	0	0		
Trade	01/08/2023 14:42	0	0		
Settle	01/10/2023	0	0		
Style		Vanilla	Vanilla	Vanilla	Vanilla
Exercise		European	European	European	European
Call/Put		Put	Put	Put	Put
Direction		Buy	Sell	Sell	Buy
Strike		24,675.16	25,973.85	25,973.85	27,272.54
Strike % Money		5.00% OTM	ATM	ATM	5.00% ITM
Shares		1.00	1.00	1.00	1.00
Expiry		04/10/2023 08:30	04/10/2023 08:30	04/10/2023 08:30	04/10/2023 08:30
Time to Expiry		91 17:48	91 17:48	91 17:48	91 17:48
Model		BS - continuous	BS - continuous	BS - continuous	BS - continuous
Vol BVOL		Mid 19.959%	Mid 18.184%	Mid 18.184%	Mid 17.025%
More Market Data					
Forward Implied		25,704.35	25,704.35	25,704.35	25,704.35
JPRate MMkt		-0.025%	-0.025%	-0.025%	-0.025%
Impl Dividends		4.227%	4.227%	4.227%	4.227%
Discounted Div Flow		267.92	267.92	267.92	267.92

Figure III.86 Butterfly strategy using puts. Source: Bloomberg®

Figure III.87 Butterfly strategy using puts. Terminal Payoff. Source: Bloomberg®

Let us now define a **combination:** it is a trading strategy that is implemented by taking a position in both call options and put options written on the same underlying. The most popular combinations are straddles, strips, straps and strangles. Here comes a brief description of each of them.

The **straddle** consists of buying a call and a put option with the same strike price K and the same maturity T.

Stock Price Range	Payoff long call (1)	Payoff long put (2)	Final Payoff
$S_T \leq K$	0	$K - S_T$	$K - S_T$
$S_T > K$	$S_T - K$	0	$S_T - K$

Table III.41 Straddle

Considering the initial investment of the strategy we have: $\Delta^{-}_{premium} = -premium_1 - premium_2$.

Stock Price Range	Payoff long call (1)	Payoff long put (2)	Final Payoff
$S_T \leq K$	$-premium_1$	$K - S_T - premium_2$	$K - S_T + \Delta^{-}_{premium}$
$S_T > K$	$S_T - K - premium_1$	$-premium_2$	$S_T - K + \Delta^{-}_{premium}$

Table III.42 Straddle with premiums

If the share price level is close to the strike price at the expiry date, the strategy leads to a loss; on the other hand, if the price of the underlying is far from the exercise price, the strategy leads to a profit. A straddle is, therefore, appropriate when an investor expects a large change in the stock price level but does not want to make assumptions about the direction. Such strategy is also called **bottom straddle** or **straddle purchase** and the reverse position, i.e. a **top straddle** or **straddle write**, is also reported in the literature. In this latter case, the strategy is created by selling a call and a put option with the same strike price and maturity.
It should be considered a decidedly risky trading strategy because if the price level is close to the strike price at maturity, the profit is substantial, but in the event of strong movements of the underlying, then the losses are potentially unlimited.

Here is an example of a Bottom Straddle:

Long position in a call, $K = 50$, $outflow = 3$
Long position in a put, $K = 50$, $outflow = 2$

Stock Price Range	Total Payoff
$S_T \leq 50$	$50 - S_T - 5 = 45 - S_T$
$S_T > 50$	$S_T - 50 - 5 = S_T - 55$

S(T)	Long Call	Initial investment	Long put	Initial Investment	Final pay-off
41	0	-3	9	-2	4
42	0	-3	8	-2	3
43	0	-3	7	-2	2
44	0	-3	6	-2	1
45	0	-3	5	-2	0
46	0	-3	4	-2	-1
47	0	-3	3	-2	-2
48	0	-3	2	-2	-3
49	0	-3	1	-2	-4
50	0	-3	0	-2	-5
51	1	-3	0	-2	-4
52	2	-3	0	-2	-3
53	3	-3	0	-2	-2
54	4	-3	0	-2	-1
55	5	-3	0	-2	0
56	6	-3	0	-2	1

Table III.43 Bottom Straddle

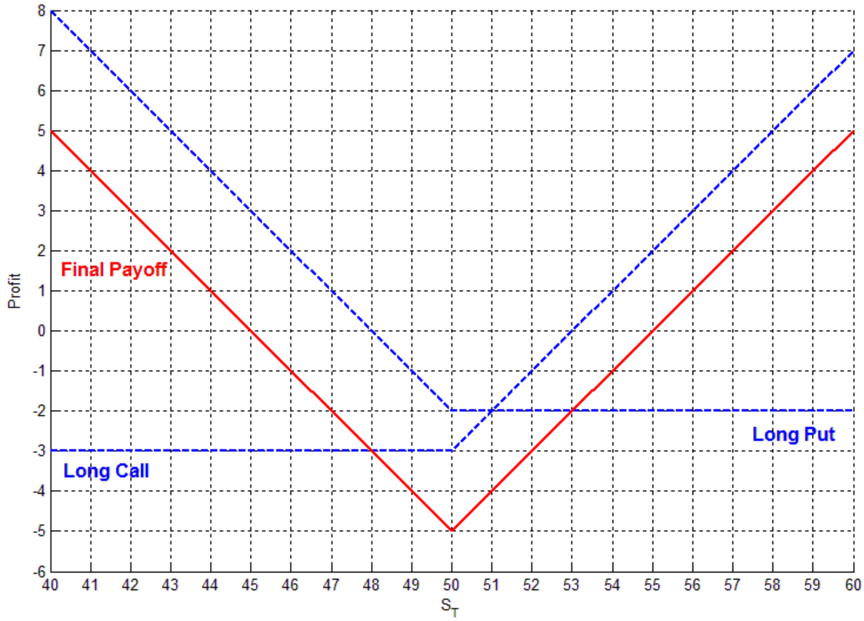


Figure III.88 Bottom Straddle

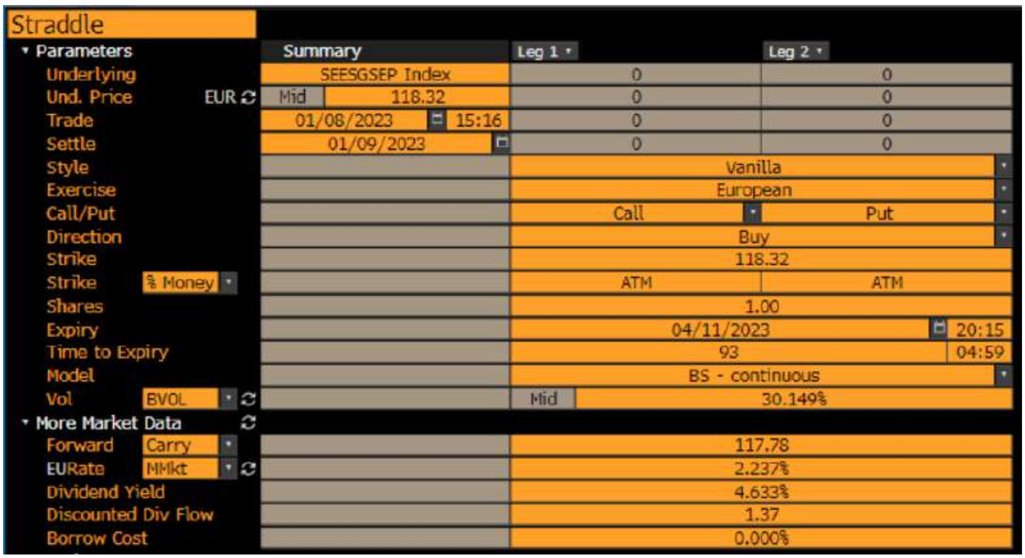


Figure III.89 Bottom Straddle. Source: Bloomberg®

Figure III.90 Bottom Straddle. Terminal Payoff. Source: Bloomberg®

Here is an example of a Top Straddle:

Short position in a call, K = 50, $inflow = 3$
Short position in a put, K = 50, $inflow = 2$

Stock Price Range	Total Payoff
$S_T \leq 50$	$S_T - 50 + 5 = S_T - 45$
$S_T > 50$	$50 - S_T + 5 = 55 - S_T$

S(T)	Short Call (1)	Initial Premium	Short Put (2)	Initial Premium	Final Pay-off
40	0	3	-10	2	-5
41	0	3	-9	2	-4
42	0	3	-8	2	-3
43	0	3	-7	2	-2
44	0	3	-6	2	-1
45	0	3	-5	2	0
46	0	3	-4	2	1
47	0	3	-3	2	2
48	0	3	-2	2	3
49	0	3	-1	2	4

Table III.44 Top Straddle

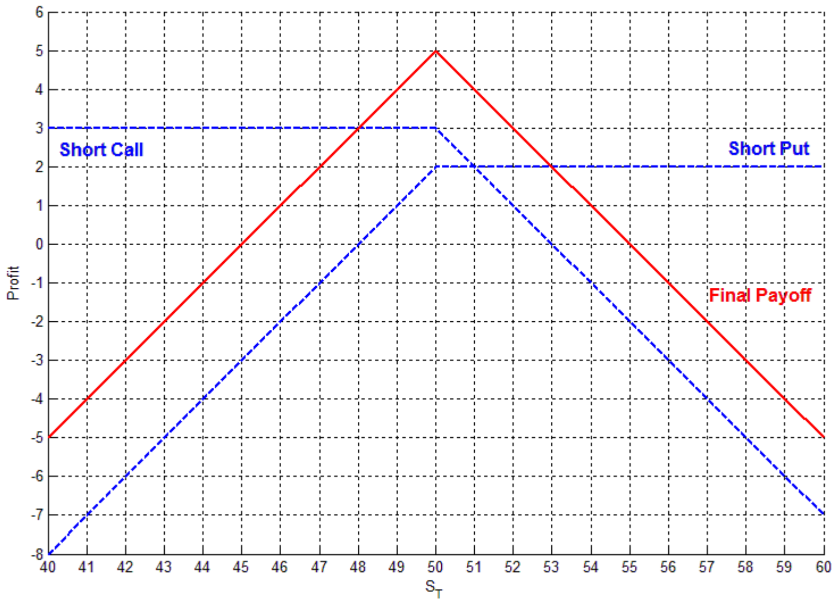


Figure III.91 Top Straddle

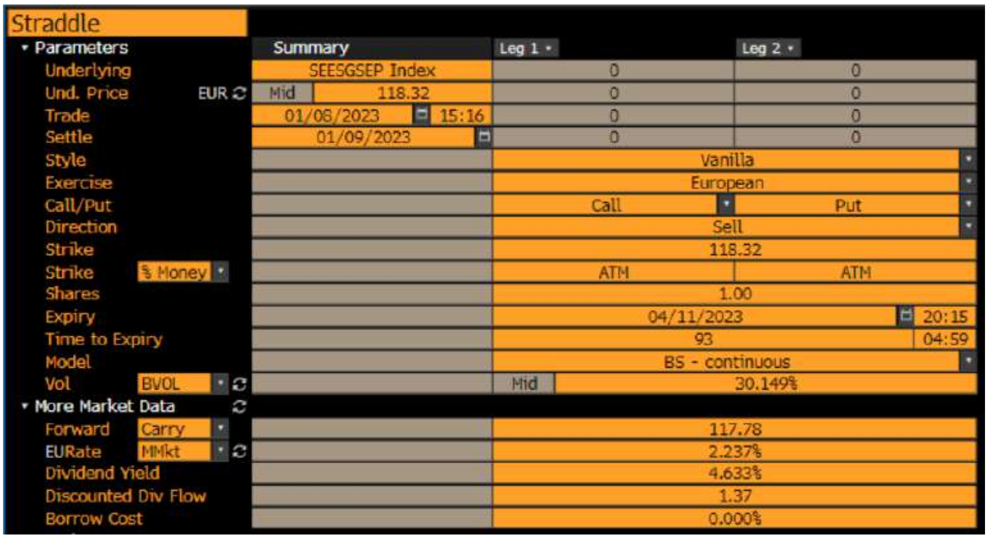


Figure III.92 Top Straddle. Source: Bloomberg®

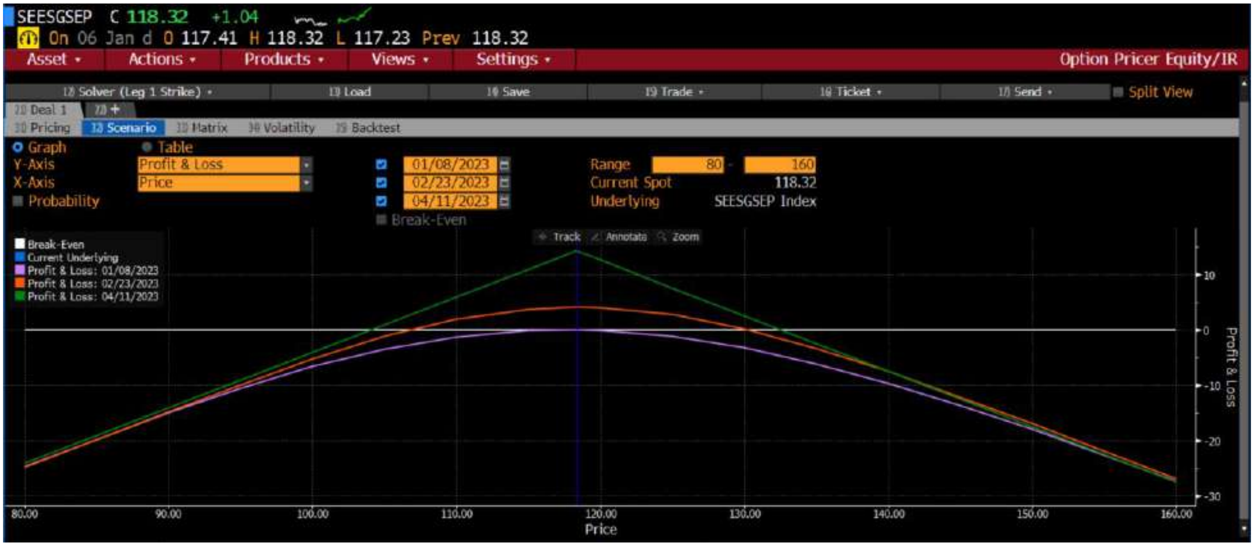


Figure III.93 Top Straddle. Terminal Payoff. Source: Bloomberg®

Another particular strategy is the **strip,** consisting of a long position in a call option and two puts with the same strike price K and the same expiry date.

Stock Price Range	Payoff long call (1)	Payoff long puts (2)	Final Payoff
$S_T \leq K$	0	$2 \cdot (K - S_T)$	$2 \cdot (K - S_T)$
$S_T > K$	$S_T - K$	0	$S_T - K$

Table III.45 Strip

Considering the initial investment, we have: $\Delta^{-}_{premium} = -premium_1 - 2 \cdot premium_2$

Stock Price Range	Payoff long call (1)	Payoff long puts (2)	Final Payoff
$S_T \leq K$	$-premium_1$	$2 \cdot (K - S_T) - 2 \cdot premium_2$	$2 \cdot (K - S_T) + \Delta^{-}_{premium}$
$S_T > K$	$S_T - K - premium_1$	$-2 \cdot premium_2$	$S_T - K + \Delta^{-}_{premium}$

Table III.46 Strip with premiums

In a strip, the investor bets that there will be a large change in the price level of the stock and believes that a decrease in price is more likely than an increase.

Let us present an example of a Strip:

Long position in a call, K $= 50, outflow = 3$
Long position in a put, K $= 50, outflow = 2$
Long position in a put, K $= 50, outflow = 2$

Stock Price Range	Total Payoff
$S_T \leq 50$	$2 \cdot (50 - S_T) - 7$
$S_T > 50$	$S_T - 50 - 7 = S_T - 57$

S(T)	Buy Call (1)	Initial Investment	Buy Puts (2)	Initial Investment	Final pay-off
44	0	-3	12	-4	5
45	0	-3	10	-4	3
46	0	-3	8	-4	1
47	0	-3	6	-4	-1
48	0	-3	4	-4	-3
49	0	-3	2	-4	-5
50	0	-3	0	-4	-7
51	1	-3	0	-4	-6
52	2	-3	0	-4	-5

Table III.47 Strip

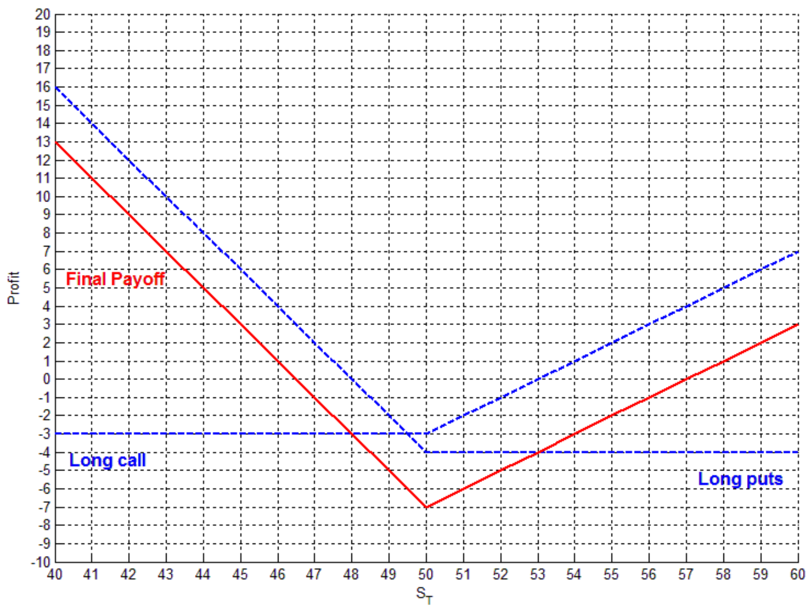


Figure III.94 Strip

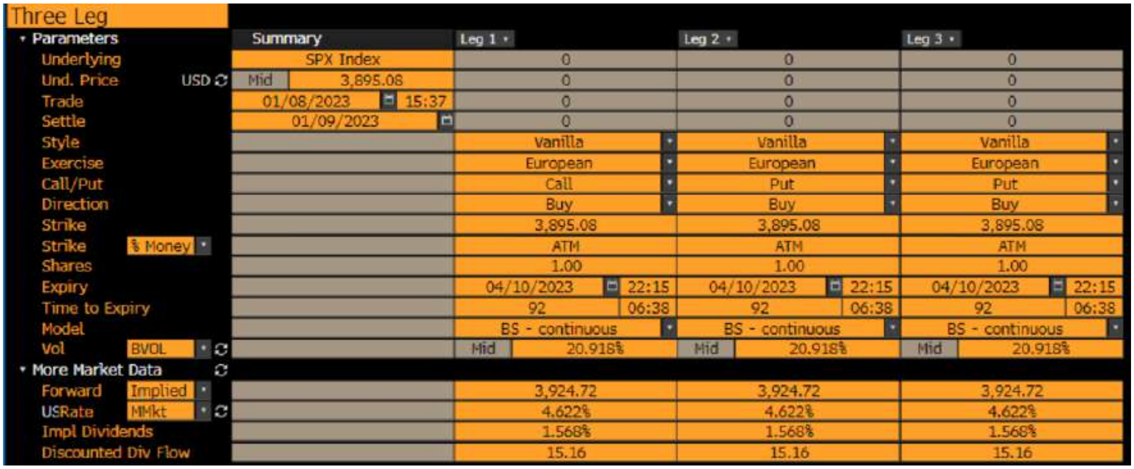


Figure III.95 Strip. Source: Bloomberg®

Figure III.96 Strip. Terminal Payoff. Source: Bloomberg®

We now introduce the strategy called **strap,** which consists of a long position in a put option and two calls, with the same strike price K and the same expiry.

Stock Price Range	Payoff long calls (1)	Payoff long put (2)	Final Payoff
$S_T \leq K$	0	$K - S_T$	$K - S_T$
$S_T > K$	$2 \cdot (S_T - K)$	0	$2 \cdot (S_T - K)$

Table III.48 Strap

Considering the initial investment, we have: $\Delta^{-}_{premium} = -2 \cdot premium_1 - premium_2$

Stock Price Range	Payoff long calls (1)	Payoff long put (2)	Final Payoff
$S_T \leq K$	$-2 \cdot premium_1$	$K - S_T - premium_2$	$K - S_T + \Delta^{-}_{premium}$
$S_T > K$	$2 \cdot (S_T - K) - 2 \cdot premium_1$	$-premium_2$	$2 \cdot (S_T - K) + \Delta^{-}_{premium}$

Table III.49 Strap with premiums

Here is an example of a Strap strategy:

Long position in a call, K $= 50$, $outflow = 3$
Long position in a call, K $= 50$, $outflow = 3$
Long position in a put, K $= 50$, $outflow = 2$

Stock Price Range	Total Payoff
$S_T \leq 50$	$50 - S_T - 8 = 42 - S_T$
$S_T > 50$	$2 \cdot (S_T - 50) - 8$

S(T)	Buy Calls (1)	Initial Investment	Buy Put (2)	Initial Investment	Final pay-off
40	0	-6	10	-2	2
41	0	-6	9	-2	1
42	0	-6	8	-2	0
43	0	-6	7	-2	-1
44	0	-6	6	-2	-2
45	0	-6	5	-2	-3
46	0	-6	4	-2	-4
47	0	-6	3	-2	-5
48	0	-6	2	-2	-6
49	0	-6	1	-2	-7
50	0	-6	0	-2	-8
51	2	-6	0	-2	-6
52	4	-6	0	-2	-4
53	6	-6	0	-2	-2
54	8	-6	0	-2	0
55	10	-6	0	-2	2
56	12	-6	0	-2	4

Table III.50 Strap

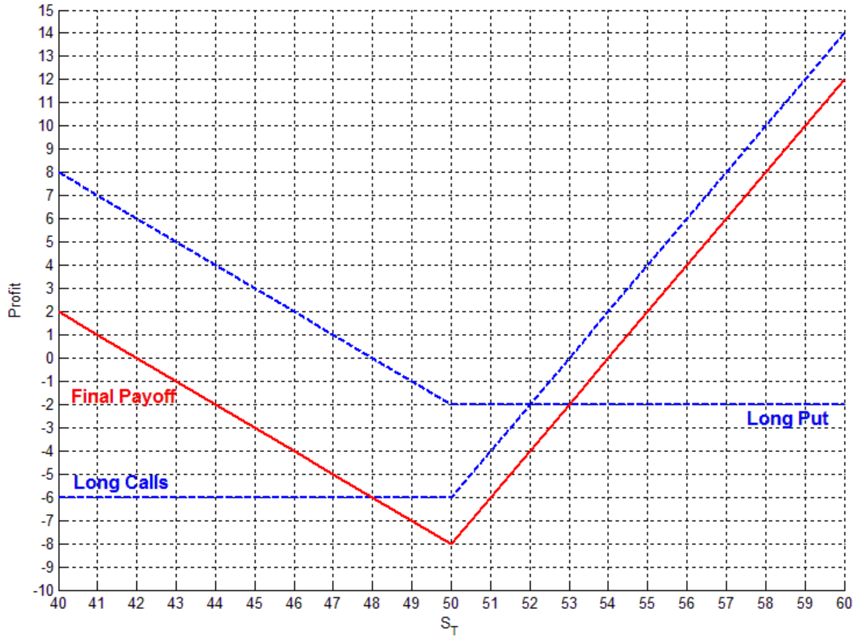
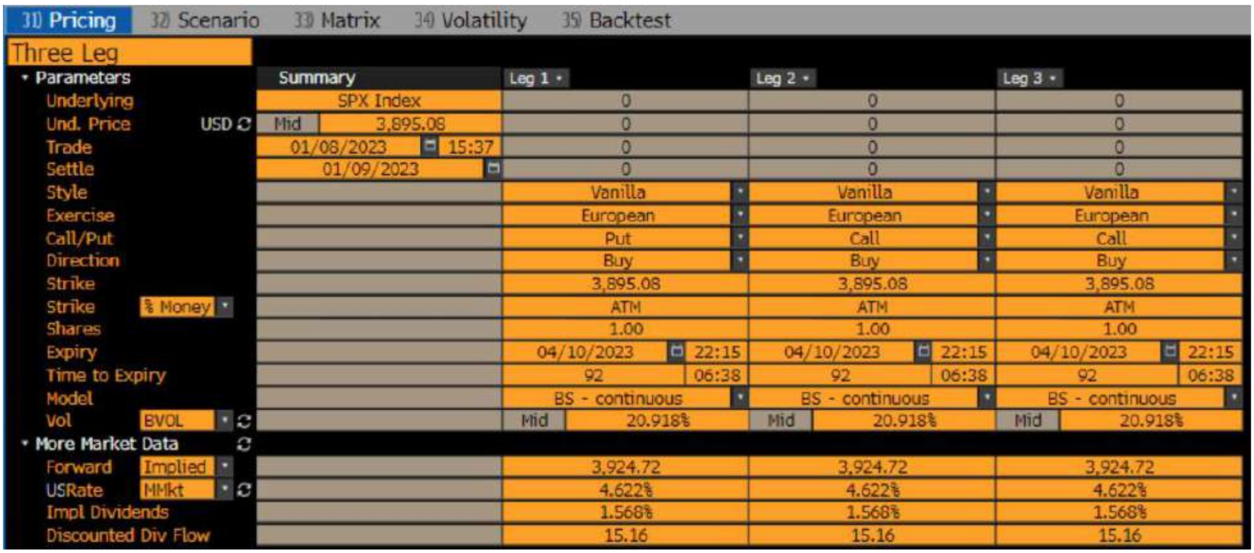


Figure III.97 Strap



31) Pricing | 32) Scenario | 33) Matrix | 34) Volatility | 35) Backtest

Three Leg

▾ Parameters	Summary	Leg 1 ▾	Leg 2 ▾	Leg 3 ▾
Underlying	SPX Index	0	0	0
Und. Price USD	Mid 3,895.08	0	0	0
Trade	01/08/2023 15:37	0	0	0
Settle	01/09/2023	0	0	0
Style		Vanilla	Vanilla	Vanilla
Exercise		European	European	European
Call/Put		Put	Call	Call
Direction		Buy	Buy	Buy
Strike		3,895.08	3,895.08	3,895.08
Strike % Money		ATM	ATM	ATM
Shares		1.00	1.00	1.00
Expiry		04/10/2023 22:15	04/10/2023 22:15	04/10/2023 22:15
Time to Expiry		92 06:38	92 06:38	92 06:38
Model		BS - continuous	BS - continuous	BS - continuous
Vol BVOL		Mid 20.918%	Mid 20.918%	Mid 20.918%
▾ More Market Data				
Forward Implied		3,924.72	3,924.72	3,924.72
USRate MMkt		4.622%	4.622%	4.622%
Impl Dividends		1.568%	1.568%	1.568%
Discounted Div Flow		15.16	15.16	15.16

Figure III.98 Strap. Source: Bloomberg®

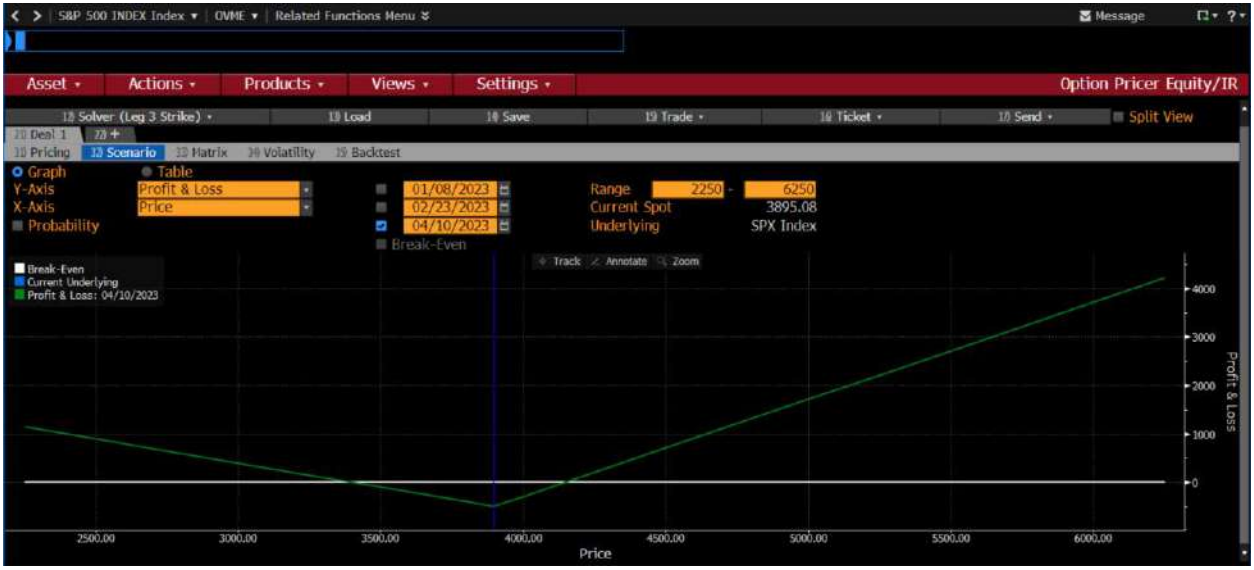


Figure III.99 Strap. Terminal Payoff Source: Bloomberg®

The last strategy we present is the **strangle**, also called **bottom vertical combination**, according to which the investor buys a put and a call option, with the same maturity but different strike prices. The strike price of the call, K_2 is higher than the strike of the put, K_1.

Stock Price Range	Payoff long put (1)	Payoff long call (2)	Final Payoff
$S_T \leq K_1$	$K_1 - S_T$	0	$K_1 - S_T$
$K_1 < S_T < K_2$	0	0	0
$S_T \geq K_2$	0	$S_T - K_2$	$S_T - K_2$

Table III.51 Strangle

Considering the initial investment of the strategy, we have $\Delta_{premium}^{-} = -premium_1 - premium_2$

Stock Price Range	Payoff long put (1)	Payoff long call (2)	Final Payoff
$S_T \leq K_1$	$K_1 - S_T - premium_1$	$-premium_2$	$K_1 - S_T + \Delta_{premium}^{-}$
$K_1 < S_T < K_2$	$-premium_1$	$-premium_2$	$\Delta_{premium}^{-}$
$S_T \geq K_2$	$-premium_1$	$S_T - K_2 - premium_2$	$S_T - K_2 + \Delta_{premium}^{-}$

Table III.52 Strangle with premiums

A strangle is in fact similar to a straddle: the investor bets that the stock will have a strong price change without overreacting on direction. In a strangle, however, the price will have to move more compared to the straddle in order to make a profit.

Here is an example of a Strangle strategy:

Long position in a put, $K_1 = 48, outflow = 2$
Long position in a call, $K_2 = 52, outflow = 1$

Stock Price Range	Total Payoff
$S_T \leq 48$	$48 - S_T - 3 = 45 - S_T$
$48 < S_T < 52$	$-2 - 1 = -3$
$S_T \geq 52$	$S_T - 52 - 3 = S_T - 55$

S(T)	Buy Put (1)	Initial Investment	Buy Call (2)	Initial Investment	Final Pay-off
43	5	-2	0	-1	2
44	4	-2	0	-1	1
45	3	-2	0	-1	0
46	2	-2	0	-1	-1
47	1	-2	0	-1	-2
48	0	-2	0	-1	-3
49	0	-2	0	-1	-3
50	0	-2	0	-1	-3
51	0	-2	0	-1	-3
52	0	-2	0	-1	-3
53	0	-2	1	-1	-2
54	0	-2	2	-1	-1
55	0	-2	3	-1	0
56	0	-2	4	-1	1
57	0	-2	5	-1	2
58	0	-2	6	-1	3
59	0	-2	7	-1	4
60	0	-2	8	-1	5
61	0	-2	9	-1	6
62	0	-2	10	-1	7

Table III.53 Strangle

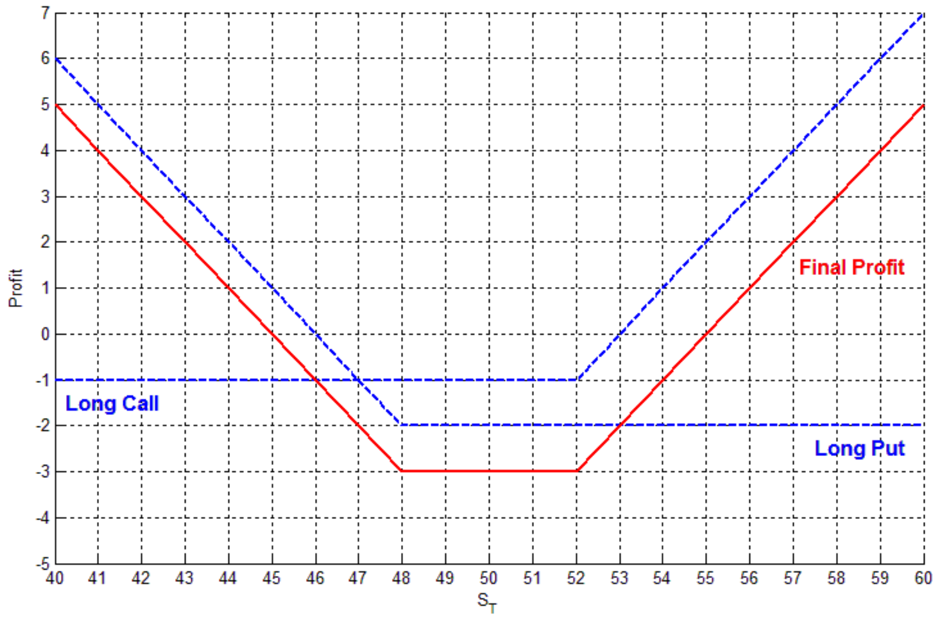


Figure III.100 Strangle

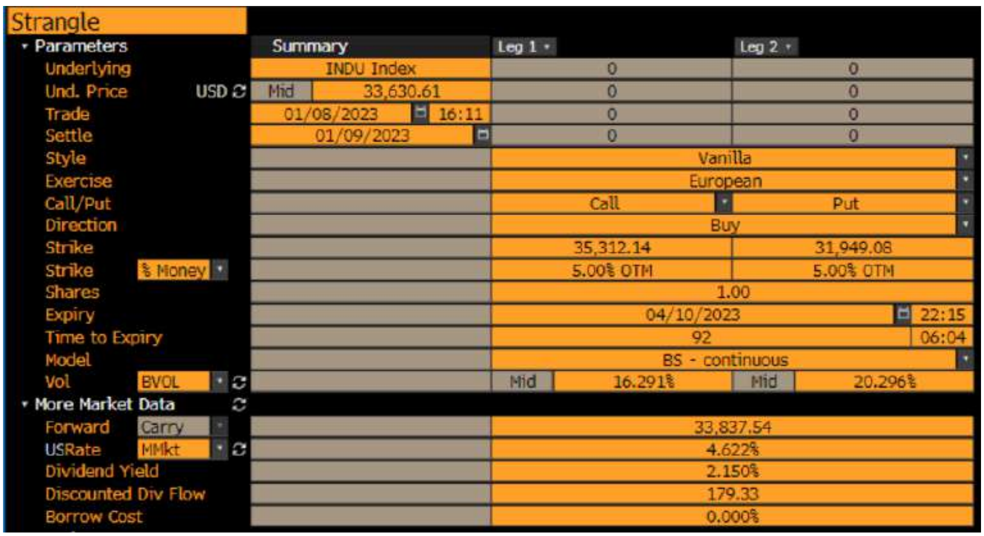


Figure III.101 Strangle. Source: Bloomberg®

Figure III.102 Strangle. Terminal Payoff. Source: Bloomberg®

On the other hand, if the price level of the stock at maturity is close to the central value, the downside risk is lower in the strangle than in a straddle. The profit pattern obtained with a strangle actually depends on how close the strike prices K_1 and K_2 are to each other: the further apart they are, the lower the downside risk and the more distant the price level of the underlying must be in order to make a profit.

In this chapter I have discussed the most widespread techniques using options to produce interesting patterns between a stock price level and profit/hedge. If European options expiring at time T were available for every single possible strike price, any generic pay-off at time T could theoretically be obtained. The easiest way to demonstrate this concept is to consider a series of butterfly spreads. The correct combination of a large number of such strategies can approximate a generic pay-off.

"With derivatives you can have almost any payoff pattern you want. If you can draw it on paper, or describe it in words, someone can design a derivative that gives you that payoff."

Fischer Black

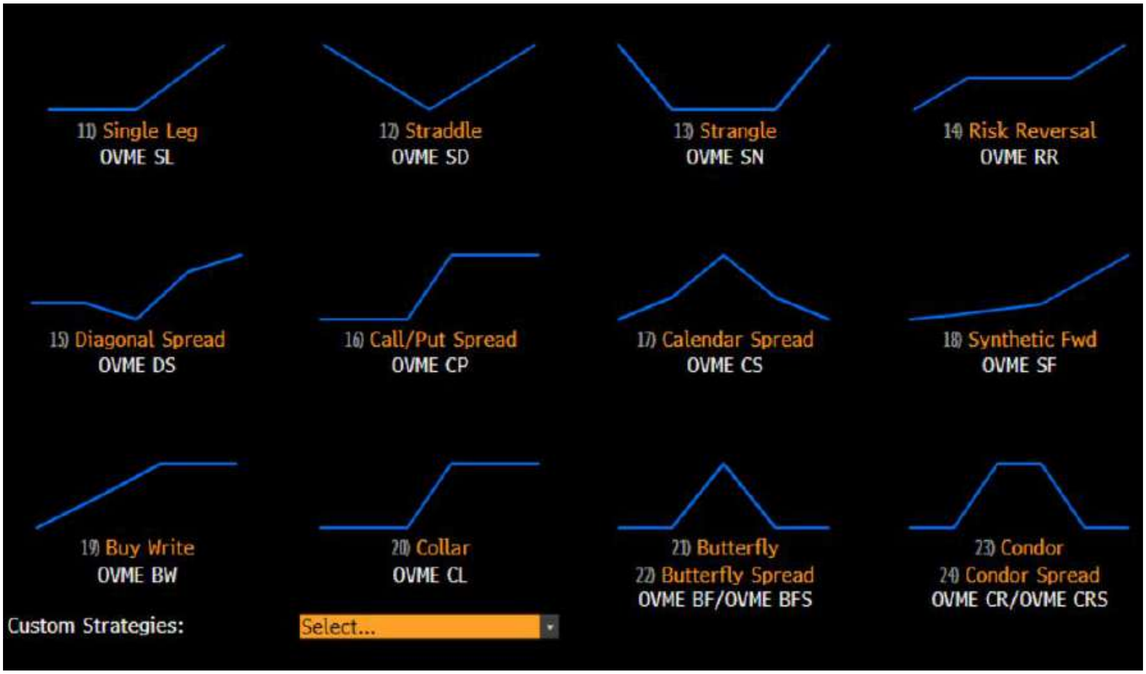


Figure III.103 Option Strategy Menu. OVME Bloomberg® module

FURTHER READINGS

Giribone P. G., Ligato S., Penone F. – "Combining robust Dynamic Neural Networks with traditional technical indicators for generating mechanic trading signals" – International Journal of Financial Engineering (2018).
Hull J. – "Options, Futures and other derivatives" – Pearson (2014).
McMillan L. G. – "Options as a strategic investment" – Prentice Hall Press (2012).
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III.4 EXOTIC OPTIONS

Derivatives such as European/American call/put options are plain-vanilla products as we have seen in the previous chapters. They have well-defined characteristics, and they are actively traded and listed by brokers on a regular basis. A different, and quite interesting aspect of the OTC market is represented by the wide variety of non-standard (or exotic) products. These products are important to investment banks, even if they represent a small part of their portfolio, because they generally allow much higher intermediary margins than those on standard products.

Such **Exotic products** have been created for several reasons. First of all, they answer genuine hedging needs of the operators, with the goal of offsetting the Greeks of their portfolio. Then, they are supported by tax, accounting, legal or regulatory reasons that make them attractive to company treasurers, and lastly, they are structured to reflect a treasurer's particular opinion on the future evolution of a market variable.

It should be noted that options can be sporadically designed with a speculative purpose by financial institutions to make them appear more interesting than they really are to the eyes of a careless treasurer.

This section describes the main types of **single-asset exotic European options** and provide analytical models (exact or approximate) for estimating their values. The analyst must take into account that if he deems the approximation of the closed valuation formula unsatisfactory for his needs, he can resort to the implementation of a numerical methodology (stochastic trees or Monte Carlo). Given the nature of exotic options, which pay-off can be subjected to strong customizations, it becomes almost a titanic undertaking to provide an exhaustive classification of all possible types. Furthermore, it is not always possible to derive a closed valuation formula in accordance with the Black-Scholes-Merton pricing framework, it often becomes necessary to use a numerical algorithm.

The notations for exotic options are consistent with those already used to describe plain-vanilla options.

We start with **Forward start options,** which are derivatives characterized by a deferred start over time, i.e., given times t_0, t_1 and T, this type of options are issued in t_0, but start to be effective from time t_1 with a residual life equal to $T - t_1$ and with strike price $K = S(t_1)$. In other words, the strike price K will be equal to the price level that the underlying will assume at time t_1.

The value of a forward start option is given by the current value of an at-the money option, with a time to maturity equal to $T - t_1$. In fact, discounting the potential final value of a Forward Start at the initial date, it can be expressed as:

$$Value = \exp[-r(T - t_0)] \cdot E\left[\max\left(S_T - S_{t_1}; 0\right)\right] \quad \text{(Eq. III.85)}$$

Analyzing the formula, it can be observed that the final value of the financial instrument depends on:

- The price level S_{t_1} assumed by the financial asset at time t_1.
- The price level S_T assumed by the underlying security at the expiry T of the deferred option.

However, this latter value is a function of the price reached by the underlying financial asset at time t_1, since the two values are linked by the relationship:

$$S_T = S_{t_1} \cdot \exp[+r \cdot (T - t_1)] \quad \textit{(Eq. III.86)}$$

Therefore, the expression of the option value can be rewritten:

$$Value = \exp[-r(T - t_0)] \cdot E(S_T) \cdot E\left(S_T | S_{t_1}\right) \cdot \left[\max\left(S_T - S_{t_1}; 0\right)\right] \quad \textit{(Eq. III.87)}$$

The terms $E(S_T)$ and $E\left(S_T|S_{t_1}\right)$ mean that the option pay-off depends on the expected value that the price of the underlying security will assume at time t_1 and at time T, i.e. respectively S_{t_1} and S_T.

The term $\left[\max\left(S_T - S_{t_1}; 0\right)\right]$ represents the pay-off of an at-the-money plain vanilla call with a time to maturity equal to $T - t_1$. Thus, at time t_1, both the price level of the underlying security and that of the strike price are equal to S_{t_1}; the latter, in turn, is a function of the current spot price as follows:

$$S_{t_1} = S_0 \cdot \exp[+r \cdot (t_1 - t_0)] \quad \textit{(Eq. III.88)}$$

Bearing these relationships in mind, the payoff of a forward start option can be rewritten:

$$Value\ Call = \exp[-r(t_1 - t_0)] \cdot E\left(S_{t_1}\right) \cdot \left[c\left(S_{t_1}, S_{t_1}, T - t_1\right)\right] \quad \textit{(Eq. III.89)}$$

$$Value\ Put = \exp[-r(t_1 - t_0)] \cdot E\left(S_{t_1}\right) \cdot \left[p\left(S_{t_1}, S_{t_1}, T - t_1\right)\right] \quad \textit{(Eq. III.90)}$$

Having obtained an analytically tractable form of pay-off, it is possible to reach a closed valuation formula for this type of derivative.

$$c = S \exp[(b - r)(t_1 - t_0)]\{\exp[(b - r) \cdot (T - t_1)]\, N(d_1) - \exp[-r(T - t_1)] N(d_2)\} \quad \textit{(Eq. III.91)}$$

$$p = S \exp[(b - r)(t_1 - t_0)]\{\exp[-r(T - t_1)] N(-d_2) - \exp[(b - r)(T - t_1)]\, N(-d_1)\} \quad \textit{(Eq. III.92)}$$

Where: $d_1 = \frac{(b + \sigma^2/2) \cdot (T - t_1)}{\sigma \sqrt{T - t_1}}$ and $d_2 = d_1 - \sigma\sqrt{T - t_1}$

With S being the current price of the underlying security, r the risk-free rate, b the cost-of-carry, t_0 the initial valuation time (if today $t_0 = 0$), t_1 the maturity date of the forward start option, σ being the volatility of the underlying, and T being the time to maturity of the option expressed in year fractions.

Let us examine a practical example, considering an investor who goes long on a call option with a 3 month delayed departure from today. The time to maturity is one year from the signing of the derivative contract (i.e. today). The market parameters and financial characteristics are the following:

$S_0 = 30, r = 2\%, b = 1\%, \sigma = 25\%, t_1 = 0.25, T = 1.$

$d_1 = \frac{(b + \sigma^2/2) \cdot (T - t_1)}{\sigma \sqrt{T - t_1}} = 0.142894,\ \ N(d_1) = 0.556813$

$$d_2 = d_1 - \sigma\sqrt{T - t_1} = -0.07361, \;\; N(d_2) = 0.470659$$

$$c = S\exp[(b - r)(t_1 - t_0)]\{\exp[(b - r)\cdot(T - t_1)]\,N(d_1) - \exp[-r(T - t_1)]N(d_2)\}$$

$$c = 29.925 \cdot \{0.992528 \cdot N(d_1) - 0.985112 \cdot N(d_2)\} = 2.6633$$

A second type of exotic option is constituted by **Cliquet options**, also known as **Rachet Options**, which pay the maximum between zero and the sum of the appreciations relating to the price of the underlying financial asset, recorded in certain time intervals over the life of the option.

Appreciations are calculated as the difference between the price of the underlying asset, effective at the end of each reporting period, and the value itself, recorded on the previous Strike Price reset date. Cliquet options are therefore options whose exercise price is determined at periodic intervals with reference to values that reflect market prices. The valuation of this financial instrument using closed formulas can be obtained as the sum of the value of a plain vanilla option, with a time to maturity equal to the first reporting period, and those relating to a series of forward start options, which are equal to the number of sub-periods, stipulated in the option contract.

Therefore, in mathematical terms, we have:

$$Call_{Cliquet} = [S\cdot\exp[-q(t_1 - t_0)]]\cdot N(d_1) - K\cdot\exp[-r(t_1 - t_0)]\,N(d_2) +$$
$$+\sum_{i=2}^{n} S[\exp[-q(t_i - t_0)]N(d_1^*) - \exp[-r(t_i - t_{i-1}) - q(t_{i-1} - t_0)]N(d_2^*)] \quad \textit{(Eq. III.93)}$$

$$Put_{Cliquet} = K\cdot\exp[-r(t_1 - t_0)]\,N(-d_2) - [S\cdot\exp[-q(t_1 - t_0)]]\cdot N(-d_1) +$$
$$+\sum_{i=2}^{n} S[\exp[-r(t_i - t_{i-1}) - q(t_{i-1} - t_0)]N(-d_2^*) - \exp[-q(t_i - t_0)]N(-d_1^*)] \quad \textit{(Eq. III.94)}$$

With:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\cdot(t_1 - t_{i-1})}{\sigma\sqrt{t_1 - t_0}}, \;\; d_2 = d_1 - \sigma\sqrt{t_1 - t_0} \quad \textit{(Eq. III.95)}$$

$$d_1^* = \frac{\left(r - q + \frac{\sigma^2}{2}\right)\cdot(t_i - t_{i-1})}{\sigma\sqrt{t_i - t_{i-1}}}, \;\; d_2^* = d_1^* - \sigma\sqrt{t_i - t_1} \quad \textit{(Eq. III.96)}$$

Unlike vanilla options, **Binary options** or **Digital options** are characterized by a discontinuous pay-off, as their final value depends on the fact that the price of the underlying asset does or does not satisfy a particular condition at maturity. European binary options are divided into three different types:

[A] Cash-or-Nothing Options.
[B] Asset-or-Nothing Options.
[C] Gap Options.

[A] **Cash-or-Nothing Options**

Cash-or-Nothing options provide for the payment of a fixed sum equal to X if, in the case of a call option, the price of the underlying security at maturity exceeds the exercise price K, otherwise the investor will receive nothing. Conversely, in the case of a purchase of a binary put option, the price of the underlying security should be lower than the Strike Price K to obtain the sum X. The pay-off of a Cash-or-Nothing Option can be represented as follows:

Call: If $S(T) \leq K \rightarrow PayOff = 0$; otherwise $PayOff = X$ *(Eq. III.97)*

Put: If $S(T) \geq K \rightarrow PayOff = 0$; otherwise $PayOff = X$ *(Eq. III.98)*

The price of this derivative can be calculated analytically using the Reiner and Rubinstein formulas:

$$Call = X \cdot \exp(-rT)N(d) \quad \textit{(Eq. III.99)}$$

$$Put = X \cdot \exp(-rT)N(-d) \quad \textit{(Eq. III.100)}$$

$$d = \frac{\ln\left(\frac{S}{K}\right) + (b - \sigma^2/2)T}{\sigma\sqrt{T}} \quad \textit{(Eq. III.101)}$$

Where S is the current price of the underlying, r is the risk-free rate, $b = r - q$ represents the cost of carry, σ is the underlying volatility, T is the time to maturity in years, K is the strike price and X is the amount of money of the pay-off.

We now present an example regarding the Reiner and Rubinstein model. We wish to compute the value of a Cash-or-Nothing put option with 9 months to maturity. The spot level is 100, the strike price is 80, the cash payout is 10, the risk-free rate is 1.5% per annum and the annualized volatility is 25%.

Thus: $S = 100, K = 80, X = 10, b = r = 0.015, \sigma = 0.25, T = 0.75$

$$d = \frac{\ln\left(\frac{S}{K}\right) + (b - \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{100}{80}\right) + (0.015 - 0.25^2/2)\cdot 0.75}{0.25\cdot\sqrt{0.75}} = 0.974364$$

$$N(-d) = N(-0.974364) = 0.164938$$

$$Put = X \cdot \exp(-rT)N(-d) = 10 \cdot \exp(-0.015 \cdot 0.75) \cdot 0.164938 = 1.630927$$

[B] **Asset-or-Nothing Options**

Asset-or-Nothing options provide for the payment of the value of a predetermined underlying security in favor of the buyer if, upon expiry of the call, its price level is higher than the Strike Price. For the put option, the underlying price must be lower than the Strike Price. The Pay-off for this category of derivatives can be represented as follows:

Call: If $S(T) \leq K \rightarrow PayOff = 0$; otherwise $PayOff = S(T)$ *(Eq. III.102)*

Put: If $S(T) \geq K \rightarrow PayOff = 0$; otherwise $PayOff = S(T)$ *(Eq. III.103)*

The Cox and Rubinstein analytical pricing formulas are obtained by replacing the sum of money X with the price of the asset on the expiry date, in the equations of case [A].

$$Call = S \cdot \exp[(b-r)T] \cdot N(d)\,,\; Put = S \cdot \exp[(b-r)T] \cdot N(-d) \quad \text{(Eq. III.104)}$$

$$d = \frac{\ln\left(\frac{S}{K}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{(Eq. III.105)}$$

Here is an example of the use of the Cox and Rubinstein model for calculating the value of an Asset-or-Nothing put option with 9 months to maturity. The price level is 100, the strike price is 80, the risk-free rate is 1.5% per annum and the annualized volatility is 25%.

Thus: $S = 100, K = 80, b = r = 0.015, \sigma = 0.25, T = 0.75$

$$d = \frac{\ln\left(\frac{S}{K}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{100}{80}\right) + (0.015 + 0.25^2/2) \cdot 0.75}{0.25 \cdot \sqrt{0.75}} = 1.190871$$

$$N(-d) = N(-1.190871) = 0.116852$$

$$Put = S \cdot \exp[(b-r) \cdot T]N(-d) = 100 \cdot \exp(-0.015 \cdot 0.75) \cdot 0.116852 = 11.5545$$

[C] Gap Options

A call gap option provides for the payment of the difference between the value of the underlying financial asset $S(T)$ and a specific level Z, with $K \neq Z$, under the condition that the underlying price exceeds the strike price set equal to K. Conversely, in the case of a purchase of a put gap option, the investor will have the right to receive the difference between a predetermined level Z of the underlying asset and its price at maturity, conditional on the fact that the final price is lower than the strike price K. In short, the Pay-Off of a Gap Option can be represented as:

Call: If $S(T) \leq K \rightarrow PayOff = 0$; otherwise $PayOff = S(T) - Z$ *(Eq. III.106)*

Put: If $S(T) \geq K \rightarrow PayOff = 0$; otherwise $PayOff = Z - S(T)$ *(Eq. III.107)*

The analytical pricing formulas of this derivative were developed by Reiner and Rubinstein and they are quite similar to the traditional Black-Scholes-Merton formula:

$$Call = S\exp[(b-r)T]N(d_1) - Z\exp(-rT)\,N(d_2) \quad \text{(Eq. III.108)}$$

$$Put = Z\exp(-rT)\,N(-d_2) - S\exp[(b-r)T]N(-d_1) \quad \text{(Eq. III.109)}$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}},\;\; d_2 = d_1 - \sigma\sqrt{T} \quad \text{(Eq. III.110)}$$

Let us now examine an example involving the Reiner and Rubinstein model considering a gap call option that expires in six months. The spot price of the share on which the exotic option is written is 50, the first strike

price is 50, the pay-off strike price is 54, the risk-free rate is 1% p.a., and the volatility is 20% p.a. In short, $S = 50, K = 50, Z = 54, b = r = 0.01, \sigma = 0.20, T = 0.5$.
We calculate and obtain:

$$d_1 = 0.106066, d_2 = -0.03536, N(d_1) = 0.542235, N(d_2) = 0.485898,\ c = 1.004.$$

It should be highlighted that the pay-off of this exotic option can be negative, depending on the price levels of K and Z. Repricing the previous contract with $Z = 57$, we obtain a negative value: $c = -0.44631$.

Another particular kind of exotic option is the **Chooser Option** (also called **As-you-like-it option**), which allows the buyer to choose whether the option in his possession is a call or a put, at a certain date t_1. In other words, the buyer of such a derivative initially pays a premium to decide at a later time t_1 whether to take a bullish (a long call position) or a bearish (a long put position) position on the underlying security. The pay-off of a chooser option can be summarized as follows:

$$PayOff = \max(Call, Put) \quad \textit{(Eq. III.111)}$$

By making the value explicit, as a function of the underlying asset $S(t_1)$, the residual time to maturity $(T_C - t_1)$ and $(T_P - t_1)$, the Strike Price K_C and K_P, we obtain:

$$PayOff = \max[Call(S(t_1), T_C - t_1, K_C), Put(S(t_1), T_P - t_1, K_P)] \quad \textit{(Eq. III.112)}$$

The holder of a chooser option therefore comes into possession of a right which expires at time t_1 and which allows him to choose between a call option with strike price equal to K_C and expiry T_C or a put option with strike price K_P and expiry T_P . The simplest case occurs when the strike price and the time to maturity price coincide for both the Call and the Put options: $K_C = K_P = K$ and $T_C = T_P = T$. The derivative that satisfies this characteristic is called Simple Chooser Option. The analytical pricing formula for the valuation of such a contract was developed by Rubinstein:

$$Value = S \cdot \exp[(b-r)T]N(d) - K \cdot \exp(-rT) \cdot N\left(d - \sigma\sqrt{T}\right) +$$

$$-S \cdot \exp[(b-r)T]\, N(-y) + K \cdot \exp(-rT)\, N(-y + \sigma\sqrt{t_1}) \quad \textit{(Eq. III.113)}$$

Where:

$$d = \frac{\ln\left(\frac{S}{K}\right) + (b + \sigma^2/2) \cdot T}{\sigma\sqrt{T}} \quad \textit{(Eq. III.114)}$$

$$y = \frac{\ln\left(\frac{S}{K}\right) + bT + \sigma^2 t_1/2}{\sigma\sqrt{t_1}} \quad \textit{(Eq. III.115)}$$

We now present a practical example using the Rubinstein model for the valuation of a simple chooser option with a six month maturity. After three months, the holder must decide whether the purchased option is a call or a put. The spot price level of the underlying equity is 50, the strike price is 50 and the risk-free rate is 2% p.a., and volatility is 25% p.a. In short: $S = 50, K = 50, T = 0.5, t_1 = 0.25, r = 0.02, b = 0.02, \sigma = 0.25$

The calculations aimed at deciding which is best between call or put are the following:

$$d = \frac{\ln\left(\frac{S}{K}\right)+(b+\sigma^2/2)\cdot T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{50}{50}\right)+(0.02+0.25^2/2)\cdot 0.5}{0.25\sqrt{0.5}} = 0.144957$$

$$y = \frac{\ln\left(\frac{S}{K}\right)+bT+\sigma^2 t_1/2}{\sigma\sqrt{t_1}} = \frac{\ln\left(\frac{50}{50}\right)+0.02\cdot 0.5+0.25^2\cdot 0.25/2}{0.25\sqrt{0.25}} = 0.1425$$

$$N(d) = 0.5576,\ N\left(d-\sigma\sqrt{T}\right) = 0.4873,\ N(-y) = 0.4433,\ N(-y+\sigma\sqrt{t_1}) = 0.4930$$

$$Value = S\cdot\exp[(b-r)T]N(d) - K\cdot\exp(-rT)\cdot N\left(d-\sigma\sqrt{T}\right) +$$

$$-S\cdot\exp[(b-r)T]\,N(-y) + K\cdot\exp(-rT)\,N(-y+\sigma\sqrt{t_1}) = 5.99696$$

In the case that the maturity date and the strike price of the options are different, $T_C \neq T_P$ and $K_C \neq K_P$ it is not possible to apply the call-put parity and therefore the valuation in the closed formula becomes complicated. Derivatives characterized by this peculiarity are defined as **Complex Chooser Options**. Thus, a complex chooser option allows the holder to choose whether the option is a standard call, after an interval t_1, with maturity T_C and strike price K_C or a put with maturity T_P and strike price K_P. The analytical valuation formulas were first studied by Rubinstein and subsequently by Nelken:

$$Value = S\cdot\exp[(b-r)T_C]\cdot M(d_1,y_1,\rho_1) - K_C\cdot\exp(-rT_C)\,M\left(d_2,y_1-\sigma\sqrt{T_C},\rho_1\right) +$$

$$-S\cdot\exp[(b-r)T_P]\cdot M(-d_1,-y_2,\rho_2) + K_P\cdot\exp(-rT_P)\,M\left(-d_2,-y_2+\sigma\sqrt{T_P},\rho_2\right) \quad \textit{(Eq. III.116)}$$

$$d_1 = \frac{\ln\left(\frac{S}{I}\right)+(b+\sigma^2/2)\cdot t_1}{\sigma\sqrt{t_1}},\ d_2 = d_1-\sigma\sqrt{t_1},\ y_1 = \frac{\ln\left(\frac{S}{K_C}\right)+(b+\sigma^2/2)\cdot T_C}{\sigma\sqrt{T_C}},\ \ y_2 = \frac{\ln\left(\frac{S}{K_P}\right)+(b+\sigma^2/2)\cdot T_P}{\sigma\sqrt{T_P}}$$

$$\rho_1 = \sqrt{\frac{t_1}{T_C}} \text{ and } \rho_2 = \sqrt{\frac{t_1}{T_P}} \quad \textit{(Eq. III.117)}$$

Where:

$M(x,y,\rho\,)$ is the bivariate cumulative normal probability distribution.

I is the solution of the following equation:

$$I\cdot\exp[(b-r)(T_C-t_1)]\cdot N(z_1) - K_C\cdot\exp[-r\cdot(T_C-t_1)]\cdot N\left(z_1-\sigma\sqrt{T_C-t_1}\right) +$$

$$I\cdot\exp[(b-r)(T_P-t_1)]\cdot N(-z_2) - K_P\cdot\exp[-r(T_P-t_1)]\cdot N\left(-z_2-\sigma\sqrt{T_P-t_1}\right) = 0 \quad \textit{(Eq. III.118)}$$

With:

$$z_1 = \frac{\ln\left(\frac{I}{K_C}\right)+(b+\sigma^2/2)\cdot(T_C-t_1)}{\sigma\sqrt{T_C-t_1}} \text{ and } z_2 = \frac{\ln\left(\frac{I}{K_P}\right)+(b+\sigma^2/2)\cdot(T_P-t_1)}{\sigma\sqrt{T_P-t_1}} \quad \textit{(Eq. III.119)}$$

Since the expression is not analytically invertible, it must be solved by implementing a numerical root-finding algorithm, typically the Newton-Raphson iterative procedure. Let us present an example using the Rubinstein-Nelken model for the valuation of a complex chooser option which gives the holder the right to choose whether the option is a call with a maturity of 6 months and a strike price of 55 or a put with 7 months to maturity and a strike price of 48. The time for the investor to decide the nature of the option is 3 months. In short, the market data are: price level of the underlying equal to 50, risk-free rate of 10%, dividend yield equal to 5% p.a., volatility is 35%. Referring to the set of formulas presented, we have: $S = 50$, $K_C = 55$, $K_P = 48$, $T_C = 0.5$, $T_P = 0.5833$, $t_1 = 0.25$, $r = 0.10$, $\sigma = 0.35$ and $b = r - q = 0.1 - 0.05 = 0.05$

The first step is to find the critical value for I, and for this pricing problem it is $I = 51.1158$.

$$I \cdot \exp[(b-r)(T_C - t_1)] \cdot N(z_1) - K_C \cdot \exp[-r \cdot (T_C - t_1)] \cdot N\left(z_1 - \sigma\sqrt{T_C - t_1}\right) +$$
$$I \cdot \exp[(b-r)(T_P - t_1)] \cdot N(-z_2) - K_P \cdot \exp[-r(T_P - t_1)] \cdot N\left(-z_2 - \sigma\sqrt{T_P - t_1}\right) = 0$$
$$I \exp[(-0.05)(0.25)] N(z_1) - 55 \exp[-0.1 \cdot (0.25)] \cdot N\left(z_1 - 0.35\sqrt{0.25}\right) +$$
$$I \exp[(-0.05)(0.3333)] N(-z_2) - 48 \exp[-0.1(0.3333)] N\left(-z_2 - 0.35\sqrt{0.3333}\right) = 0$$

$$z_1 = \frac{\ln\left(\frac{I}{K_C}\right) + (b + \sigma^2/2) \cdot (T_C - t_1)}{\sigma\sqrt{T_C - t_1}} = \frac{\ln\left(\frac{I}{55}\right) + (0.05 + 0.35^2/2) \cdot (0.25)}{0.35\sqrt{0.25}}$$

$$z_2 = \frac{\ln\left(\frac{I}{K_P}\right) + (b + \sigma^2/2) \cdot (T_P - t_1)}{\sigma\sqrt{T_P - t_1}} = \frac{\ln\left(\frac{I}{48}\right) + (0.05 + 0.35^2/2) \cdot (0.3333)}{0.35 \cdot \sqrt{0.3333}}$$

Implementing a root-finding algorithm we obtain $I = 51.1158$. We proceed and calculate:

$$d_1 = \frac{\ln\left(\frac{S}{I}\right) + (b + \sigma^2/2) \cdot t_1}{\sigma\sqrt{t_1}} = \frac{\ln\left(\frac{50}{50.1158}\right) + (0.05 + 0.35^2/2) \cdot 0.25}{0.35 \cdot \sqrt{0.25}} = 0.032811$$
$$d_2 = 0.032811 - 0.35 \cdot \sqrt{0.25} = -0.14219$$

$$y_1 = \frac{\ln\left(\frac{S}{K_C}\right) + (b + \sigma^2/2) \cdot T_C}{\sigma\sqrt{T_C}} = \frac{\ln\left(\frac{50}{55}\right) + (0.05 + 0.35^2/2) \cdot 0.5}{0.35 \cdot \sqrt{0.5}} = -0.16035$$

$$y_2 = \frac{\ln\left(\frac{S}{K_P}\right) + (b + \sigma^2/2) \cdot T_P}{\sigma\sqrt{T_P}} = \frac{\ln\left(\frac{50}{48}\right) + (0.05 + 0.35^2/2) \cdot 0.5833}{0.35 \cdot \sqrt{0.5833}} = 0.3955$$

$$\rho_1 = \sqrt{\frac{t_1}{T_C}} = \sqrt{\frac{0.25}{0.5}} = 0.7071,\ \rho_2 = \sqrt{\frac{t_1}{T_P}} = \sqrt{\frac{0.25}{0.5833}} = 0.6547,\ M(d_1, y_1, \rho_1) = 0.3464,$$
$$M\left(d_2, y_1 - \sigma\sqrt{T_C}, \rho_1\right) = 0.2660,\ M(-d_1, -y_2, \rho_2) = 0.2725,\ M\left(-d_2, -y_2 + \sigma\sqrt{T_P}, \rho_2\right) = 0.3601$$

$$Value = S \cdot \exp[(b-r)T_C] \cdot M(d_1, y_1, \rho_1) - K_C \cdot \exp(-rT_C)\, M\left(d_2, y_1 - \sigma\sqrt{T_C}, \rho_1\right) +$$
$$-S \cdot \exp[(b-r)T_P] \cdot M(-d_1, -y_2, \rho_2) + K_P \cdot \exp(-rT_P)\, M\left(-d_2, -y_2 + \sigma\sqrt{T_P}, \rho_2\right) =$$
$$Value = 50 \cdot \exp[(0.05 - 0.1) \cdot 0.5] \cdot 0.3464 - 55 \cdot \exp(-0.1 \cdot 0.5) \cdot 0.2660 +$$
$$-50 \exp[(0.05 - 0.1) \cdot 0.5833] \cdot 0.2725 + 48 \exp(-0.05 \cdot 0.5833) \cdot 0.3601 = 6.0479$$

Yet another type of exotic options is constituted by **Compound options** which are options written on other options, i.e. the underlying asset is an option rather than a stock or an index. In other words, Compound Options give the holder the right to buy (Call) or sell (Put) at maturity at a price K_{CO} (compound strike) defined ex-ante, an option on a certain underlying financial asset, which starts at a deferred time t_1 and expiring at a later time T, upon payment of a strike price equal to K (plain vanilla strike price). There are four different types of possible compositions:

- Call on call c_{CALL},
- Call on put c_{PUT},
- Put on call p_{CALL}
- Put on put p_{PUT}.

Indicating with t_1 the expiration date of the compound option, with T the maturity date of the underlying option, with $t_1 < T$ and with K_{CO} and K the strike price of the compound option and of the underlying option, then, respectively the pay-offs for the aforementioned four categories of compound options are defined as follows:

$$Payoff\ Call\ on\ call = \max \begin{cases} 0\ ; if\ the\ underlying\ call\ value\ is \leq K_{CO} \\ (underlying\ call\ value - K_{CO})\ ; otherwise \end{cases}$$

$$Payoff\ Call\ on\ put = \max \begin{cases} 0\ ; if\ the\ underlying\ put\ value\ is\ \leq K_{CO} \\ (underlying\ put\ value\ - K_{CO})\ ; otherwise \end{cases}$$

$$Payoff\ Put\ on\ put = \max \begin{cases} 0\ ; if\ the\ underlying\ put\ value\ is\ \geq K_{CO} \\ (K_{CO} - underlying\ put\ value)\ ; otherwise \end{cases}$$

$$Payoff\ Put\ on\ call = \max \begin{cases} 0\ ; if\ the\ underlying\ call\ value\ is\ \geq K_{CO} \\ (K_{CO} - underlying\ call\ value\ )\ ; otherwise \end{cases}$$

From the analysis of the potential payoff values relating to the four categories of compound options, we can see that their value is a function of the value of the underlying options. Therefore, to value a compound option it is necessary to know the critical price $I = S^*(t_1)$ of the financial asset covered by the option underlying the compound option such as to make the latter at-the-money with respect to the strike price K_{CO}. In formal terms, if the value of the underlying option is expressed as a function of the value of the underlying stock $I = S^*(t_1)$, the residual life $T - t_1$ and the strike price K, the above mentioned break-even condition can be written as:

$$Underlying\ Option\ Value(I, T - t_1, K) = K_{CO} \quad \textit{(Eq. III.120)}$$

Based on this condition it follows that, in the case of a call option on call (c_{CALL}) and put on put (p_{PUT}), if it occurs that $S(t_1) > S^*(t_1)$ then the two compound options will be exercised at expiry t_1. On the other hand, in the case of a call option on put (c_{PUT}) and a put option on call (p_{CALL}), if it occurs that $S(t_1) < S^*(t_1)$ then the two compound options will be exercised at expiry t_1. Therefore, the expiry value of a compound option can be written as a function of the value that S will assume on expiry t_1 of the compound option. The valuation formulas for this type of exotic derivative were initially published by Geske in 1977 and 1979. The solution was subsequently addressed and discussed by researchers Hodges and Selby in 1987 and further explored by Rubinstein in 1991.

$$c_{CALL} = S\exp[(b-r)T]\, M(z_1, y_1, \rho) - K\exp(-rT)\, M(z_2, y_2, \rho) - K_{CO}\exp(-rt_1)\, N(y_2) \quad \textit{(Eq. III.121)}$$

$$p_{CALL} = K\exp(-rT)\, M(z_2, -y_2, -\rho) - S\exp[(b-r)T]\, M(z_1, -y_1, -\rho) + K_{CO}\exp(-rt_1)\, N(-y_2) \quad \textit{(Eq. III.122)}$$

$$c_{PUT} = K\exp(-rT)\, M(-z_2, -y_2, \rho) - S\exp[(b-r)T]\, M(-z_1, -y_1, \rho) - K_{CO}\exp(-rt_1)\, N(-y_2) \quad \textit{(Eq. III.123)}$$

$$p_{PUT} = S\exp[(b-r)T]\, M(-z_1, y_1, -\rho) - K\exp(-rT)\, M(-z_2, y_2, -\rho) + K_{CO}\exp(-rt_1)\, N(y_2) \quad \textit{(Eq. III.124)}$$

Where:

$$z_1 = \frac{\ln\left(\frac{S}{K}\right) + (b + \sigma^2/2)\cdot T}{\sigma\sqrt{T}},\ \ z_2 = z_1 - \sigma\sqrt{T},\ \ \rho = \sqrt{\frac{t_1}{T}} \text{ and } y_1 = \frac{\ln\left(\frac{S}{I}\right) + (b + \sigma^2/2)\cdot t_1}{\sigma\sqrt{t_1}},\ \ y_2 = y_1 - \sigma\sqrt{t_1}$$

The value of I is computed through a root-finding algorithm, such as the Newton-Raphson method:

- For c_{CALL} or p_{CALL}: $Underlying\ Call(I, K, T - t_1) = K_{CO}$ *(Eq. III.125)*
- For c_{PUT} or p_{PUT}: $Underlying\ Put(I, K, T - t_1) = K_{CO}$ *(Eq. III.126)*

We now analyze an example using the Geske model, considering a call put that gives the investor the right to sell a call option at 50 (K_{CO}) three months from today (t_1). The strike price on the underlying call is 520 (K) and its time to expiry is 6 months from today (T). The price of the underlying stock index is 500 (S), the risk-free rate is 8% (r), the dividend-yield of the index is 3% (q) and the volatility is 35% (σ). In short, $S = 500$, $K = 520$, $K_{CO} = 50$, $t_1 = 0.25$, $T = 0.5$, $r = 0.08$, $b = 0.08 - 0.03 = 0.05$, $\sigma = 0.35$.

We start by computing the critical value I:

$$Underlying\ Call(I, K, T - t_1) = K_{CO}$$

$$Underlying\ Call(I, 520, 0.5 - 0.25) = 50$$

$$I \cdot \exp[(b-r)\cdot T] \cdot N(d_1) - K \cdot \exp(-rT) \cdot N(d_2) = 50$$

$$d_1 = \frac{\ln\left(\frac{I}{K}\right) + (b + \sigma^2/2) \cdot (T - t_1)}{\sigma\sqrt{T - t_1}} = \frac{\ln\left(\frac{I}{520}\right) + (0.05 + 0.35^2/2) \cdot 0.25}{0.35 \cdot \sqrt{0.25}}$$

$$d_2 = d_1 - \sigma\sqrt{T - t_1} = d_1 - 0.35 \cdot \sqrt{0.25}$$

The value of I that solves the equation is 538.3165.

Consequently, we compute:

$$y_1 = \frac{\ln\left(\frac{S}{I}\right) + (b + \sigma^2/2) \cdot t_1}{\sigma\sqrt{t_1}} = \frac{\ln\left(\frac{500}{538.3165}\right) + (0.05 + 0.35^2/2) \cdot 0.25}{0.35 \cdot \sqrt{0.25}} = -0.26301$$

$$y_2 = y_1 - \sigma\sqrt{t_1} = -0.26301 - 0.35 \cdot \sqrt{0.25} = -0.43801$$

$$z_1 = \frac{\ln\left(\frac{S}{K}\right) + (b + \sigma^2/2) \cdot T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{500}{520}\right) + (0.05 + 0.35^2/2) \cdot 0.5}{0.35 \cdot \sqrt{0.5}} = 0.06628$$

$$z_2 = z_1 - \sigma\sqrt{T} = 0.06628 - 0.35 \cdot \sqrt{0.5} = -0.1812$$

$$\rho = \sqrt{\frac{t_1}{T}} = 0.7071,\ M(z_2, -y_2, -\rho) = 0.1736,\ M(z_1, -y_1, -\rho) = 0.1996,\ N(-y_2) = 0.6693$$

$$p_{CALL} = K \exp(-rT)\, M(z_2, -y_2, -\rho) - S \exp[(b - r)T]\, M(z_1, -y_1, -\rho) + K_{CO} \exp(-rt_1)\, N(-y_2)$$

$$p_{CALL} = 520 \exp(-0.08 \cdot 0.5) \cdot 0.1736 - 500 \exp[-0.03 \cdot 0.5] \cdot 0.1996 + 50 \exp(-0.25) \cdot 0.6693$$

The value of the put on call considered is therefore equal to $p_{CALL} = 21.221$.

The options whose pay-off depends on a set of values assumed by the underlying during the life of the derivative are called **path-dependent options**. The most common categories belonging to this class are:

- **Asian** options.
- **Barrier** options.
- **Lookback** options.

We start by examining **Asian options,** which are options with a final value depending on the average price of the underlying asset observed during the life of the option. The final value (pay-off) of a call written on the **average price call** is $\max(S_{AVG} - K, 0)$ and the pay-off of a put written on the average price (**average price put**) is $\max(K - S_{AVG}, 0)$, where S_{AVG} is the average price of the underlying asset calculated over a predetermined period. Average price options are generally less expensive than ordinary options and are perhaps better suited than ordinary options to meet certain needs of treasurers.

Let us now suppose the treasurer of a US company expects USD 100 million in cash flow from the Australian subsidiary, spread evenly over the next year. It is probable that the treasurer is interested in an option that guarantees him an annual average exchange rate above a certain level. An average price put can achieve this more effectively than ordinary puts.

Another type of Asian option is the option with an average strike price: the pay-off of an **average strike call** is: $\max(S_T - S_{AVG}, 0)$ and that of an **average strike put** is $\max(S_{AVG} - S_T, 0)$. Average strike price options can ensure that the average price paid for a frequently traded asset over a period of time is not greater than the final price. Alternatively, they can ensure that the average price received for a frequently traded asset over a period of time is not lower than the final price.

While in practice most Asian options traded are based on arithmetic averages, it is also true that the application of the geometric mean in the valuation of Asian European options plays an important role. This is due to the fact that, unlike the arithmetic mean, for the geometric mean exact closed formulas can be obtained for their valuation. Starting from the hypothesis that the asset underlying the option is distributed according to a log-normal distribution, the geometric mean: $\prod_{i=1}^{N}(S_i)^{\frac{1}{N}}$ will itself be distributed log-normally, thus the Asian option based on the geometric mean will have pay-off: $\max\left(\prod_{i=1}^{N}(S_i)^{\frac{1}{N}} - K, 0\right)$ if call, $\max\left(K - \prod_{i=1}^{N}(S_i)^{\frac{1}{N}}, 0\right)$ if put. Considering this type of average, the financial instrument can correctly be priced starting from the generalized Black and Scholes formula (GBS formula), introducing a small adjustment on the volatility (σ) and on the cost-of-carry (b). Therefore:

$$c = S_0 \exp\left[\left(b_{ADJ} - r\right)T\right]N(d_1) - K\exp(-rT)\,N(d_2) \quad \textit{(Eq. III.127)}$$

$$p = K\exp(-rT)\,N(-d_2) - S_0 \exp\left[\left(b_{ADJ} - r\right)T\right]N(-d_1) \quad \textit{(Eq. III.128)}$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(b_{ADJ} + \sigma_{ADJ}^2/2\right)T}{\sigma_{ADJ}\sqrt{T}}, d_2 = d_1 - \sigma_{ADJ}\sqrt{T} \quad \textit{(Eq. III.129)}$$

With the adjusted volatility $\sigma_{ADJ} = \frac{\sigma}{\sqrt{3}}$ and the adjusted cost-of-carry $b_{ADJ} = \frac{1}{2}\left(b - \frac{\sigma^2}{6}\right)$.

Let us, for example, price a continuous geometric-average put option with a 9-month maturity considering a strike price of 85, an initial price of the underlying equal to 80, a risk-free rate of 2%, a cost-of-carry equal to 5% and a volatility of 20%. In short, we have:

$S_0 = 80$, $K = 85$, $T = 0.25$, $r = 0.02$, $b = 0.05$ and $\sigma = 0.20$.

We thus calculate:

$$\sigma_{ADJ} = \frac{\sigma}{\sqrt{3}} = \frac{0.20}{\sqrt{3}} = 0.11547,\ b_{ADJ} = \frac{1}{2}\left(0.05 - \frac{0.2^2}{6}\right) = 0.021667$$

$$d_1 = \frac{\ln\left(\frac{80}{85}\right) + \left(0.021667 + 0.11547^2/2\right)\cdot 0.75}{0.11547\sqrt{0.75}} = -0.39375 \rightarrow N(-d_1) = 0.653116$$

$$d_2 = -0.39375 - 0.11547\sqrt{0.75} = -0.49375 \rightarrow N(-d_2) = 0.689257$$

$$p = 85\exp(-0.02\cdot 0.75)\cdot 0.689257 - 80\exp[(0.021667 - 0.02)\cdot 0.75]\cdot 0.653116 = 5.400009$$

When Asian options are defined in terms of arithmetic averages, analytical valuation formulas are not available though. This depends on the fact that the distribution of the arithmetic mean of a set of variables distributed according to a lognormal does not have properties that make it analytically tractable. The way to valuate options written on arithmetic means is through an analytical approximation: it is a question of calculating exactly the first two moments of the probabilistic distribution of the arithmetic mean in a risk-neutral world and therefore assuming that this distribution is log-normal. Such approach was proposed by Turnbull and Wakeman in 1991. Let us examine this approach in mathematical terms, considering a newly issued Asian option whose final value at time T is based on an arithmetic mean between time zero and time T. In a risk-neutral world, the first moment, M_1, and the second moment, M_2, of the mean are respectively:

$$M_1 = \frac{\exp(bT)-1}{bT}, \quad M_2 = \frac{2\exp\{[2b+\sigma^2]T\}}{(b+\sigma^2)(2b+\sigma^2)T^2} + \frac{2}{bT^2}\left[\frac{1}{2b+\sigma^2} - \frac{\exp[bT]}{b+\sigma^2}\right] \quad \textit{(Eq. III.130)}$$

The calculation of the two moments is used as an adjustment factor within the expressions of volatility, σ_{ADJ} and Cost of Carry, b_{ADJ}, of the generalized formula of Black and Scholes for call and put options:

$$b_{ADJ} = \frac{\ln(M_1)}{T}, \; \sigma_{ADJ} = \sqrt{\frac{\ln(M_2)}{T} - 2b_{ADJ}} \quad \textit{(Eq. III.131)}$$

Therefore, the approximate value of an option is given by:

$$c \approx S_0 \exp\left[\left(b_{ADJ} - r\right)T\right]N(d_1) - K\exp(-rT)\,N(d_2) \quad \textit{(Eq. III.132)}$$

$$p \approx K\exp(-rT)\,N(-d_2) - S_0 \exp\left[\left(b_{ADJ} - r\right)T\right]N(-d_1) \quad \textit{(Eq. III.133)}$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(b_{ADJ} + \sigma_{ADJ}^2/2\right)T}{\sigma_{ADJ}\sqrt{T}}, \; d_2 = d_1 - \sigma_{ADJ}\sqrt{T} \quad \textit{(Eq. III.134)}$$

We now present an example using the Turnbull and Wakeman model for the valuation of the previous put option, assuming that the arithmetic rather than the geometric mean of the prices is used. The set of data is: $S_0 = 80$, $K = 85$, $T = 0.75$, $r = 0.02$, $b = 0.05$ and $\sigma = 0.20$, and we compute as follows:

$$M_1 = \frac{\exp(bT)-1}{bT} = \frac{\exp(0.05 \cdot 0.75)-1}{0.05 \cdot 0.75} = 1.0189866$$

$$M_2 = \frac{2\exp\{[2 \cdot 0.05 + 0.2^2] \cdot 0.75\}}{(0.05+0.2^2)(2 \cdot 0.05+0.2^2) \cdot 0.75^2} + \frac{2}{0.05 \cdot 0.75^2}\left[\frac{1}{2 \cdot 0.05 + 0.2^2} - \frac{\exp[0.05 \cdot 0.75]}{0.05+0.2^2}\right] = 1.0488941$$

$$b_{ADJ} = \frac{\ln(1.0189866)}{0.75} = 0.0250781, \; \sigma_{ADJ} = \sqrt{\frac{\ln(1.0488941)}{0.75} - 2 \cdot 0.0250781} = 0.1161562$$

$$d_1 = \frac{\ln\left(\frac{80}{85}\right) + \left(0.0250781 + 0.1161562^2/2\right) \cdot 0.75}{0.1161562 \cdot \sqrt{0.75}} = -0.365393 \rightarrow N(-d_1) = 0.642591$$

$$d_2 = -0.365393 - 0.1161562\sqrt{0.75} = -0.465987 \rightarrow N(-d_2) = 0.6793877$$

$$p \approx 85\exp(-0.02 \cdot 0.75) \cdot 0.6793877 - 80\exp[(0.0250781 - 0.02)0.75] \cdot 0.642591 = 5.2847579.$$

We now introduce **Barrier options,** i.e., options whose final value depends on whether the price of the underlying asset reaches a predetermined level in a certain period of time or not. Several barrier options are regularly traded in the OTC market: they generally are less expensive than ordinary options, as they cease to exist (or begin to be valid) if the limit imposed by the barrier is exceeded by the underlying asset. Barrier options can be divided into **knock-out** options and **knock-in** options. The first type ceases to exist when the price of the underlying asset reaches a certain barrier, while the second type only begins to exist when the price of the underlying asset reaches the predetermined barrier.

Down-and-out call options are knock-out options: they are ordinary calls that cease to exist when the price of the underlying asset falls under a certain barrier level H, with $H < S_0$. The corresponding knock-in options are represented by **down-and-in** calls, i.e., ordinary call options that start to exist only when the price of the underlying asset falls to H, with $H < S_0$. Thus, if $H \leq K$, the current value c_{DI} for a down-and-in call option is:

$$c_{DI} = S_0 \cdot \exp(-qt) \cdot \left(\frac{H}{S_0}\right)^{2\lambda} N(y) - K\exp(-rt) \cdot \left(\frac{H}{S_0}\right)^{2\lambda-2} N\left(y - \sigma\sqrt{T}\right) \quad \textit{(Eq. III.135)}$$

$$\lambda = \frac{r-q+\sigma^2/2}{\sigma^2},\ y = \frac{\ln\left(\frac{H^2}{S_0 K}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \quad \textit{(Eq. III.136)}$$

Given that an ordinary call option (c) equals the sum of the corresponding down-and-in and down-and-out calls, the current value, c_{DO}, of a down-and-out call is: $c_{DO} = c - c_{IN}$.

On the other hand, if $H > K$, the value of a **down-and-out call** option is the following:

$$c_{DO} = S_0 \cdot \exp(-qt) \cdot N(x_1) - K\exp(-rt) \cdot N\left(x_1 - \sigma\sqrt{T}\right) +$$

$$-S_0 \cdot \exp(-qt)\left(\frac{H}{S_0}\right)^{2\lambda} N(y_1) + K \cdot \exp(-rt)\left(\frac{H}{S_0}\right)^{2\lambda-2} N\left(y_1 - \sigma\sqrt{T}\right) \quad \textit{(Eq. III.137)}$$

$$x_1 = \frac{\ln\left(\frac{S_0}{H}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}\ ,\ y_1 = \frac{\ln\left(\frac{H}{S_0}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \quad \textit{(Eq. III.138)}$$

Similarly to the previous case, the value of a **down-and-in call** is given by $c_{DI} = c - c_{DO}$.

Up-and-out calls are also knock-out options. In this case, as the name suggests, they are ordinary call options that cease to exist when the price of the underlying asset rises to H, with $H > S_0$.
The corresponding knock-in options are **up-and-in calls**. They are ordinary calls that begin to exist when the price of the underlying asset rises to H, with $H > S_0$.
By definition, if $H \leq K$, then the current value of an up-and-out call option, c_{UO} is null and the current value of an up-and-in call, c_{UI}, is equal to a vanilla call c.

In case $H > K$, the value for an up-and-in call is:

$$c_{UI} = S_0 \cdot \exp(-qt) \cdot N(x_1) - K \exp(-rt) \cdot N\left(x_1 - \sigma\sqrt{T}\right) +$$
$$-S_0 \cdot \exp(-qt) \left(\frac{H}{S_0}\right)^{2\lambda} [N(-y) - N(-y_1)] +$$
$$+K \cdot \exp(-rt) \left(\frac{H}{S_0}\right)^{2\lambda-2} \left[N\left(-y + \sigma\sqrt{T}\right) - N\left(-y_1 + \sigma\sqrt{T}\right)\right] \quad \textit{(Eq. III.139)}$$

While the current value of an up-and-out call is $c_{UO} = c - c_{UI}$.

Barrier put options are defined similarly to their corresponding calls. Standard up-and-out put options cease to exist when the price of the underlying asset rises to H, with $H > S_0$.

Standard up-and-in puts only start to exist when the price of the underlying asset rises to H, with $H > S_0$. If $H > K$, the current value of an up-and-in put, p_{UI} is:

$$p_{UI} = -S_0 \cdot \exp(-qT) \left(\frac{H}{S_0}\right)^{2\lambda} N(-y) + K \cdot \exp(-rT) \left(\frac{H}{S_0}\right)^{2\lambda-2} N\left(-y + \sigma\sqrt{T}\right) \quad \textit{(Eq. III.140)}$$

While the current value of an up-and-out put is $p_{UO} = p - p_{UI}$. In case $H \leq K$, the current value of an up-and-out put option is equal to:

$$p_{UO} = -S_0 \cdot \exp(-qT)\, N(-x_1) + K \cdot \exp(-rT)\, N\left(-x_1 + \sigma\sqrt{T}\right) +$$
$$+S_0 \cdot \exp(-qT) \left(\frac{H}{S_0}\right)^{2\lambda} N(-y_1) - K \cdot \exp(-rT)\, N\left(-y_1 + \sigma\sqrt{T}\right) \left(\frac{H}{S_0}\right)^{2\lambda} \quad \textit{(Eq. III.141)}$$

While the current value of an up-and-in put is $p_{UI} = p - p_{UO}$.

Down-and-out puts cease to exist when the price of the underlying asset falls to H, with $H < S_0$. **Down-and-in puts** only start to exist when the price of the underlying asset falls to H, with $H < S_0$. Thus, if $H \geq K$, the current value, p_{DO}, of a down-and-out put is zero and the current value, p_{DI}, of a down-and-in put is equal to p. Conversely, if $H < K$, the current value of a down-and-in put option , p_{DI} is:

$$p_{DI} = -S_0 \cdot \exp(-qT)\, N(-x_1) + K \cdot \exp(-rT)\, N\left(-x_1 + \sigma\sqrt{T}\right) +$$
$$+S_0 \cdot \exp(-qT) \left(\frac{H}{S_0}\right)^{2\lambda} [N(y) - N(y_1)] +$$
$$-K \cdot \exp(-rT) \left(\frac{H}{S_0}\right)^{2\lambda-2} \left[N\left(y - \sigma\sqrt{T}\right) - N\left(y_1 - \sigma\sqrt{T}\right)\right] \quad \textit{(Eq. III.142)}$$

And the current value of a down-and-out put is $p_{DO} = p - p_{DI}$.

All the valuation formulas for the barrier options presented above are based on the assumption that the probabilistic distribution of the share price in a future instant of time is log-normal.

An important aspect of barrier options is how often the underlying asset price S is observed to see if the barrier is reached. The equations we presented assume that S is continuously observed, but often the contractual

conditions state that the underlying is observed in a discrete way, typically once a day. Broadie, Glasserman and Kou have developed an approximation of the formula to take into account the discretization of the observation frequency. The correction factor proposed by these researchers is based on the modification to be made, for each observation, on the level of the barrier with: $H_D = H \cdot \exp(\beta\sigma\sqrt{\Delta t})$ if the barrier is an upper-bound for the asset underlying the option. In a similar way, if the barrier represents a lower-bound, the adjustment is: $H_D = H \cdot \exp(-\beta\sigma\sqrt{\Delta t})$, where Δt is the time that elapses between the instants of observation of the barrier. It has been shown that
$\beta = \frac{\varsigma(0.5)}{\sqrt{2\pi}} \approx 0.5826$, where $\varsigma(\cdot)$ is the Riemann zeta-function.

In order for them to be implemented in a programming environment, the previous formulas of Reiner and Rubinstein should be rearranged according to the classification proposed by Rich. This pricing procedure provides for the use of the cost-of-carry, b, and the possibility of receiving a predefined amount of money (rebate cash R) in the worst cases or when the knock-in option has not been activated or when the knock-out option has failed to reach its natural expiry.

$$A = \phi S \cdot \exp[(b-r)T]N(\phi x_1) - \phi K \cdot \exp(-rT)N(\phi x_1 - \phi\sigma\sqrt{T}) \quad \textit{(Eq. III.143)}$$

$$B = \phi S \cdot \exp[(b-r)T]N(\phi x_2) - \phi K \cdot \exp(-rT)N(\phi x_2 - \phi\sigma\sqrt{T}) \quad \textit{(Eq. III.144)}$$

$$C = \phi S \cdot \exp[(b-r)T]\left(\frac{H}{S}\right)^{2(\mu+1)} N(\eta y_1) - \phi K \cdot \exp(-rT)\left(\frac{H}{S}\right)^{2\mu} N(\eta y_1 - \eta\sigma\sqrt{T}) \quad \textit{(Eq. III.145)}$$

$$D = \phi S \cdot \exp[(b-r)T]\left(\frac{H}{S}\right)^{2(\mu+1)} N(\eta y_2) - \phi K \cdot \exp(-rT)\left(\frac{H}{S}\right)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T}) \quad \textit{(Eq. III.146)}$$

$$E = R \cdot \exp(-rT)\left[N(\eta x_2 - \eta\sigma\sqrt{T}) - \left(\frac{H}{S}\right)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T})\right] \quad \textit{(Eq. III.147)}$$

$$F = R \cdot \left[\left(\frac{H}{S}\right)^{\mu+\lambda} N(\eta z) + \left(\frac{H}{S}\right)^{\mu-\lambda} N(\eta z - 2\eta\lambda\sigma\sqrt{T})\right] \quad \textit{(Eq. III.148)}$$

Where: $x_1 = \frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}$ $\quad x_2 = \frac{\ln\left(\frac{S}{H}\right)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}$ $\quad y_1 = \frac{\ln\left(\frac{H^2}{SK}\right)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}$

$y_2 = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}$ $\quad z = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$ $\quad \mu = \frac{b-\sigma^2/2}{\sigma^2}$ $\quad \lambda = \sqrt{\mu^2 + \frac{2r}{\sigma^2}}$

Let us analyze the different cases:

Down-and-in call $S > H$

Pay-off: $\max(S-K;0)$ if $S \leq H$ before T, otherwise R at maturity

$c_{K>H}^{DI} = C + E \qquad \eta = +1, \phi = +1$
$c_{K<H}^{DI} = A - B + D + E \qquad \eta = +1, \phi = +1$

Up-and-in call $S < H$

Pay-off: $\max(S - K; 0)$ if $S \geq H$ before T, otherwise R at maturity

$c^{UI}_{K>H} = A + E \qquad \eta = -1, \phi = +1$
$c^{UI}_{K<H} = B - C + D + E \qquad \eta = -1, \phi = +1$

Down-and-in put $S > H$
Pay-off: $\max(K - S; 0)$ if $S \leq H$ before T, otherwise R at maturity

$p^{DI}_{K>H} = B - C + D + E \qquad \eta = +1, \phi = -1$
$p^{DI}_{K<H} = A + E \qquad \eta = +1, \phi = -1$

Up-and-in put $S < H$
Pay-off: $\max(K - S; 0)$ if $S \geq H$ before T, otherwise R at maturity

$p^{UI}_{K>H} = A - B + D + E \qquad \eta = -1, \phi = -1$
$p^{UI}_{K<H} = C + E \qquad \eta = -1, \phi = -1$

Down-and-out call $S > H$
Pay-off: $\max(S - K; 0)$ if $S > H$ before T, otherwise R at the hit

$c^{DO}_{K>H} = A - C + F \qquad \eta = +1, \phi = +1$
$c^{DO}_{K<H} = B - D + F \qquad \eta = +1, \phi = +1$

Up-and-out call $S < H$
Pay-off: $\max(S - K; 0)$ if $S < H$ before T, otherwise R at the hit

$c^{UO}_{K>H} = F \qquad \eta = -1, \phi = +1$
$c^{UO}_{K<H} = A - B + C - D + F \qquad \eta = -1, \phi = +1$

Down-and-out put $S > H$
Pay-off: $\max(K - S; 0)$ if $S > H$ before T, otherwise R at the hit

$p^{DO}_{K>H} = A - B + C - D + F \qquad \eta = +1, \phi = -1$
$p^{DO}_{K<H} = F \qquad \eta = +1, \phi = -1$

Up-and-out put $S < H$
Pay-off: $\max(K - S; 0)$ if $S < H$ before T, otherwise R at the hit

$p^{UO}_{K>H} = B - D + F \qquad \eta = -1, \phi = -1$
$p^{UO}_{K<H} = A - C + F \qquad \eta = -1, \phi = -1$

Let us now illustrate a partial example, supposing that we want to determine the fair value of a standard barrier

option having the following financial parameters: $S = 100, R = 4, T = 0.75, r = 0.06, b = 0.03$.
Table III.54 shows the valuations corresponding to different strike prices $K \in \{90,100,110\}$, different barrier levels $H \in \{95,100,105\}$ and different values for the annualized volatility of the underlying $\sigma \in \{0.25,0.30,0.35\}$.

Type	K	H	$\sigma = 0.25$	$\sigma = 0.30$	$\sigma = 0.35$	Type	K	H	$\sigma = 0.25$	$\sigma = 0.30$	$\sigma = 0.35$
c_{DO}	90	95	9.6825	9.5472	9.4483	p_{DO}	90	95	3.2296	3.3806	3.4872
c_{DO}	100	95	7.9084	8.1239	8.2725	p_{DO}	100	95	3.2378	3.3855	3.4903
c_{DO}	110	95	6.3130	6.8068	7.1645	p_{DO}	110	95	3.4248	3.4964	3.5611
c_{DO}	90	100	4.0000	4.0000	4.0000	p_{DO}	90	100	4.0000	4.0000	4.0000
c_{DO}	100	100	4.0000	4.0000	4.0000	p_{DO}	100	100	4.0000	4.0000	4.0000
c_{DO}	110	100	4.0000	4.0000	4.0000	p_{DO}	110	100	4.0000	4.0000	4.0000
c_{UO}	90	105	3.4499	3.4690	3.4986	p_{UO}	90	105	5.0546	5.4978	5.8407
c_{UO}	100	105	3.2668	3.3595	3.4283	p_{UO}	100	105	6.5836	6.8793	7.0990
c_{UO}	110	105	3.2598	3.3554	3.4258	p_{UO}	110	105	8.2888	8.3662	8.4250
c_{DI}	90	95	9.3859	10.8911	12.4111	p_{DI}	90	95	4.1035	5.3223	6.6368
c_{DI}	100	95	5.5016	6.9462	8.4529	p_{DI}	100	95	7.9968	9.5093	11.0597
c_{DI}	110	95	3.1721	4.3468	5.6729	p_{DI}	110	95	13.4449	15.0418	16.6610
c_{DI}	90	100	15.1259	16.4865	17.9006	p_{DI}	90	100	3.3905	4.7511	6.1653
c_{DI}	100	100	9.4674	11.1183	12.7666	p_{DI}	100	100	7.2921	8.9429	10.5913
c_{DI}	110	100	5.5425	7.2018	8.8787	p_{DI}	110	100	12.9271	14.5864	16.2633
c_{UI}	90	105	15.6206	16.9692	18.3592	p_{UI}	90	105	2.2806	3.2051	4.2818
c_{UI}	100	105	10.1453	11.7107	13.2955	p_{UI}	100	105	4.6532	6.0155	7.4495
c_{UI}	110	105	6.2274	7.7982	9.4102	p_{UI}	110	105	8.5830	10.1720	11.7955

Table III.54 Standard Barrier Option

Still in the wide category of exotic options, we mention **lookback options** which value depends on the minimum or maximum price reached by the underlying during the life of the option.

The final value of a **floating-strike lookback call** is equal to the difference between the final price of the stock and the minimum price reached by the stock during the life of the option:

$$c(S, S_{min}, T) = \max(S - S_{min}; 0) = S_T - S_{min} \quad \textit{(Eq. III.149)}$$

The final value of a **floating-strike lookback put** option, on the other hand, is equal to the difference between the maximum price reached by the underlying asset during its life and its final price:

$$p(S, S_{max}, T) = \max(S_{max} - S; 0) = S_{max} - S_T \quad \textit{(Eq. III.150)}$$

For this type of financial instrument, the valuation formulas of Goldman-Sosin and Gatto (1979) and those of Garman (1989) are proposed. Thus, we indicate with $N(\cdot)$ the standard cumulative normal distribution function and with $n(\cdot)$ the standard normal distribution function.

Let us examine the **Floating-Strike Lookback Call** from a mathematical perspective:

If $b \neq 0$, then:

$$c = S \cdot \exp[(b-r)T]\, N(a_1) - S_{min} \cdot \exp(-rT) N(a_2) +$$
$$+ S \cdot \exp(-rT)\ \frac{\sigma^2}{2b}\left[\left(\frac{S}{S_{min}}\right)^{-\frac{2b}{\sigma^2}} N\left(-a_1 + \frac{2b}{\sigma}\sqrt{T}\right) - \exp(bT)\, N(-a_1)\right] \quad \textit{(Eq. III.151)}$$

If $b = 0$, then:

$$c = S \cdot \exp(-rT)\, N(a_1) - S_{min} \cdot \exp(-rT)\, N(a_2) +$$
$$+ S \cdot \exp(-rT)\, \sigma\sqrt{T}\{n(a_1) + a_1[N(a_1) - 1]\} \quad \textit{(Eq. III.152)}$$

Where:

$$a_1 = \frac{\ln\left(\frac{S}{S_{min}}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad a_2 = a_1 - \sigma\sqrt{T}$$

For the **Floating-Strike Lookback Put,** we have:

If $b \neq 0$, then:

$$p = S_{max} \cdot \exp(-rT)\, N(-b_2) - S \cdot \exp[(b-r)T]\, N(-b_1)$$
$$+ S \cdot \exp(-rT)\ \frac{\sigma^2}{2b}\left[-\left(\frac{S}{S_{max}}\right)^{-\frac{2b}{\sigma^2}} N\left(b_1 - \frac{2b}{\sigma}\sqrt{T}\right) + \exp(bT)\, N(b_1)\right] \quad \textit{(Eq. III.153)}$$

If $b = 0$, then:

$$p = S_{max} \cdot \exp(-rT)\, N(-b_2) - S \cdot \exp[(b-r)T]\, N(-b_1) +$$

$$+ S \cdot \exp(-rT)\, \sigma\sqrt{T}\{n(b_1) + N(b_1) b_1\} \quad \textit{(Eq. III.154)}$$

$$b_1 = \frac{\ln\left(\frac{S}{S_{max}}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad b_2 = b_1 - \sigma\sqrt{T}$$

The formulas just presented assume that the price of the underlying asset is observed continuously.

Let us present an example using the Goldman-Sosin-Gatto and Garman model to valuate a lookback call option that expires in 6 months. We assume that this option gives the right to buy the underlying stock index at the lowest price recorded during its lifetime, which is currently 50. The spot price level of the index is 52. The risk-free rate is 2%, the dividend yield is 1%. The annualized implied volatility is 20%. In short, we have: $S = 52$, $S_{min} = 50, T = 0.5, r = 0.02, q = 0.01 \rightarrow b = 0.01, \sigma = 0.2$, and we implement the calculations:

$$a_1 = \frac{\ln\left(\frac{S}{S_{min}}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{52}{50}\right) + (0.01 + 0.2^2/2) \cdot 0.5}{0.2 \cdot \sqrt{0.5}} = 0.383398$$

$$a_2 = a_1 - \sigma\sqrt{T} = 0.383398 - 0.2 \cdot \sqrt{0.5} = 0.241977,\ N(a_1) = 0.649288$$

$$N(-a_1) = 0.350712, N(a_2) = 0.595601, N\left(-a_1 + \frac{2b}{\sigma}\sqrt{T}\right) = 0.377259$$

$$c = S \cdot \exp[(b - r)T]\, N(a_1) - S_{min} \cdot \exp(-rT) N(a_2) +$$
$$+ S \cdot \exp(-rT)\ \frac{\sigma^2}{2b}\left[\left(\frac{S}{S_{min}}\right)^{-\frac{2b}{\sigma^2}} N\left(-a_1 + \frac{2b}{\sigma}\sqrt{T}\right) - \exp(bT)\, N(-a_1)\right] = 5.908888$$

In a **fixed-strike lookback call**, the strike price is already set when the contract is entered into. At maturity, the option pays the maximum between the difference of the highest price observed during the life of the option, S_{max}, and the strike K, and zero. In mathematical terms:

$$c(S, S_{max}, T) = \max(S_{max} - K; 0) \quad \textit{(Eq. III.155)}$$

In a **fixed-strike lookback put**, the strike price is already set when the contract is entered into. At maturity, the option pays the maximum between the difference of the strike price K and the lowest price observed during the life of the option, S_{min}, and zero. In mathematical terms:

$$p(S, S_{min}, T) = \max(K - S_{min}; 0) \quad \textit{(Eq. III.156)}$$

Fixed-strike lookback options can be valued using the formulas of Conze and Viswanathan (1991), as we show hereafter. For a Fixed-Strike Lookback Call, the calculations are:

If $K > S_{max}$, then:

$$c = S \cdot \exp[(b - r)T]\, N(d_1) - K \cdot \exp(-rT) N(d_2) +$$
$$+ S \cdot \exp(-rT)\ \frac{\sigma^2}{2b}\left[-\left(\frac{S}{K}\right)^{-\frac{2b}{\sigma^2}} N\left(d_1 - \frac{2b}{\sigma}\sqrt{T}\right) - \exp(bT)\, N(d_1)\right] \quad \textit{(Eq. III.157)}$$

Where: $d_1 = \frac{\ln\left(\frac{S}{K}\right) + (b + \sigma^2/2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$

If $K \leq S_{max}$, then:

$$c = \exp(-rT) \cdot (S_{max} - K) + S \cdot \exp[(b - r)T]\, N(e_1) - S_{max} \cdot \exp(-rT) N(e_2) +$$

$$+S \cdot \exp(-rT)\ \frac{\sigma^2}{2b}\left[-\left(\frac{S}{S_{max}}\right)^{-\frac{2b}{\sigma^2}} N\left(e_1 - \frac{2b}{\sigma}\sqrt{T}\right) + \exp(bT)\, N(e_1)\right] \quad \textit{(Eq. III.158)}$$

Where: $e_1 = \frac{\ln\left(\frac{S}{S_{max}}\right)+(b+\sigma^2/2)T}{\sigma\sqrt{T}}$, $e_2 = e_1 - \sigma\sqrt{T}$

While for a Fixed-Strike Lookback Put, we reach the following result:

If $K < S_{min}$, then:

$$p = K \cdot \exp(-rT)\, N(-d_2) - S \cdot \exp[(b-r)T]\, N(-d_1) +$$

$$+S \cdot \exp(-rT)\ \frac{\sigma^2}{2b}\left[\left(\frac{S}{K}\right)^{-\frac{2b}{\sigma^2}} N\left(-d_1 + \frac{2b}{\sigma}\sqrt{T}\right) - \exp(bT)\, N(-d_1)\right] \quad \textit{(Eq. III.159)}$$

If $K \geq S_{min}$, then:

$$p = \exp(-rT) \cdot (K - S_{min}) - S \cdot \exp[(b-r)T]\, N(-f_1) - S_{min} \cdot \exp(-rT) N(-f_2) +$$

$$+S \cdot \exp(-rT)\ \frac{\sigma^2}{2b}\left[\left(\frac{S}{S_{min}}\right)^{-\frac{2b}{\sigma^2}} N\left(-f_1 + \frac{2b}{\sigma}\sqrt{T}\right) - \exp(bT)\, N(-f_1)\right] \quad \textit{(Eq. III.160)}$$

Where: $f_1 = \frac{\ln\left(\frac{S}{S_{min}}\right)+(b+\sigma^2/2)T}{\sigma\sqrt{T}}$, $f_2 = f_1 - \sigma\sqrt{T}$.

The following table shows fixed-strike lookback call/put prices for various maturities T, strike prices K and σ.

		Call			Put		
T	K	$\sigma = 0.10$	$\sigma = 0.20$	$\sigma = 0.30$	$\sigma = 0.10$	$\sigma = 0.20$	$\sigma = 0.30$
0.5	95	11.2666	17.1372	23.2797	1.3845	5.7022	10.3527
0.5	100	6.4144	12.2850	18.4274	4.7033	9.8406	14.7608
0.5	105	2.6686	8.0752	14.0397	9.5556	14.6928	19.6131
1	95	14.1272	22.4378	31.3023	2.5273	8.7414	15.0143
1	100	9.4183	17.7290	26.5935	6.0722	12.9488	19.4233
1	105	5.4111	13.4551	22.2099	10.7810	17.6577	24.132

Table III.55 Fixed-strike lookback call/put prices

In the literature, closed formulas are also reported when the monitoring period of the observed extreme value is limited to a time interval shorter than the life of the option contract. These types of derivatives are called Partial-Time Fixed-Strike Lookback Option and Partial-Time Floating-Strike Lookback Option, whereas in the case of more complex pay-offs it is necessary to resort to numerical valuation methods.

FURTHER READINGS

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III.5 BINOMIAL TREES

Black-Scholes analytical formulas (BS closed formulas) are unable to provide a reasonable valuation for all types of options traded in the financial markets. In particular, they are unable to provide a fair value for options with non-standard characteristics such as the possibility of exercising them before expiry (Bermuda/American options) or with particularly complex pay-offs (exotic options). In such cases, a numerical methodology has to be implemented for the valuation of the derivative. The literature includes in fact numerous mathematical techniques that allow to obtain a price in line with the principles of the Black-Scholes framework. Among them, the most used by quantitative analysts are the following:

- Stochastic trees.
- Numerical integration schemes for partial differential equations.
- Fourier transforms.
- Monte Carlo methodology.

With the exception of the Monte Carlo methodology, the other algorithms are deterministic, i.e., using the same inputs, we always obtain the same outputs. It is also worth noting that all these approaches are internally consistent with the Black-Scholes pricing framework: in fact, with the same financial inputs, if an analyst wants to price a vanilla European option with one of the numerical methodologies reported, he will obtain a value which, at the continuous limit, converges to the fair value obtained using the traditional set of closed formulas. Therefore, excluding a potential error introduced by numerical discretization, these algorithms can faithfully replicate the pricing obtained through closed formulas. Although the approaches lead to the same price, it is possible to list a preference of use, mostly linked to the computational efficiency and the speed of convergence to the theoretical price.
Thus, as a general statement, the use of a closed valuation formula is preferred when it exists, otherwise a deterministic calculation algorithm can be used and, as the last viable alternative, a stochastic technique (Monte Carlo method). We will now introduce and discuss the most popular stochastic binomial tree: the **CRR (Cox-Ross-Rubinstein) model**. This technique has had a wide-spread diffusion because it is characterized by interesting aspects. First of all, its simplicity of construction, since it does not require highly specialized knowledge of advanced mathematical concepts. Secondly, its design flexibility, as the definition of non-standard pay-offs and the early exercise clause can be implemented quite flexibly. The third aspect consists of its numerical efficiency, programming in a development environment is actually not onerous, both in terms of lines of code and in terms of computational time. Then, the error attributable to the discretization can be monitored, so that the analyst is able to estimate the error with which he communicates the theoretical price of the derivative. Furthermore, this methodology is easily traceable since the process described is of the Markovian type. Lastly, the algorithm is deterministic. In fact, the binomial model can be seen as a time-discretized version of the Black-Scholes formula.

The first formulation of this method dates back to 1979 by John C. Cox, Stephen A. Ross and Mark E. Rubinstein. The technique essentially involves dividing the time between the option's valuation date (today) and its expiration date into many sub-periods, assuming that in the time between each interval, there may be two possible changes in the value of the share on which the derivative is written. By operating in this way, it is possible to obtain the fair value of the option by building a portfolio composed of shares and risk-free zero-coupons that replicate the dynamics of the option value over time. This way of proceeding turns out to be much more intuitive and far simpler than the mathematical operators present in the Black-Scholes equation. Furthermore, as already mentioned, it is to be considered a more powerful tool than the closed formula approach as it allows the valuation of Bermuda, American and a large number of exotic options.

Let us proceed with the discussion examining a binomial model with a single interval (one-step binomial model) which can be considered as the founding block of the logic underlying a CRR Tree. Let us assume we have a single European call option written on a stock with a single time interval before its expiration and characterized by a strike price K. We initially assume that the underlying is not profitable, i.e. that it does not pay cash dividends during the life of the option. The binomial model assumes that the price level of the underlying S follows a simple binomial stationary process. In each time frame, the price can rise by $u \cdot S$ (with probability q) or fall by $d \cdot S$ (with probability $1 - q$). This tree interprets a geometric multiplicative binomial process because the price movements do not have a fixed amplitude, but they are proportional to the value of the share at the node from which the binomial step is generated. Let C be the current value of a call option on that stock, C_u be its value at the end of the period in which the stock price level rises to $u \cdot S$ and C_d its value at the end of the period in which the stock price level falls to $d \cdot S$. Since only one time step has been assumed before the European option expires, we know that:

$$C_u = \max[u \cdot S - K, 0]\,;\; C_d = \max[d \cdot S - K, 0] \quad \textit{(Eq. III.161)}$$

We can see from the tree in the figure below that the unknown variable is C, while C_u and C_d are known. The value of the European call option C is equal to:

$$C = \frac{C_u \cdot \Pi + C_d \cdot (1 - \Pi)}{1+R} \quad \textit{(Eq. III.162)}$$

With $\Pi = \frac{1+R-d}{u-d}$ and R the risk-free rate in the considered time interval.

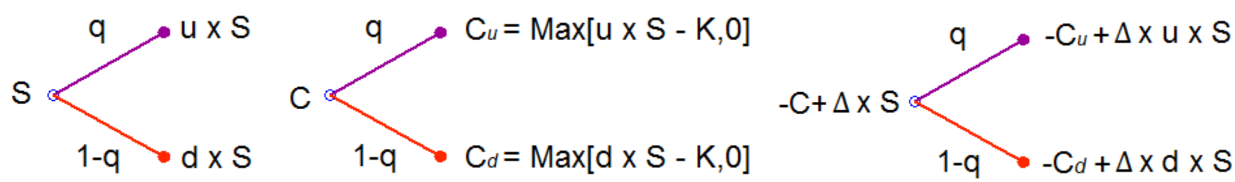

Figure III.104 The single period binomial tree

The expression of the value of the European Call C can be demonstrated in two ways that lead to the same result: by means of a delta-hedging strategy or through synthetic options.

Let us first consider the portfolio strategy called "delta-hedging" which consists in writing a call option and buying Δ units of the underlying asset of the derivative. The value of the portfolio thus formed is equal to:

	Portfolio value at t=0	Portfolio value at t=1	
Write one call	$-C$	$S_{t=1} = u \cdot S$	$S_{t=1} = d \cdot S$
		$-C_u$	$-C_d$
Buy shares of stock	$\Delta \cdot S$	$\Delta \cdot u \cdot S$	$\Delta \cdot d \cdot S$
Total	$-C + \Delta \cdot S$	$-C_u + \Delta \cdot u \cdot S$	$-C_d + \Delta \cdot d \cdot S$

Table III.56 CRR Tree: Delta hedging strategy

We decide to choose the portion of shares to buy Δ, in such a way that the portfolio is completely risk-free. In this way, the same value of the portfolio would be obtained regardless of the price level assumed by S at the end of the first period:

$$-C_u + \Delta \cdot u \cdot S = -C_d + \Delta \cdot d \cdot S \rightarrow \Delta = \frac{C_u - C_d}{S \cdot u - S \cdot d} = \frac{C_u - C_d}{S \cdot (u - d)} \quad \text{(Eq. III.163)}$$

The meaning of Δ can be understood from this ratio: in fact, it measures how much the value of the Call varies according to how much the price level of the underlying share varies. An investor holding this portfolio is not exposed to any type of risk since the final value at maturity is known with certainty. At the end of the period, therefore, he should expect an amount capitalized at the risk-free rate:

$$(-C + \Delta \cdot S) \cdot (1 + R) = -C_d + \Delta \cdot d \cdot S \quad \text{(Eq. III.164)}$$

By rearranging the members of the equation, after a few algebraic passages we determine C

$$(-C + \Delta \cdot S) \cdot (1 + R) = -C_d + \Delta \cdot d \cdot S \rightarrow -C + \Delta \cdot S = \frac{-C_d + \Delta \cdot d \cdot S}{1+R} \quad \text{(Eq. III.165)}$$

$$C = \Delta \cdot S + \frac{C_d - \Delta \cdot d \cdot S}{1+R} = \frac{\Delta \cdot S \cdot (1+R) + C_d - \Delta \cdot d \cdot S}{1+R} \quad \text{(Eq. III.166)}$$

Remembering that $\Delta = \frac{C_u - C_d}{S \cdot (u-d)}$,

$$C = \frac{\frac{C_u - C_d}{S\cdot(u-d)}\cdot S\cdot(1+R) + C_d - \frac{C_u - C_d}{S\cdot(u-d)}\cdot d\cdot S}{1+R} \text{ (Eq. III.167)}$$

$$C = \frac{\frac{(C_u - C_d)\cdot(1+R) + C_d\cdot(u-d) - d\cdot(C_u - C_d)}{u-d}}{1+R} = \frac{C_u\cdot\left(\frac{1+R-d}{u-d}\right) + C_d\cdot\left(\frac{u-R-1}{u-d}\right)}{1+R} \text{ (Eq. III.168)}$$

Defining the quantity $\prod = \frac{1+R-d}{u-d}$, the formula becomes equal to the previous one:

$$C = \frac{C_u\cdot\prod + C_d\cdot(1-\prod)}{1+R} \text{ (Eq. III.169)}$$

As we have seen, the basic idea of delta-hedging is thus to create a risk-free portfolio using a specific combination of the stock and the option on which the security is written. Alternatively, the investor could also consider combining the risk-free asset with the underlying of the option, in such a way as to replicate the pay-off of the option itself. This combination is called a "synthetic option". In order to find the number of shares to buy (Δ) and the amount of money to initially invest in the risk-free asset (B) to synthesize the option, it is necessary to simultaneously solve the equations:

$$\begin{cases} \Delta\cdot S\cdot u + B\cdot(1+R) = C_u \\ \Delta\cdot S\cdot d + B\cdot(1+R) = C_d \end{cases} \text{ (Eq. III.170)}$$

The solution for the 2x2 system is:

$$\Delta = \frac{C_u - C_d}{S\cdot u - S\cdot d} \text{ (Eq. III.171)}$$

$$B = \frac{1}{1+R}\cdot\left[\frac{u\cdot C_d - d\cdot C_u}{u-d}\right] \text{ (Eq. III.172)}$$

The cost for the option synthesis is therefore the cost of the portfolio made up of Δ units of shares and of investing an amount equal to B in the risk-free bond: $C = S\cdot\Delta + B$.

$$C = S\cdot\Delta + B = \frac{C_u - C_d}{u-d} + \frac{u\cdot C_d - d\cdot C_u}{(u-d)\cdot(1+R)} \text{ (Eq. III.173)}$$

$$C = \frac{1}{1+R}\cdot\left[\frac{C_u\cdot(1+R-d) - C_d\cdot(u-1-R)}{(u-d)}\right] \text{ (Eq. III.174)}$$

$$C = \frac{1}{1+R}\cdot\left[\frac{1+R-d}{u-d}C_u + \frac{u-1-R}{u-d}C_d\right] = \frac{C_u\cdot\prod + C_d\cdot(1-\prod)}{1+R} \text{ (Eq. III.175)}$$

We have just shown in two independent ways how much a European call option must be worth in a one-period binomial model. We now focus on analyzing the formula in more detail.

$$C = \frac{C_u\cdot\prod + C_d\cdot(1-\prod)}{1+R} \text{ (Eq. III.176)}$$

An intuitive way to read this relationship is to view the option price as the discounted value of a weighted average of the possible call states at expiration. From this point of view, $\prod$ and $(1-\prod)$ are interpreted as the probability of occurrence. We should also note the interesting aspect that the initial probabilities linked to the movement of the price of the underlying S, i.e. q and $(1-q)$, are not used in the formula for determining the fair value of the call. Thus, the option price is independent from the expected future return of the stock on which it is written. This happens because the option can be replicated synthetically and therefore its price does not depend on the subjective preferences of an investor or on the personal view of the market.
$\prod$ can therefore be interpreted as a risk-neutral probability and this method of valuing derivatives is called risk-neutral valuation. This valuation approach can be extended to any derivative that can be synthetically replicated with elementary financial instruments.

Let us now present an example, bearing in mind that the single period model envisages a single interval which starts from the initial instant t_0 and ends at instant t_1. Let us assume that a share S, having a value of 100 in t_0, can only have two values in t_1:
$S_u = 110$ and $S_d = 91$, consequently $u = \frac{110}{100} = 1.1$ and $d = \frac{91}{100} = 0.91$.
Let us now consider, for simplicity, that the achievement in t_1 of either one of the two values has the same probability. The call option whose value in t_0 is to be determined has maturity in t_1 and strike price K equal to 100.
Its value at maturity is equal to its pay-off $\max(S-K,0)$:

10 if the price of S in t_1 is equal to 110, that is $S_u = 110$, $C_u = 10$.

0 if the price of S in t_1 is equal to 91, that is $S_d = 110$, $C_d = 0$.

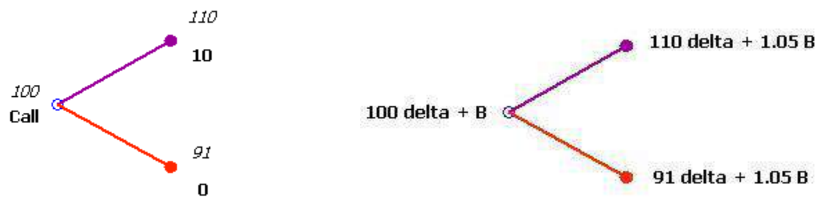

Figure III.105 The single period binomial tree. Example

Assuming then that the interest rate for the period considered is equal to 5% ($R = 0.05$), it is possible to construct in t_0 a financial portfolio consisting of shares and bonds, in the proportions Δ and B respectively, which exactly replicates the value of the call option at time t_1. Operating in a risk-free world, it is therefore possible to define the quantities Δ and B for which the value of the financial portfolio assumes the same value as the option at maturity. To find these quantities, we need to solve the 2 x 2 system:

$$\begin{cases} 110 \cdot \Delta + 1.05 \cdot B = 10 \\ 91 \cdot \Delta + 1.05 \cdot B = 0 \end{cases} \rightarrow \begin{cases} \Delta = +0.526315 \\ B = -45.61403 \end{cases}$$

And the cost for building such a portfolio is: $S \cdot \Delta + \mathrm{B} = 7.018$.

Since the financial portfolio thus constructed is worth exactly as much as the option at maturity, its cost in t_0 must be exactly equal to the option premium. By directly using the formulas obtained from the previous general discussion, the same price is obtained.

$$\prod = \frac{1+R-d}{u-d} = \frac{1.05-0.91}{1.1-0.91} = 0.736842$$

$$C = \frac{C_u \cdot \prod + C_d \cdot (1-\prod)}{1+R} = \frac{10 \cdot 0.736842 + 0 \cdot (1-0.736842)}{1.05} = 7.018$$

This approach must be extended to several periods: therefore, we begin to deal with the binomial pricing model with two time periods between the valuation date of the derivative and its maturity. Using the same procedure, at the end of the second period, we will have three possible states since the tree is recombining (**recombining tree**) and the factors u and d are constant. The figure shows the two trees with the possible price levels: on the left, those of the share and on the right those of the call option. We have:

$C_{uu} = \max[u^2 \cdot S - K, 0]$; $C_{dd} = \max[d^2 \cdot S - K, 0]$ and $C_{du} = \max[d \cdot u \cdot S - K, 0]$ *(Eq. III.177)*

Figure III.106 The binomial pricing model with two time periods

Using the results obtained in the single-period formulation of the model, the value of the option in period 1 can be determined:

$$C_u = \frac{C_{uu} \cdot \prod + C_{du} \cdot (1-\prod)}{1+R};\ C_d = \frac{C_{ud} \cdot \prod + C_{dd} \cdot (1-\prod)}{1+R} \quad \textit{(Eq. III.178)}$$

Considering now period 0, knowing the value at time 1, we operate in a similar way by applying the same result.

$$C = \frac{C_u \cdot \prod + C_d \cdot (1-\prod)}{1+R} \quad \textit{(Eq. III.179)}$$

$$C = \frac{\left(\frac{C_{uu} \cdot \prod + C_{du} \cdot (1-\prod)}{1+R}\right) \cdot \prod + \left(\frac{C_{ud} \cdot \prod + C_{dd} \cdot (1-\prod)}{1+R}\right) \cdot (1-\prod)}{1+R} \quad \textit{(Eq. III.180)}$$

$$C = \frac{(\prod)^2 C_{uu} + 2 \cdot \prod \cdot (1-\prod) \cdot C_{du} + (1-\prod)^2 \cdot C_{dd}}{(1+R)^2} \quad \text{(Eq. III.181)}$$

Following the same logic, an analyst can derive the pricing formula for a three-period binomial tree.

Let us examine a practical example considering a stock with an initial value of $S = 100$. In each period, the price level can either rise to $1.1 \cdot S$ (with a probability q), or fall to $S/1.1$ (with a probability $1 - q$). The flat risk-free rate is 5%. This requires the valuation of a European at-the-money call option ($S = K = 100$) using a three-period binomial tree.

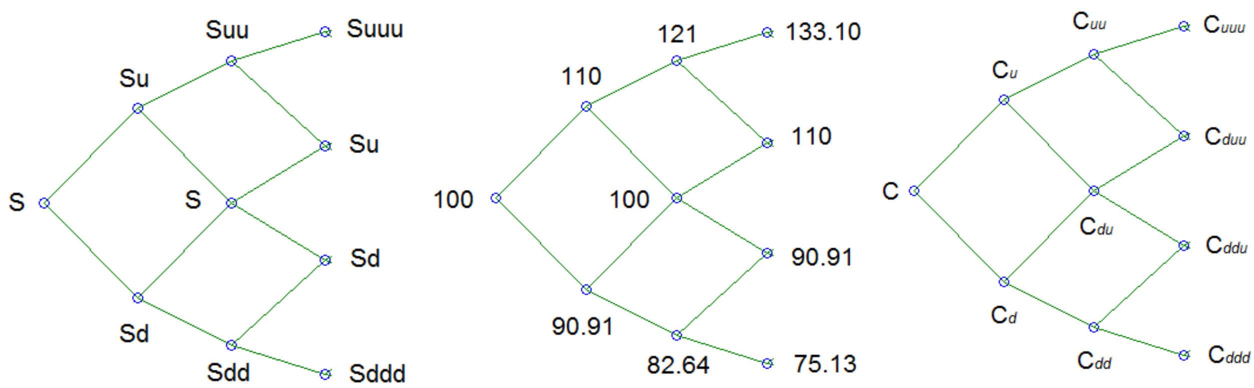

Figure III.107 The multiperiod binomial tree. Example

The first step consists of building the binomial tree over three periods which interprets the dynamics of the underlying S. The second step is to estimate the pay-off of the call option at the end nodes of the chain:

$$C_{uuu} = \max[S \cdot u^3 - K; 0] = 33.10 \qquad C_{duu} = \max[S \cdot d \cdot u^2 - K; 0] = 10$$
$$C_{ddu} = \max[S \cdot d^2 \cdot u - K; 0] = 0 \qquad C_{duu} = \max[S \cdot d^3 - K; 0] = 0$$

The third step consists in tracing the nodes of the tree backwards (**backwardation**), starting from the end, and estimating the option prices in previous periods. To do this, it is necessary to calculate the risk-neutral probability $\prod = \frac{1+R-d}{u-d} = \frac{1.05-\frac{1}{1.1}}{1.1-\frac{1}{1.1}} = 0.738095.$

We then proceed to calculate the fair value in the second time period:

$C_{uu} = \frac{0.738095 \cdot 33.10 + 0.261905 \cdot 10}{1.05} = 25.761905$

$C_{du} = \frac{0.738095 \cdot 10 + 0.261905 \cdot 0}{1.05} = 7.029478$

$C_{dd} = \frac{0.738095 \cdot 0 + 0.261905 \cdot 0}{1.05} = 0$

The procedure is repeated backwards for interval 1, obtaining:

$C_{u} = \frac{0.738095 \cdot 25.761905 + 0.261905 \cdot 7.029478}{1.05} = 19.862660$

$C_{d} = \frac{0.738095 \cdot 7.029478 + 0.261905 \cdot 0}{1.05} = 4.941357$

The price of the derivative is read at time 0 and is equal to:

$C = \frac{0.738095 \cdot 19.862660 + 0.261905 \cdot 4.941356}{1.05} = 15.194952.$

The pricing instrument of a European call option thus described can be generalized to a generic tree consisting of N sub-periods.

$$C = \frac{\sum_{j=0}^{N} \frac{N!}{j! \cdot (N-j)!} \cdot (\prod)^{j} \cdot (1-\prod)^{N-j} \cdot \max\left[0; u^{j} \cdot d^{N-j} \cdot S - K\right]}{(1+R)^{N}} \quad \textit{(Eq. III.182)}$$

We repeat the valuation of the previous example directly using the generalized pricing formula. Substituting into the expression: $N = 3$, $S = K = 100$, $R = 5\%$, $u = 1.1$, $d = \frac{1}{1.1}$ and $\prod = 0.738095$.

$$C = \frac{\sum_{j}^{3} \frac{3!}{j! \cdot (3-j)!} \cdot 0.738095^{j} \cdot (1-0.738095)^{3-j} \cdot \max\left[0; 1.1^{j} \cdot \left(\frac{1}{1.1}\right)^{3-j} \cdot S - K\right]}{(1+0.05)^{3}}$$

$$= \frac{1}{1.05^{3}} \left\{ \begin{array}{l} \frac{3!}{0! \cdot 3!} \cdot 0.738095^{0} \cdot (1-0.738095)^{3-0} \cdot \max\left[0; 1.1^{0} \cdot \left(\frac{1}{1.1}\right)^{3} \cdot S - K\right] \\ + \frac{3!}{1! \cdot 2!} \cdot 0.738095^{1} \cdot (1-0.738095)^{3-1} \cdot \max\left[0; 1.1^{1} \cdot \left(\frac{1}{1.1}\right)^{2} \cdot S - K\right] \\ + \frac{3!}{2! \cdot 1!} \cdot 0.738095^{2} \cdot (1-0.738095)^{3-2} \cdot \max\left[0; 1.1^{2} \cdot \left(\frac{1}{1.1}\right)^{1} \cdot S - K\right] \\ + \frac{3!}{3! \cdot 0!} \cdot 0.738095^{3} \cdot (1-0.738095)^{3-3} \cdot \max\left[0; 1.1^{3} \cdot \left(\frac{1}{1.1}\right)^{0} \cdot S - K\right] \end{array} \right\}$$

$$C = \frac{1}{1.05^{3}} \left\{ \begin{array}{c} 1 \cdot 1 \cdot 0.017965 \cdot \max[0; 1 \cdot 0.751315 \cdot 100 - 100] \\ +3 \cdot 0.738095 \cdot 0.068594 \cdot \max[0; 1.1 \cdot 0.826446 \cdot 100 - 100] \\ +3 \cdot 0.544785 \cdot 0.261905 \cdot \max[0; 1.21 \cdot 0.909091 \cdot 100 - 100] \\ +1 \cdot 0.402103 \cdot 1 \cdot \max[0; 1.331 \cdot 1 \cdot 100 - 100] \end{array} \right\}$$

$$C = \frac{1}{1.05^{3}} \{0 + 0 + 4.280457 + 13.30961\} = 15.1949$$

The generalized formula can be expressed computationally more efficiently. Let us denote with a the minimum number of upward movements that the stock must make in the following N periods in such a way that it ends in-the-money. Translated into mathematical terms, a must be the smallest non-negative integer such that the following condition is satisfied: $u^a \cdot d^{N-a} \cdot S \geq K$. Taking the natural logarithm from both sides, we obtain: $\ln(u^a \cdot d^{N-a} \cdot S) \geq \ln(K)$

Recalling the properties of logarithms and implementing a few algebraic steps, we obtain the condition on the index a:

$$\ln(u^a \cdot d^{N-a} \cdot S) \geq \ln(K)$$

$$\ln(u^a) + \ln(d^{N-a}) + \ln(S) \geq \ln(K)$$

$$a \cdot \ln(u) + (N-a) \cdot \ln(d) + \ln(S) \geq \ln(K)$$

$$a \cdot \ln(u) + N \cdot \ln(d) - a \cdot \ln(d) + \ln(S) \geq \ln(K)$$

$$a \cdot [\ln(u) - \ln(d)] \geq -N \cdot \ln(d) - \ln(S) + \ln(K)$$

$$a \geq \frac{-N \cdot \ln(d) - \ln(S) + \ln(K)}{[\ln(u) - \ln(d)]} \rightarrow a \geq \frac{\ln\left(\frac{K}{S \cdot d^N}\right)}{\ln\left(\frac{u}{d}\right)}$$

For any index j smaller than a, the value of the European call is zero: $\max[0; u^j \cdot d^{N-j} \cdot S - K] = 0$.

For any index j greater than a, the value of the European call is: $u^j \cdot d^{N-j} \cdot S - K$.

By setting the condition on the summation index, we can speed up the calculations, making the iterations more computationally efficient.

$$C = \frac{\sum_{j=a}^{N} \frac{N!}{j! \cdot (N-j)!} \cdot (\prod)^j \cdot (1-\prod)^{N-j} \cdot (u^j \cdot d^{N-j} \cdot S - K)}{(1+R)^N} \text{ Such that } a \geq \frac{\ln\left(\frac{K}{S \cdot d^N}\right)}{\ln\left(\frac{u}{d}\right)}.$$

Taking into consideration the previous example, we obtain $a \geq \frac{\ln\left(\frac{100}{100 \cdot 0.909091^3}\right)}{\ln\left(\frac{1,1}{0.909091}\right)} \geq 1.5 \rightarrow 2$.

$$C = \frac{\sum_{j=2}^{3} \frac{3!}{j! \cdot (3-j)!} \cdot 0.738095^j \cdot (1-0.738095)^{3-j} \cdot \left(1.1^j \cdot \left(\frac{1}{1.1}\right)^{3-j} \cdot 100 - 100\right)}{(1+0.05)^3} = \frac{1}{1.05^3}\{4.280457 + 13.30961\} = 15.1949\,.$$

The same logic for the valuation of a European call option can be replicated for the pricing of a put, as follows:

$$P = \frac{\sum_{j=0}^{N} \frac{N!}{j! \cdot (N-j)!} \cdot (\prod)^j \cdot (1-\prod)^{N-j} \cdot \max[0; K - u^j \cdot d^{N-j} \cdot S]}{(1+R)^N} \quad \textit{(Eq. III.183)}$$

The routine can be optimized in this case as well.

$$P = \frac{\sum_{j=0}^{a-1} \frac{N!}{j!\cdot(N-j)!} \cdot (\prod)^j \cdot (1-\prod)^{N-j} \cdot (K-u^j \cdot d^{N-j} \cdot S)}{(1+R)^N} \text{ Such that: } a \geq \frac{\ln\left(\frac{K}{S\cdot d^N}\right)}{\ln\left(\frac{u}{d}\right)}$$

The advantage of implementing this numerical pricing method consists in the possibility of determining the fair value of options that provide for **early exercise** (Bermuda and American). This feature adds further complexity to the binomial model: in fact, instead of valuing the exercise value at maturity and going backwards by applying the standard backwardation algorithm, a further convenience check must be performed at each time period in order to test whether the option is worth more while alive or exercised. The price of the derivative will be the higher of these two values. Thus, at each node in the chain, the value of the option can be expressed as:

$$C_t = \max[C_{dead}; C_{alive}] = \max\left[S_t - K; \frac{C_u \cdot \prod + C_d \cdot (1-\prod)}{1+R}\right] \text{ *(Eq. III.184)*}$$

$$P_t = \max[P_{dead}; P_{alive}] = \max\left[K - S_t; \frac{P_u \cdot \prod + P_d \cdot (1-\prod)}{1+R}\right] \text{ *(Eq. III.185)*}$$

We now consider, as an example, a put option with a strike price $K = 100$, written on an underlying stock, whose spot price is $S = 100$. In the following period, the value of the underlying can be $S_u = 1.1 \cdot S = 110$ or $S_d = 0.95 \cdot S = 95$. In the second time interval, the share price may assume the following three states:

$S_{uu} = 1.1 \cdot 1.1 \cdot S = 121$, $S_{ud} = S_{du} = 0.95 \cdot 1.1 \cdot S = 104.5$ and $S_{dd} = 0.95 \cdot 0.95 \cdot S = 90.25$

The risk-free rate is assumed to be 5% per period.

If this option is considered as a **European put option**, at maturity the final value will be null if the underlying stock rises to 121 or 104.5 and will be worth 9.75 if the stock falls to 90.25 in the last period. Using the principles of evaluation of the binomial tree, the following values are obtained:

$$P_u = \frac{P_{uu} \cdot \prod + P_{ud} \cdot (1-\prod)}{1+R} = 0$$

$$P_d = \frac{P_{ud} \cdot \prod + P_{dd} \cdot (1-\prod)}{1+R} = \frac{P_{ud} \cdot \frac{1+R-d}{u-d} + P_{dd} \cdot \left(1 - \frac{1+R-d}{u-d}\right)}{1+R} = \frac{0 \cdot \frac{1+0.05-0.95}{1.1-0.95} + 9.75 \cdot \left(1 - \frac{1+0.05-0.95}{1.1-0,95}\right)}{1+0.05} = 3.095238$$

$$P = \frac{P_u \cdot \prod + P_d \cdot (1-\prod)}{1+R} = \frac{0 \cdot \frac{1+0.05-0.95}{1.1-0.95} + 3.095238 \cdot \left(1 - \frac{1+0.05-0.95}{1.1-0.95}\right)}{1.05} = 0.982615.$$

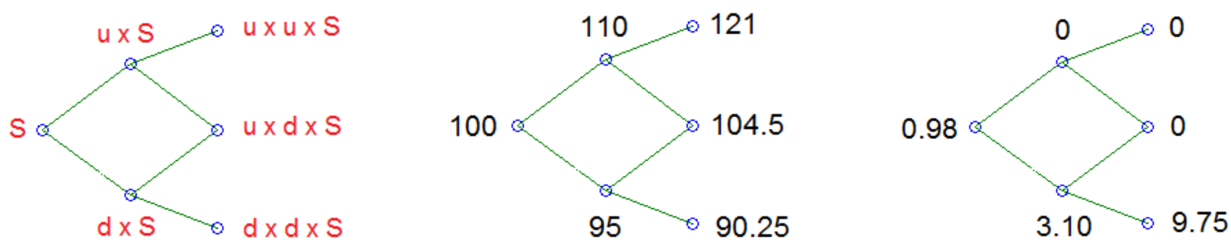

Figure III.108 The multiperiod binomial tree. European Put Option

If on the other hand the option is considered as an **American put option**, at maturity the final value will be equal to that of the European option, but in time interval 1, the value differs:

$$P_{1,u} = \max\left[K - S_u; \frac{P_{uu}\cdot\prod + P_{ud}\cdot(1-\prod)}{1+R}\right] = \max[100 - 110; 0] = 0$$

$$P_{1,d} = \max\left[K - S_d; \frac{P_{ud}\cdot\prod + P_{dd}\cdot(1-\prod)}{1+R}\right] = \max\left[100 - 95; \frac{0\cdot\frac{1+0.05-0.95}{1.1-0.95} + 9.75\cdot\left(1-\frac{1+0.05-0.95}{1.1-0,95}\right)}{1+0.05}\right] =$$
$$= \max[5; 3.095238] = 5$$

$$P_0 = \max\left[K - S_{t=0}; \frac{P_{1,u}\cdot\prod + P_{1,d}\cdot(1-\prod)}{1+R}\right] = \max\left[100 - 100; \frac{0\cdot\frac{1+0.05-0.95}{1.1-0.95} + 5\cdot\left(1-\frac{1+0.05-0.95}{1.1-0.95}\right)}{1.05}\right] = 1.587302.$$

The binomial method was able to quantify the benefit derived from early exercise.

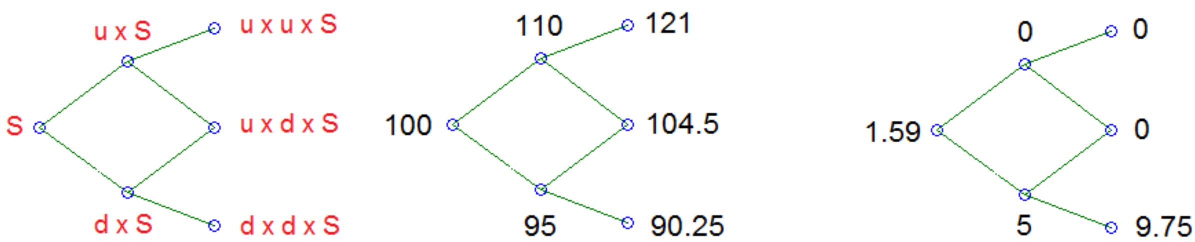

Figure III.109 The multiperiod binomial tree. American Put Option

To generalize the procedure in a multi-period context, it is necessary to use a double indexing in order to identify both the reference time interval (index j), and also the possible value assumed by the financial instrument for the same period (index i).

The generalized expression to consider the possible early exercise of an option in a generic node (j, i) of the binomial tree is given by:

$$P_{j,i} = \max\left\{K - S\cdot u^i \cdot d^{j-1}, \frac{P_{j+1,i+1}\cdot\prod + P_{j+1,i}\cdot(1-\prod)}{1+R}\right\} \quad \textit{(Eq. III.186)}$$

$$C_{j,i} = \max\left\{S\cdot u^i \cdot d^{j-1} - K, \frac{C_{j+1,i+1}\cdot\prod + C_{j+1,i}\cdot(1-\prod)}{1+R}\right\} \quad \textit{(Eq. III.187)}$$

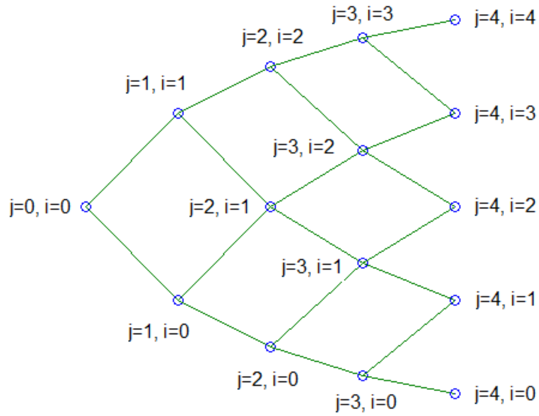

Figure III.110 Binomial tree indexing

As the arborescence of the stochastic tree increases, the estimate of the price of the derivative improves. Common practice when the discretization time interval Δt is reasonably small is to perform a continuous compounding of the interest rate. Under this working hypothesis, the formulas discussed above become:

For the European Option:

$$C = \exp(-rT) \cdot \sum_{j=a}^{N} \frac{N!}{j! \cdot (N-j)!} \cdot (\prod)^{j} \cdot (1-\prod)^{N-j} \cdot \left(u^{j} \cdot d^{N-j} \cdot S - K\right) \quad \textit{(Eq. III.188)}$$

$$P = \exp(-rT) \cdot \sum_{j=0}^{a-1} \frac{N!}{j! \cdot (N-j)!} \cdot (\prod)^{j} \cdot (1-\prod)^{N-j} \cdot \left(K - u^{j} \cdot d^{N-j} \cdot S\right) \quad \textit{(Eq. III.189)}$$

For the American Option:

$$P_{j,i} = \max\left\{K - S \cdot u^{i} \cdot d^{j-1}, \exp(-r \cdot T) \cdot \left[P_{j+1,i+1} \cdot \prod + P_{j+1,i} \cdot (1-\prod)\right]\right\} \quad \textit{(Eq. III.190)}$$

$$C_{j,i} = \max\left\{S \cdot u^{i} \cdot d^{j-1} - K, \exp(-r \cdot T) \cdot \left[C_{j+1,i+1} \cdot \prod + C_{j+1,i} \cdot (1-\prod)\right]\right\} \quad \textit{(Eq. III.191)}$$

With $\prod = \frac{\exp(r \cdot \Delta t) - d}{u-d}$

Arbitrary values for the growth and decrease factors of the share were used in our discussion so far. In order to have a matching with the stochastic dynamic hypothesized by the Black Scholes framework, Cox – Ross – Rubinstein proposed to choose the parameters u and d so that, for each discretization time interval Δt, the hypothesized future values of the asset were consistent with the theoretical mean and variance of the continuous model. Given the assumption that traders are risk neutral, the expected rate of return on the stock is equal to the risk-free interest rate r. Thus, the expected value of the stock price at the end of interval Δt is equal to $S \cdot \exp(r\Delta t)$, where S is the stock price at the beginning of the interval. It follows that:

$$S \cdot \exp(r \cdot \Delta t) = \prod \cdot S \cdot u + (1 - \prod) \cdot S \cdot d \rightarrow \exp(r \cdot \Delta t) = \prod \cdot u + (1 - \prod) \cdot d \quad \text{(Eq. III.192)}$$

The stochastic process assumed for the stock price implies that the variance of its rate of change in a short interval of length Δt is $\sigma^2 \Delta t$.

Since the variance of a random variable Q is defined as $E(Q^2) - E(Q)^2$, where $E(\cdot)$ denotes the expected value, we can obtain the second equation which relates the second moment of the stochastic process to the evolution followed by the binomial tree:

$$\prod \cdot u^2 + (1 - \prod) \cdot d^2 - [\prod \cdot u + (1 - \prod) \cdot d]^2 = \sigma^2 \Delta t \quad \text{(Eq. III.193)}$$

Obtaining $\prod$ from the equation of the first moment and substituting the value of $\prod$ within the relation of the second moment, we obtain:

$$\exp(r \cdot \Delta t) = \prod \cdot u + (1 - \prod) \cdot d \rightarrow \exp(r \cdot \Delta t) - d = \prod \cdot (u - d) \rightarrow \prod = \frac{\exp(r \cdot \Delta t) - d}{u - d} \quad \text{(Eq. III.194)}$$

$$\exp(r \cdot \Delta t) \cdot (u + d) - u \cdot d - \exp(2 \cdot r \cdot \Delta t) = \sigma^2 \Delta t \quad \text{(Eq. III.195)}$$

Recalling that Cox, Ross and Rubinstein had assumed for their model that $u = 1/d$, we obtain a 3 x 3 system which allows us to express the parameters $\prod$, u and d in terms of r, σ and Δt:

$$\begin{aligned} &\exp(r\Delta t) = \prod \cdot u + (1 - \prod) \cdot d \\ &\exp(r\Delta t) \cdot (u + d) - u \cdot d - \exp(2 \cdot r \cdot \Delta t) = \sigma^2 \Delta t \\ &u = \frac{1}{d} \end{aligned} \quad \rightarrow \quad \begin{cases} \Pi = \dfrac{\exp(r \cdot \Delta t) - d}{u - d} \\ u = \exp\left(\sigma\sqrt{\Delta t}\right) \\ d = \exp\left(-\sigma\sqrt{\Delta t}\right) \end{cases}$$

The set of parameters allows to construct a binomial stochastic tree in complete agreement with the Black-Scholes pricing framework.

Therefore, if a certain number of time discretization intervals tending to infinity $N \rightarrow \infty$ were considered, a theoretical convergence to the closed valuation formula for European vanilla options would be obtained.

Let us prove this statement experimentally by considering pricing a European option having the following financial characteristics: S=50, K=55, T=0.5, r=5%, σ=30%.

The closed formulas would give a value equal to $C_E = 2.7935$ and $P_E = 6.4356$.

Using 20 time-intervals, the call option price converges to 2.7377. The graphical representation of the binomial tree shown in the figure below was programmed in Matlab.

Two possible chain paths have been traced on the chain: for each of them the value assumed by the share at each step and the relative call price are shown in the following figures.

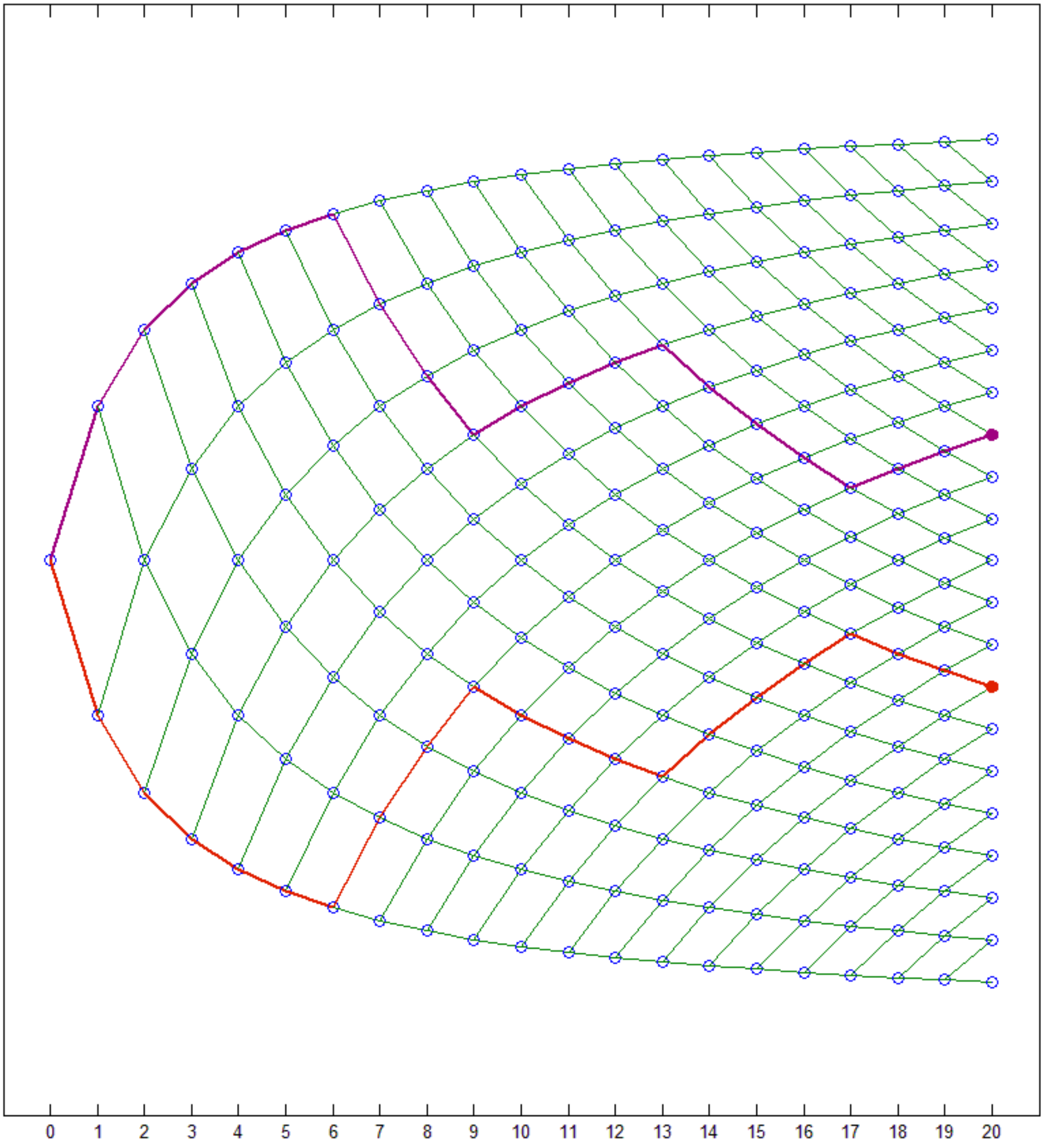

Figure III.111 Binomial tree generated using Matlab

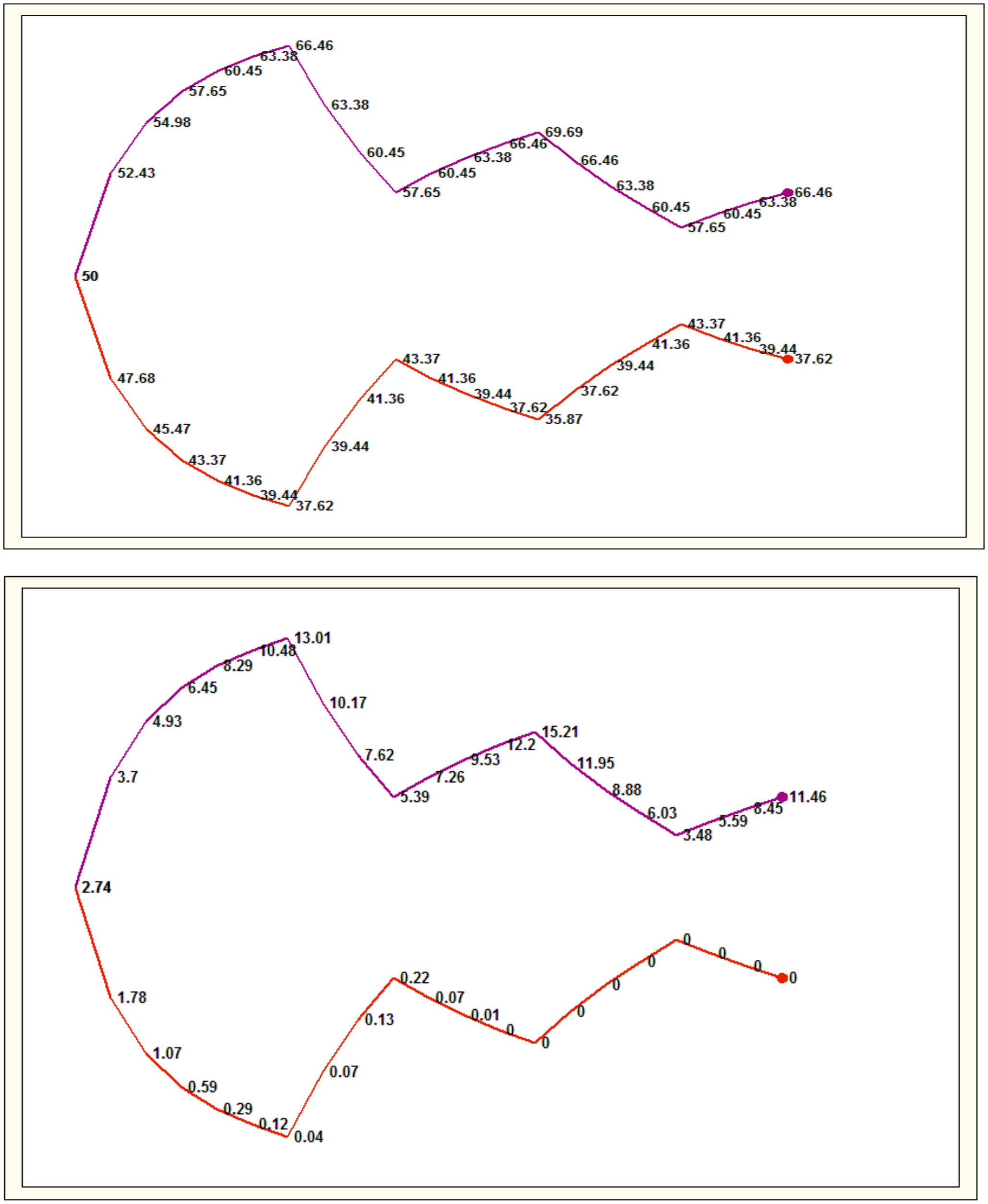

Figure III.112 Underlying and Option Trees generated by Matlab (Call option)

By increasing the number of discretization steps, we experience the convergence to the theoretical value of the Black-Scholes analytic formula.

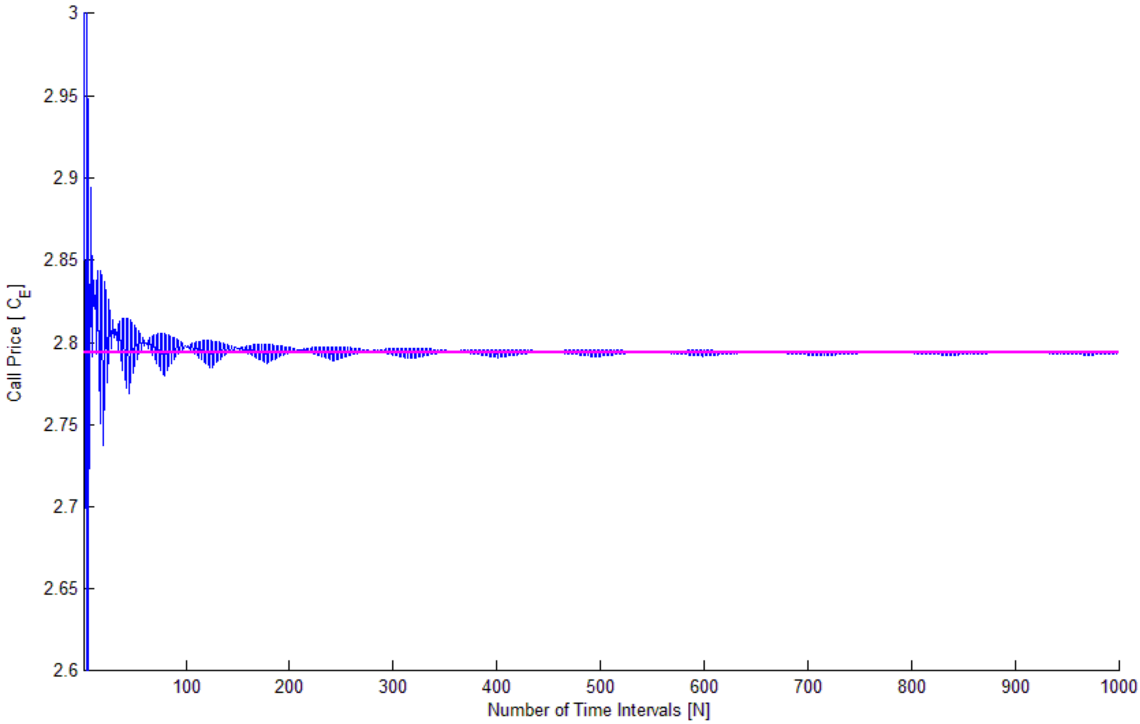

Figure III.113 Convergence to the exact BS Call price close formula

The same procedure has been conducted for the valuation of the European put. In this case, the pricing with the CRR tree having 20 discretization steps leads to a fair value of the derivative equal to 6.3797.

The following figures show the values of the European put assumed in the various nodes of the binomial tree and the verification of the convergence to the Black-Scholes analytic formula.

For $N = 1000$ we obtain a price of 6.4352.

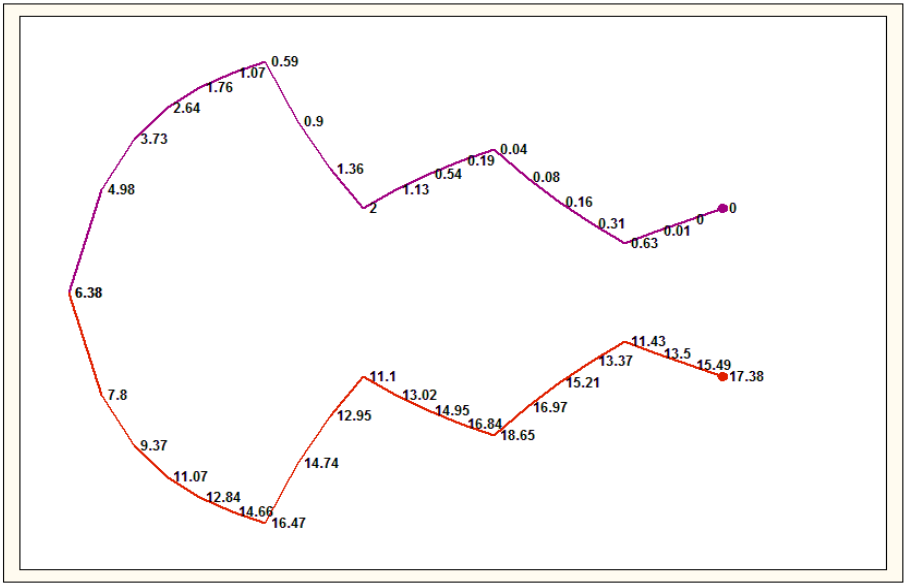

Figure III.114 Option Tree generated by Matlab (Put option)

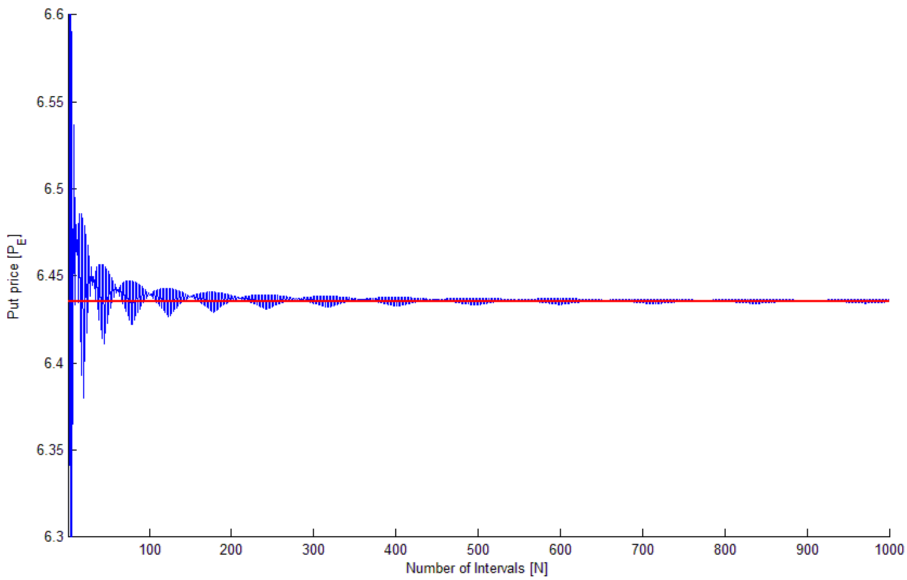

Figure III.115 Convergence to the exact BS Put price close formula

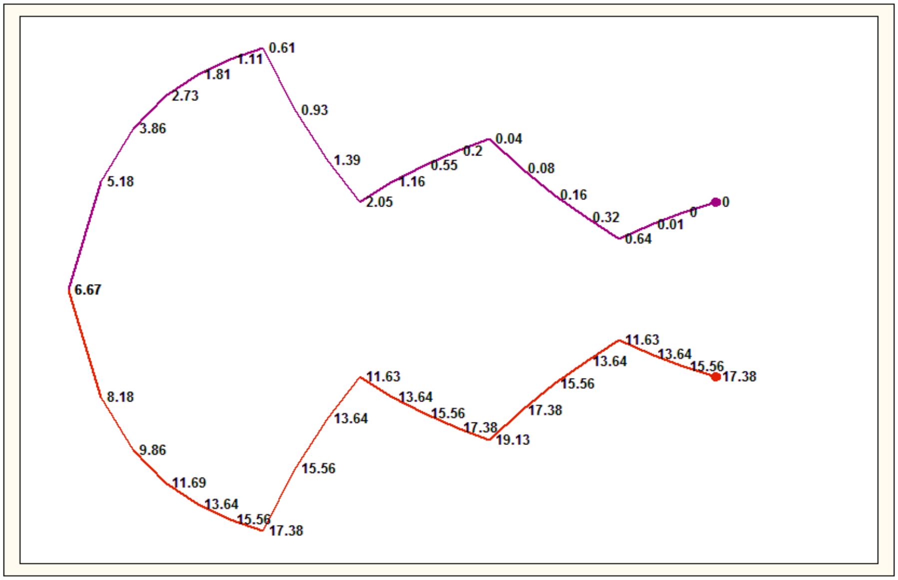

Figure III.116 Option Tree generated by Matlab (American Put option)

The price of the American put option is equal to $P_A = 6.6942$.

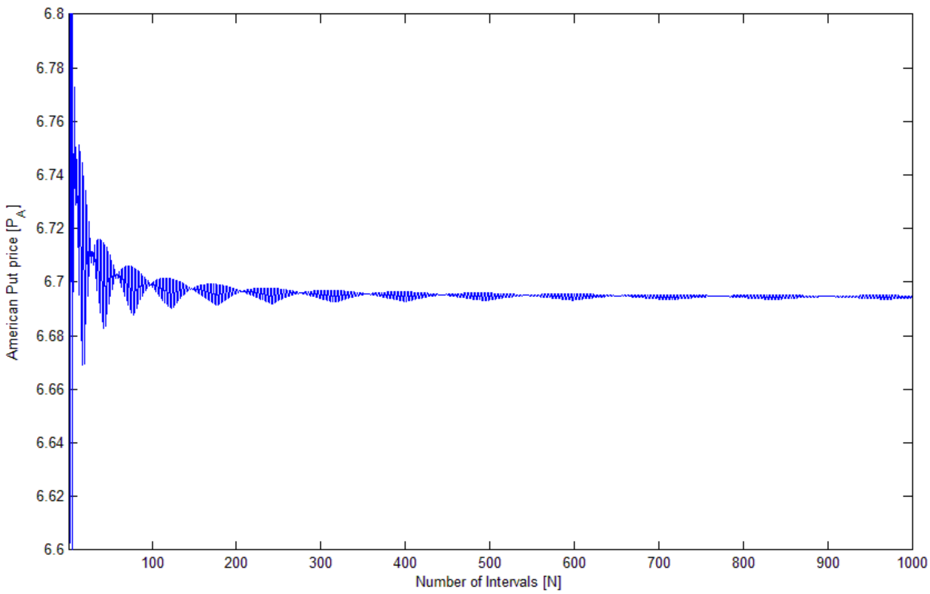

Figure III.117 Convergence plot for an American Put option

The price diagram generated for $N = 20$ time steps to determine the value of an American call option is identical to that used to value the European call option with the same financial characteristics. This means that at each node of the binomial tree it has never been convenient to exercise the option in advance: the derivative is therefore always worth more while it is alive. This result is absolutely in line with the theory, since it satisfies the property of the options which affirms the parity of value between the American call option and the European call option, if they are written on an underlying share that does not pay a dividend and in normal market conditions (risk-free rate positive, in this case $r = 5\%$).

For $N = 1000$, it is possible to determine a price of the derivative equal to $C_E = C_A = 2.7932$ with an estimation error lower than the third decimal place. Similarly to the logical steps discussed for the analytical formulas of Black-Scholes, the numerical formulas of the binomial approach can be extended to further underlyings by introducing the parameter b called **cost-of-carry**. Depending on the value assumed by the parameter b, we reach a pricing framework that can be used for a large number of underlyings on which the call/put option can be written. The adjustment to be made is on the definition of the risk-neutral probability Π

$$\prod = \frac{\exp(b \cdot \Delta t) - d}{u - d} \quad \textit{(Eq. III.196)}$$

Let us analyze the different cases:
- If $b = r$, the definition is suitable for the pricing of options written on shares that pay no dividend.
- If $b = r - q$, the definition is suitable for the pricing of options written on shares/indexes with a continuous dividend yield q.
- If $b = 0$, the definition is suitable for the pricing of options on futures.
- If $b = r - r_{FOR}$, the definition is suitable for the pricing of currency options.

The total number of nodes making up the binomial tree is: $\frac{(N+1)\cdot(N+2)}{2}$.
It should be highlighted that a low volatility and a relatively high cost-of-carry can lead to negative risk-neutral probabilities. In particular, this phenomenon occurs when the condition that $\sigma < \left|b\sqrt{T}\right|$ is satisfied. The drawback in obtaining negative probabilities of occurrence is in and of itself inconsistent with the basic axioms of statistics, but for pricing purposes it does not lead to obtaining a model output divergent from that expected from the theory.
The real problem arises though when, faced with the occurrence of this condition, it is not possible to generate a set of probability states large enough to cover all relevant events. To deal with this problem, it is worth highlighting that the parameters characterizing the binomial chain CRR Π, u and d, chosen in such a way as to agree with the first and second moment of the distribution of the underlying, are not the only triad that satisfies this condition.
There are several formulations in the literature, and the most popular setting is $\Pi = 0.5$, which is the so-called **Equal-Probability trees** (EQP Tree). Rendleman-Bartter actually formulated the first approach in the same year of discussion of the CRR Tree, proposing the following triad of parameters:

$$\prod = 0.5, u = \exp\left[(b - \sigma^2/2)\Delta t + \sigma \cdot \sqrt{\Delta t}\right],\ d = \exp\left[(b - \sigma^2/2)\Delta t - \sigma \cdot \sqrt{\Delta t}\right] \quad \textit{(Eq. III.197)}$$

Similarly to the Cox-Ross-Rubinstein method, the Rendleman-Bartter model also allows convergence to the theoretical values of the closed formulas of the Black-Scholes framework.

As a practical implementation of this model, the derivatives proposed in the previous example are recalculated with the EQP Tree: S=50, K=55, T=0.5, r=5%, σ=30%.

- With $N = 20$, we calculate: $P_A = 6.6686$, $P_E = 6.3876$ and $C_E = C_A = 2.7456$.

- With $N = 1000$, we calculate: $P_A = 6.6940$, $P_E = 6.4351$ and $C_E = C_A = 2.7930$.

The numerical examples proposed so far have assumed the possibility of exercising the option at any moment while it is alive. This right that can be exercised continuously makes it an American-style option. For the valuation of Bermuda-type options, i.e. those types of derivatives that can be exercised only in specific periods, the binomial models presented here can however be usefully employed. In fact, in this case the formulas for the test of convenience to exercise the option in advance no longer occurs for all the nodes, but only for those involved. An analyst typically has the foresight to adopt a time discretization designed in such a way as to match with the dates on which the option holder can exercise his right, and the above formulas can be used only for the nodes of the chain present in that stage. For all the others, the standard backwardation process is adopted.

In short:

If j belongs to an interval in which the Bermuda option can be exercised, for all indices i the following is adopted:

$$P_{j,i} = \max\left\{K - S \cdot u^i \cdot d^{j-1}, \frac{P_{j+1,i+1} \cdot \prod + P_{j+1,i} \cdot (1 - \prod)}{1+R}\right\} \text{ (Eq. III.198)}$$

$$C_{j,i} = \max\left\{S \cdot u^i \cdot d^{j-1} - K, \frac{C_{j+1,i+1} \cdot \prod + C_{j+1,i} \cdot (1 - \prod)}{1+R}\right\} \text{ (Eq. III.199)}$$

Otherwise:

$$P_{j,i} = \frac{P_{j+1,i+1} \cdot \prod + P_{j+1,i} \cdot (1 - \prod)}{1+R} \text{ (Eq. III.200)}$$

$$C_{j,i} = \frac{C_{j+1,i+1} \cdot \prod + C_{j+1,i} \cdot (1 - \prod)}{1+R} \text{ (Eq. III.201)}$$

As an example, let us consider the valuation of the put option discussed above, assuming that it can be exercised every month, at the end of the month. The financial characteristics of the Bermuda put option thus are: S=50, K=55, T=0.5, r=5%, σ=30%, exercisable monthly. Considering months with 30 days and adopting a daily discretization of the binomial tree ($N = 180$), the condition of early exercise will be tested in correspondence of all the nodes generated for each j multiple of 30.

The price of the Bermuda put, calculated in this way is equal to:

- $P_{BERMUDA} = 6.6531$ using the EQP Tree.

- $P_{BERMUDA} = 6.6519$ using the CRR Tree.

Months	Year fraction	Days	j exercise
1	0.083333	30	30
2	0.166667	60	60
3	0.25	90	90
4	0.333333	120	120
5	0.416667	150	150
6	0.5	180	180

Table III.57 CRR Tree: Bermuda Option pricing

If it is impossible to make the time index coincide exactly with the date on which the option can be exercised, the check is implemented in the j-closest to this event.

The binomial tree can easily be generalized for the pricing of exotic options, whose pay-off is not path-dependent, i.e. dependent on the values of the underlying assumed during the life of the derivative. The most general expression of the European binomial model is:

$$C = \exp(-rT) \cdot \sum_{j=a}^{N} \frac{N!}{j! \cdot (N-j)!} \cdot (\prod)^j \cdot (1 - \prod)^{N-j} \cdot g[S(T), K] \quad \textit{(Eq. III.202)}$$

$$P = \exp(-rT) \cdot \sum_{j=0}^{a-1} \frac{N!}{j! \cdot (N-j)!} \cdot (\prod)^j \cdot (1 - \prod)^{N-j} \cdot g[S(T), K] \quad \textit{(Eq. III.203)}$$

Where $S(T) = S \cdot u^i \cdot d^{n-1}$ and $g[S(T), K]$ indicates a generic function that expresses the payoff of the exotic option at maturity. This formulation makes it clear that the numerical model can be extremely powerful: it is able to valorize any European option written on a single asset, whose pay-off is not path-dependent. For example, if we wanted to obtain the price of a power option, whose pay-off at expiry is equal to: $\max[S^2 - K, 0]$, the pricing formula would be: $g[S(T), K] = \max[(S \cdot u^j \cdot d^{n-j})^2 - K, 0]$.

Further examples of exotic options that can be treated with the generalized European binomial tree are:

Power Contract (type A): S^m

Capped power contract: $\min[S^m, Cap]$

Power Contract (type B): $\left(\frac{S}{K}\right)^m$

Power Contract (type C): $(S - K)^m$

Standard Power Option: $\max[\text{z} \cdot (S^m - K), 0]$

Capped Standard Power Option: $\min\{\max[\text{z} \cdot (S^m - K), 0], Cap\}$

Powered Option: $\max[\text{z} \cdot (S - K), 0]^m$

Capped Powered Option: $\min\{\max[z \cdot (S-K), 0]^m, Cap\}$

Log Contract: $\ln\left(\frac{S}{K}\right)$

Log Option: $\max\left[\ln\left(\frac{S}{K}\right), 0\right]$

Square root contract: $\max\sqrt{[z \cdot (S-K), 0]}$

With $z = +1$ if call and $z = -1$ if put. $m \in \mathbb{R}$

For completeness, it should be noted that the simplest path-dependent options can be valuated using stochastic trees, but certain non-trivial adjustments to the basic formula are necessary. For derivatives that cannot be managed through trees (or through any other deterministic valuation algorithm), the Monte Carlo methodology is used, since it can be used with any financial instrument, despite having the drawback of being a stochastic technique.

We now focus on how to estimate sensitivity measures (Greeks) using the binomial tree. As we know, the Delta of an option Δ, is the first derivative of the price of the derivative with respect to the price of the underlying: $\Delta = \frac{\Delta f}{\Delta S}$, where ΔS is a small change in the share price and Δf is the corresponding small change in the fair-value of the option.

At time Δt, we have an estimate f_{11} of the option price when the underlying price is $S_0 \cdot u$ and an estimate $f_{10,}$ of the option price when the value of the underlying is $S_0 \cdot d$. In other words, when $\Delta S = S_0 \cdot u - S_0 \cdot d$, the value of Δf is equal to $f_{11} - f_{10}$.

Therefore, an estimation of Δ at time Δt is given by:

$$\Delta = \frac{f_{11} - f_{10}}{S_0 \cdot u - S_0 \cdot d} \text{ (Eq. III.204)}$$

To determine Γ, i.e. the second derivative of the price of the derivative with respect to the price of the underlying, we note that two estimates of the Greek Δ are available at time $2\Delta t$.

When $S = \frac{S_0 \cdot u \cdot d + S_0 \cdot d^2}{2}$ (halfway between the first and second node), the delta is equal to $\Delta^D = \frac{f_{21} - f_{20}}{S_0 - S_0 \cdot d^2}$.

The difference between the two values of S is h, where $h = \frac{S_0 \cdot u^2 - S_0 \cdot d^2}{2}$.

By definition, Gamma represents the first derivative of Delta with respect to S, so the incremental ratio that provides an estimate of Γ is:

$$\Gamma = \frac{\Delta^U - \Delta^D}{h} = \frac{\frac{f_{22} - f_{21}}{S_0 \cdot u^2 - S_0 \cdot u \cdot d} - \frac{f_{21} - f_{20}}{S_0 \cdot u \cdot d - S_0 \cdot d^2}}{h}, \Delta = \frac{f_{11} - f_{10}}{S_0 \cdot u - S_0 \cdot d} \text{ (Eq. III.205)}$$

Another widely used measure of sensitivity that can be obtained directly from the tree is Theta, Θ, which is the derivative of the option price with respect to time. It can be estimated by calculating the ratio:

$$\Theta = \frac{f_{21} - f_{00}}{2 \cdot \Delta t} \text{ (Eq. III.206)}$$

The remaining sensitivity measures cannot be calculated starting from the values read on the nodes of the tree, and it is necessary to calculate the value of the option twice according to the general formulas:

$$\vartheta = \frac{f(\sigma + \Delta\sigma) - f(\sigma - \Delta\sigma)}{2 \cdot \Delta\sigma} \text{ (Eq. III.207)}; \rho = \frac{f(r + \Delta r) - f(r - \Delta r)}{2 \cdot \Delta r} \text{ (Eq. III.208)}$$

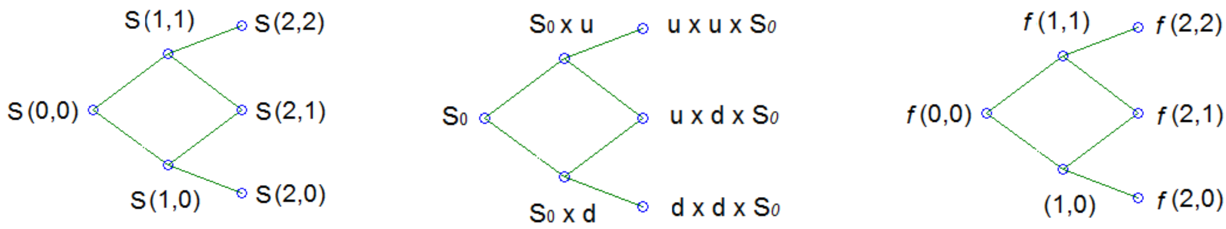

Figure III.118 Binomial trees indexing for Greek computation

Below is the estimate of the sensitivities in the case of the example provided above for the European call C_E. In short, inputs are: S=50, K=55, T=0.5, r=5%, σ=30%. The Greeks were computed using a 20-interval CRR Tree:

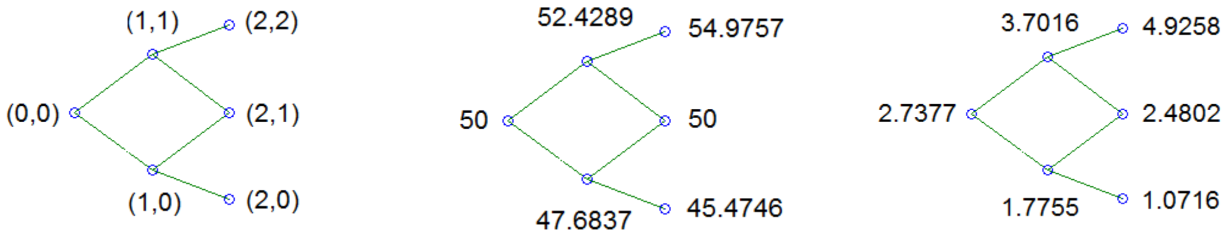

Figure III.119 Binomial trees for Greek computation

$$\Delta = \frac{f_{11} - f_{10}}{S_{11} - S_{10}} = \frac{3.7016 - 1.7755}{52.4289 - 47.6837} = 0.405905$$

$$\Gamma = \frac{\frac{f_{22} - f_{21}}{S_{22} - S_{21}} - \frac{f_{21} - f_{20}}{S_{21} - S_{20}}}{\frac{S_{22} - S_{20}}{2}} = \frac{\frac{4.9258 - 2.4802}{54.9757 - 50} - \frac{2.4802 - 1.0716}{50 - 47.6837}}{\frac{54.9757 - 45.4746}{2}} = 0.037942$$

$$\Theta = \frac{f_{21}-f_{00}}{2\cdot\Delta t} = \frac{f_{21}-f_{00}}{2\cdot\frac{T}{N}} = \frac{2.4802-2.7377}{2\cdot\frac{0.5}{20}} = -5.15 \rightarrow \Theta = -\frac{5.15}{360} = -0.01431.$$

The Vega (ϑ) and the Rho (ρ) cannot be estimated by reading the values directly on the nodes of the tree: it is necessary to use the definition:

$$\rho = \frac{Change\ in\ value\ of\ option}{Change\ in\ interest\ rates} = \frac{f(r+\Delta r)-f(r-\Delta r)}{2\cdot\Delta r} \quad \textit{(Eq. III.209)}$$

$$\vartheta = \frac{Change\ in\ value\ of\ option}{Change\ in\ volatility} = \frac{f(\sigma+\Delta\sigma)-f(\sigma-\Delta\sigma)}{2\cdot\Delta\sigma} \quad \textit{(Eq. III.210)}$$

It is also necessary to implement a double valuation of the fair value in correspondence with the deviations of the monitored parameter, obviously without modifying the other input parameters of the algorithm.

For $\Delta r = 1\ bp$:

$$\rho = \frac{f(0.05+0.0001)-f(0.05-0.0001)}{2\cdot 0.0001} = \frac{2.7386-2.7368}{2\cdot 0.0001} = 9$$

For $\Delta\sigma = 1\%$:

$$\vartheta = \frac{f(0.3+0.01)-f(0.3-0.01)}{2\cdot 0.01} = \frac{2.8838-2.6146}{2\cdot 0.01} = 13.46$$

Using a CRR Tree of 1,000 time intervals, the following values are obtained:

$S_{00} = 50, S_{10} = 49.6657\ S_{11} = 50.3365, S_{20} = 49.3337\ S_{21} = 50\ \ S_{22} = 50.6753$

$f_{00} = 2.7932\ f_{10} = 2.6554\ f_{11} = 2.9310\ f_{20} = 2.5227\ f_{21} = 2.7881\ f_{22} = 3.0738$

Greeks estimated with such a dense arborescence are very close to those calculated with the closed analytical formulas of Black-Scholes, as shown below:

$$\Delta_{\text{CRR}} = \frac{f_{11}-f_{10}}{S_{11}-S_{10}} = \frac{2.9310-2.6554}{50.3365-49.6657} = 0.410853$$

$$\Delta_{FORM} = N(d_1) = N\left(\frac{\ln\left(\frac{50}{55}\right)+\left(0.05+0.3^2/2\right)\cdot 0.5}{0.3\cdot\sqrt{0.5}}\right) = N(-0.22538) = 0.410842$$

$$\Gamma_{\text{CRR}} = \frac{\frac{f_{22}-f_{21}}{S_{22}-S_{21}}-\frac{f_{21}-f_{20}}{S_{21}-S_{20}}}{\frac{S_{22}-S_{20}}{2}} = \frac{\frac{3.0738-2.7881}{50.6753-50}-\frac{2.7881-2.5227}{50-49.3337}}{\frac{50.6753-49.3337}{2}} = 0.036899$$

$$\Gamma_{\text{FORMULA}} = \frac{n(d_1)}{S\sigma\sqrt{T}} = \frac{n(-0.22538)}{50\cdot 0.3\cdot\sqrt{0.5}} = 0.036669$$

$$\Theta_{\mathrm{CRR}} = \frac{f_{21} - f_{00}}{2 \cdot \frac{T}{N}} = \frac{2.7881 - 2.7932}{2 \cdot \frac{0.5}{1000}} = -5.1 \rightarrow \Theta_{\mathrm{CRR}} = -\frac{5.1}{360} = -0.014167$$

$$\Theta_{\mathrm{FORMULA}} = -\frac{\frac{S \cdot \sigma}{2 \cdot \sqrt{T}} \cdot n(d_1) - Ke^{-rT} \cdot r \cdot N(d_1 - \sigma\sqrt{T})}{360} = -0.0139$$

$$\rho_{CRR} = \frac{f(r+\Delta r) - f(r-\Delta r)}{2 \cdot \Delta r} = \frac{f(0.05+0.0001) - f(0.05-0.0001)}{2 \cdot 0.0001} = \frac{2.7940435 - 2.7922689}{0.0002} = 8.8730$$

$$\rho_{\mathrm{FORMULA}} = T \cdot K \cdot e^{-rT} \cdot N(d_1 - \sigma\sqrt{T}) = 8.8743$$

$$\vartheta_{CRR} = \frac{f(\sigma+\Delta\sigma) - f(\sigma-\Delta\sigma)}{2 \cdot \Delta\sigma} = \frac{f(0.3+0.01) - f(0.3-0.01)}{2 \cdot 0.01} = \vartheta_{CRR} = \frac{2.931026 - 2.656952}{0.02} = 13.704$$

$$\vartheta_{FORMULA} = S \cdot \sqrt{T} \cdot n(d_1) = 13.7510.$$

FURTHER READINGS

Bottasso A., Bruno L., Giribone P. G. – "The impact of negative interest rates on the pricing of options written on equity: a technical study for a suitable estimate of early termination" – Risk Management Magazine Vol. 17, N. 3 (2022).

Fabbri M., Giribone P. G. – "Progettazione di un sistema di pricing e di gestione del rischio per il prodotto strutturato EAKO - European American Knock-Out option" – Risk Management Magazine Vol. 15, N. 2 (2020).

Giribone P. G., Ventura S. – "Studio della convergenza dei modelli di pricing discreti multinomiali azionari: teoria e applicazioni con tecniche di controllo dell'errore" – AIFIRM Magazine Vol. 6, N. 1 (2011).

III.6 MONTE CARLO

The value of a derivative is closely linked to the price changes of the underlying financial asset $S(t)$ in the period of time between the signing of the contract and the maturity date $t \in [0, T]$. For this purpose, it is necessary to mathematically describe a dynamic that represents the possible future trajectories that can be assumed by the asset on which the option is written. The stochastic process commonly adopted and consistent with the Black-Scholes-Merton pricing framework is called Geometric Brownian motion and it is represented by the following Stochastic Differential Equation (SDE):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_t$$ *(Eq. III.211)*

μ is the annualized expected return earned by an investor over time period dt. In a risk-neutral context it is set equal to the risk-free rate r.
σ is the annualized volatility of the asset.
dW_t is a Wiener process.
The SDE can be integrated using Euler's numerical scheme and implemented in a numerical processing software:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_t \rightarrow \Delta S = \mu S\Delta t + \sigma S\Delta \mathrm{W} \rightarrow$$
$$S_t = S_{t-1} + \mu S_{t-1}\Delta t + \sigma S_{t-1}\epsilon\sqrt{\Delta t}$$ *(Eq. III.212)*

Where ϵ is a draw from a standard normal distribution.

The figure below depicts a simulated daily **path** of an asset having the following characteristics: $S = 50$, $r = 5\%$ and $\sigma = 30\%$. The period of time over which the simulation was conducted is 180 days (continuous trading), thus $T = 0.5 \rightarrow \Delta t = \frac{0.5}{180}$.

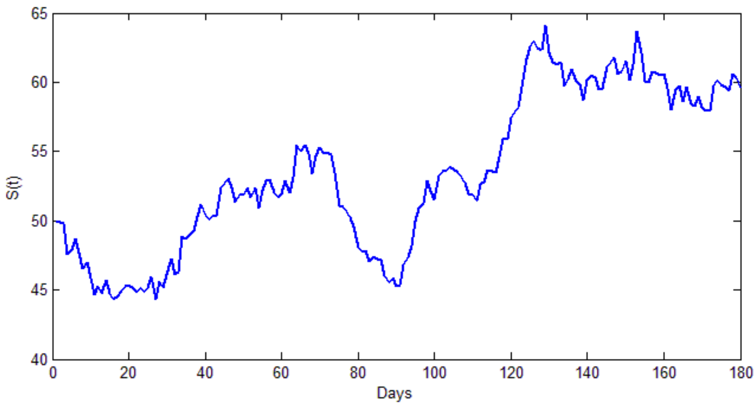

Figure III.120 A Monte Carlo path

By repeating the routine several times, numerous possible paths of the asset can be simulated. Figure III.121 actually shows 100 simulated paths with the previous inputs. As we have seen previously, to value a European call option, the $\max(S_T - K, 0)$ pay-off is directly applied with K being the strike price at maturity.

Therefore, for each path, the final value S_T is considered and the discounted pay-off is computed: $\exp(-rT)\max(S_T - K, 0)$. In the end there will be as many possible prices for the derivative as the simulations conducted. The fair value of the option will be given by the average of these prices.

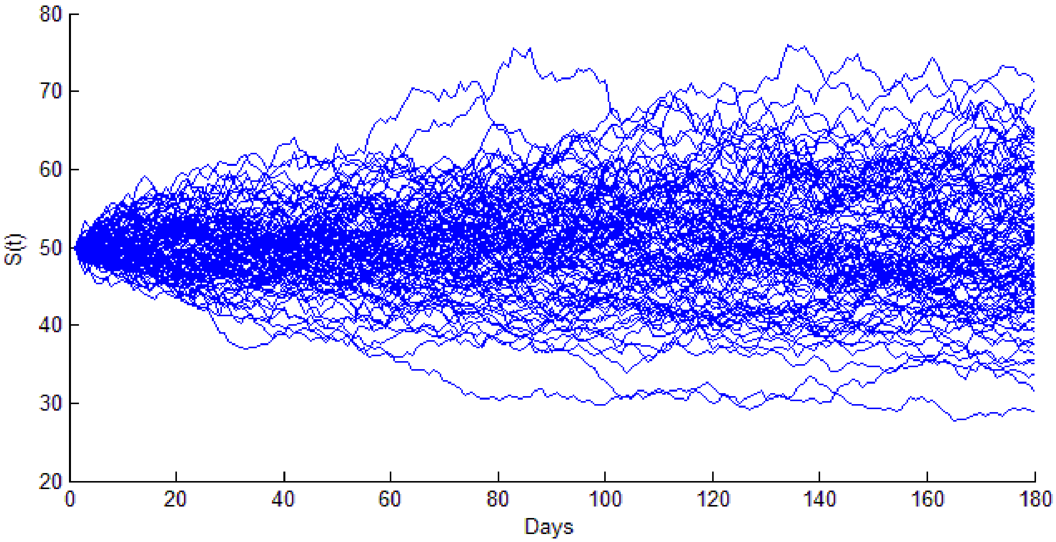

Figure III.121 100 simulated Monte Carlo paths

Assuming that the call option has an exercise price equal to $K = 55$, the valuation of the European call option is implemented (S=50, K=55, T=0.5, r=5%, σ=30%).

Simulations	100	1,000	10,000	100,000	1,000,000	10,000,000
Fair Value	2.1816	2.9605	2.7258	2.7621	2.7864	2.7953

Table III.58 Option prices varying the number of simulations

The table above shows the fair values obtained using the Monte Carlo method. Given that this approach, by definition, is stochastic (for the same input, different outputs are obtained), it is essential to be able to quantify the level of error that can be committed using a thousand simulations rather than a million.

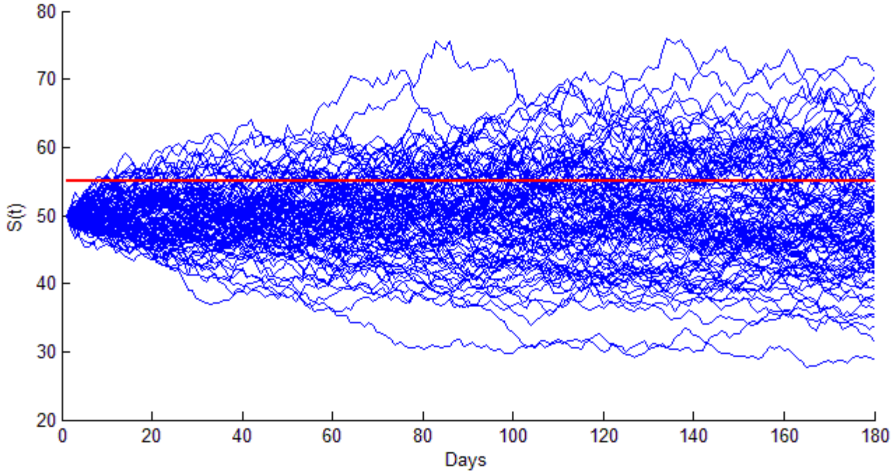

Figure III.122 100 simulated Monte Carlo paths with Option Exercise Price

It is therefore necessary to accompany the information with a measure of the divergence of the output. Typically, for this purpose, the standard deviation calculated on the outputs produced by the simulator after various launches is used. Expressing the concept in more formal terms, it is customary to estimate the speed of output convergence by calculating the standard deviation over a campaign of K replications of N runs each. In our case, a campaign of 15 replications of N runs has been conducted, as shown in the table below.

Runs	Repl.1	Repl.2	Repl.3	Repl.4	Repl.5	Repl.6	Repl.7	Repl.8	Repl.9	Repl.10	Repl.11	Repl.12
10^2	3.125	2.322	2.834	2.120	2.224	4.782	2.875	2.088	2.797	3.079	2.192	2.532
10^3	2.707	2.662	2.579	3.190	2.626	2.667	2.855	2.902	2.628	3.057	2.606	2.685
10^4	2.770	2.790	2.797	2.848	2.632	2.778	2.705	2.812	2.901	2.730	2.782	2.835
10^5	2.794	2.796	2.768	2.797	2.762	2.807	2.768	2.781	2.809	2.780	2.764	2.819
10^6	2.793	2.805	2.801	2.791	2.788	2.795	2.803	2.798	2.790	2.797	2.792	2.790
10^7	2.793	2.793	2.789	2.794	2.792	2.797	2.796	2.794	2.796	2.793	2.796	2.793

Runs	Repl.13	Repl.14	Repl.15	Great Mean	Std. Dev
10^2	4.409	2.944	3.393	2.914	0.795
10^3	2.859	2.597	2.899	2.768	0.185
10^4	2.814	2.860	2.814	2.791	0.066
10^5	2.790	2.779	2.810	2.788	0.018
10^6	2.786	2.799	2.795	2.795	0.006
10^7	2.792	2.791	2.796	2.794	0.002

Table III.59 A simulation campaign for option pricing

The great mean (i.e. the average over the K outputs) expresses the statistically most reliable measure of the fair value of the derivative. It should also be noted that the value converges to the theoretical value calculated using the traditional closed formula of the Black-Scholes framework ($C_E = 2.7935$).

Stochastic calculus allows to formulate an analytic expression for the simulation of $S(t = T)$. This result is considered of extreme practical importance since it allows to implement direct simulations of the asset at a generic future time $t = T$ without the need to know the values assumed in previous times $S(t < T)$. Starting from the hypothesis that the variable follows a stochastic process of the type: $dS(t) = \mu S(t)dt + \sigma S(t)dW_t$.

It follows from Ito's lemma that there must exist a function $G\big(S(t)\big)$ which follows the following dynamics:

$$dG(S,\ t) = \left(\frac{\partial G(S,t)}{\partial S}\mu S + \frac{\partial G(S,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 G(S,t)}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G(S,t)}{\partial S}\sigma S(t)dW_t \quad \textit{(Eq. III.213)}$$

Setting $G = \ln(S)$, we obtain:

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

$$d\ln(S(t)) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

Integrating the expression over time, we reach:

$$\int_0^T d\ln(S(t)) = \int_0^T \left(\mu - \frac{\sigma^2}{2}\right)dt + \int_0^T \sigma dW_t$$

$$\ln\left(\frac{S(T)}{S(0)}\right) = \left(\mu - \frac{\sigma^2}{2}\right)T + \sigma dW_T$$

$$S(T) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma dW_T\right]$$

The latter expression can be implemented in a numerical processing software:

$$S(T) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma dW_T\right] \rightarrow S(T) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{T}\right] \quad \textit{(Eq. III.214)}$$

Where ϵ represents an extraction from a normal distribution with zero mean and unit variance. The formula therefore allows to simulate the value of the asset underlying an option at any point in time, in a manner consistent with the BS framework. In the case of the previous valuation, the computational advantage is evident in the implementation of the analytical formula for solving the SDE rather than the numerical one. In fact, the latter simulates the final price of the asset $S(T)$ (i.e. the price level assumed on the 180th day) only after calculating the previous 179 time-steps; while the analytical formula can estimate it directly:

$$S(T) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{T}\right]$$

$$S(T) = 50\exp\left[\left(0.05 - \frac{0.3^2}{2}\right)0.5 + 0.3\epsilon\sqrt{0.5}\right] = 50\exp[0.0025 + 0.212132 \cdot \epsilon]$$

The estimation of the call fair value is significantly more efficient:

$$c = \frac{\exp(-rT)}{N}\sum_{i=1}^{N}\max\left\{S\exp\left[(r-\sigma^2/2)T+\sigma\epsilon_i\sqrt{T}\right]-K,0\right\} \quad \textit{(Eq. III.215)}$$

The formulas can be extended to different categories of underlyings by introducing the cost-of-carry parameter, b. Depending on its value, we reach a pricing framework that can be used for a large number of underlyings on which the call/put option can be written:

$$c = \frac{\exp(-rT)}{N}\sum_{i=1}^{N}\max\left\{S\exp\left[(b-\sigma^2/2)T+\sigma\epsilon_i\sqrt{T}\right]-K,0\right\} \quad \textit{(Eq. III.216)}$$

$$p = \frac{\exp(-rT)}{N}\sum_{i=1}^{N}\max\left\{K-S\exp\left[(b-\sigma^2/2)T+\sigma\epsilon_i\sqrt{T}\right],0\right\} \quad \textit{(Eq. III.217)}$$

Let us analyze the different cases depending on the cost-of-carry parameter:
- If $b = r$, the definition is suitable for pricing options written on shares that pay no dividend.
- If $b = r - q$, the definition is suitable for pricing options written on shares/indexes with a continuous dividend yield q.
- If $b = 0$, the definition is suitable for pricing options on futures.
- If $b = r - r_{FOR}$, the definition is suitable for pricing currency options.

The price graph displayed in the figure below was obtained through a campaign of one thousand replications of one million paths each. The theoretical fair value for this financial instrument, equal to the average of the prices obtained from the Monte Carlo, is 6.435588 with a standard deviation of 0.006725. The convergence value (for $NSim \to \infty$) can be calculated using the BS closed formula and it is: $P_E = 6.4355920$.

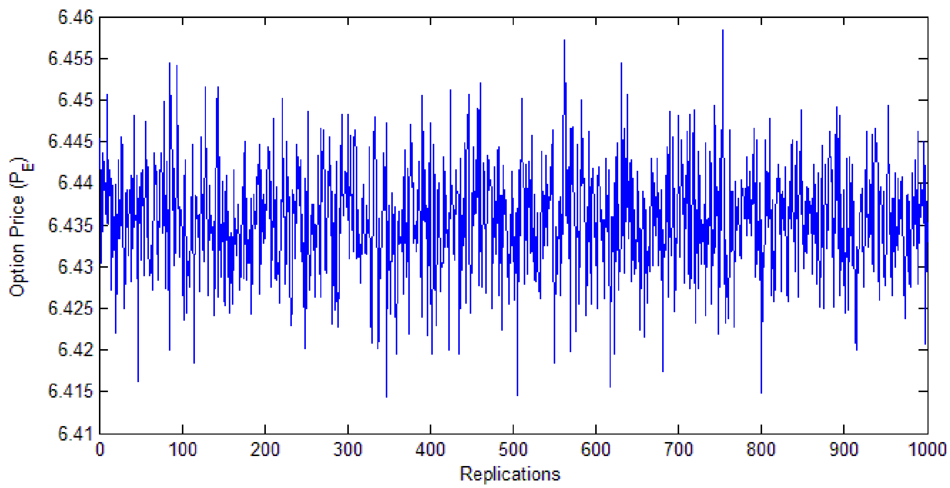

Figure III.123 Prices obtained through a campaign of one thousand replications of one million paths each

Thanks to its flexibility, the Monte Carlo methodology allows for the pricing of highly exotic derivatives characterized by non-standard and highly non linear pay-offs.

In fact, using the formula $S(T) = S \exp\left[\left(b - \frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{T}\right]$, it is possible to simulate the potential price levels of a single underlying in a generic time t and therefore evaluate the pay-off that characterizes the exotic derivative at the dates of interest.

One of the strengths of the Monte Carlo method is the possibility to evaluate path-dependent options.

As we wish to monitor the trend of the underlying over the entire life of the option, it is useful to express the closed formula as a function of Δt:

$$S(T) = \sum_{j=1}^{m} S_{j-1} \exp\left[(b - \sigma^2/2)\Delta t + \sigma\epsilon_j\sqrt{\Delta t}\right] \quad \textit{(Eq. III.218)}$$

Where m is the number of temporal intervals, $\Delta t = \frac{T}{m}$ and $S(0)$ is the initial value for the asset at $j = 1$.

Let us now consider a derivative that does not have a closed valuation formula: callable options. As we know, callable options are call options that the owner is forced to exercise if the price level of the underlying asset on which they are written exceeds a pre-set level, called the barrier (H) for a number of consecutive days ($MovingDaysN$).

Similarly, a callable put option requires that the holder exercises, if the price level of the asset on which the option is written falls below the barrier for a specified number of consecutive days.

Let us examine an example on the pricing of a callable call option with the following characteristics:

$S = 50$ is the spot price.

$K = 55$ is the strike price.

$H = 60$ is the barrier level.

$MovingDaysN = 10$ is the number of days in which the underlying price must be beyond the barrier.

$T = 0.5$ is the time to maturity.

$r = 5\%$ is the risk-free rate.

$\mathrm{q} = 2\%$ is the continuous dividend yield, thus $b = r - q = 3\%$ represents the cost-of-carry.

$\sigma = 30\%$ is the underlying volatility.

The figure below shows a price histogram obtained through a campaign of one thousand replications of one million paths each.

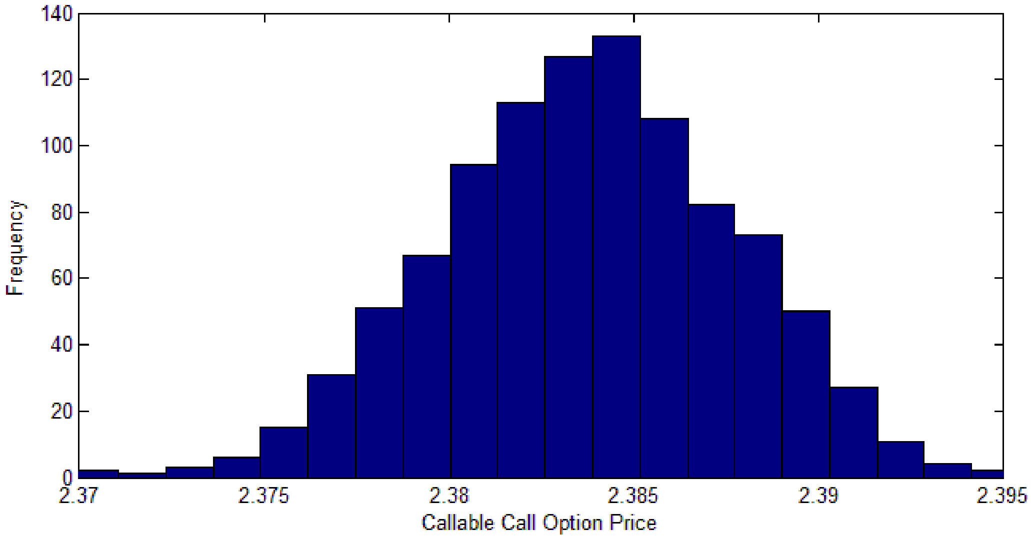

Figure III.124 Callable Option price

The theoretical fair value for this financial instrument is equal to the average of the prices obtained by the Monte Carlo method. The price of the callable call option considered is 2.3837 with a standard deviation of 0.003937.

This method is also capable of valuing options written on several underlyings, taking into account the correlation (ρ) existing between the assets. In the case of two underlyings S_1 and S_2, the formulas are

$$S_1 + \Delta S_1 = S_1 \exp\left[\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\Delta t + \sigma_1 \alpha_{1,t}\sqrt{\Delta t}\right] \quad \textit{(Eq. III.219)}$$

$$S_2 + \Delta S_2 = S_2 \exp\left[\left(\mu_2 - \frac{1}{2}\sigma_2^2\right)\Delta t + \sigma_2 \alpha_{2,t}\sqrt{\Delta t}\right] \quad \textit{(Eq. III.220)}$$

The correlation between the two underlyings is considered by setting:

$$\alpha_{1,t} = \epsilon_{1,t}; \alpha_{2,t} = \rho\epsilon_{1,t} + \epsilon_{2,t}\sqrt{1-\rho^2} \quad \textit{(Eq. III.221)}$$

We now illustrate a practical example considering the exotic path-dependent option called European arithmetic average spread option. The pay-off at maturity is defined as: $\max[z \cdot (Average1 - Average2 - K), 0]$.

If $z = 1$ call, if $z = -1$ put. $Average$ is the mean of the values of the asset during the life of the option.

The financial characteristics on which the valuation has been conducted using the Monte Carlo technique are:

$CallPutFlag = "call" \rightarrow \; z = +1$ → payoff: $\max[(Average1 - Average2 - K), 0]$

$S_1 = 100$ is the spot price for asset 1.

$S_2 = 50$ is the spot price for asset 2.

$K = 45$ is the strike price for the exotic path-dependent option.

$T = 0.5$ is the time to maturity in years.

$r = 5\%$ is the risk-free rate.

$b_1 = 2\%$ is the cost-of-carry for asset 1.

$b_2 = 3\%$ is the cost-of-carry for asset 2.

$\sigma_1 = 30\%$ is the volatility of asset 1.

$\sigma_2 = 25\%$ is the volatility of asset 2.

$\rho = 0.25$ is the correlation between asset 1 and asset 2.

$nSteps = 180$ is the number of time step intervals, thus $\Delta t = \frac{T}{nSteps} = 0.002778$.

$nSimulations = 10^6$ is the number of simulations (or paths) for computing a price.

The price histogram shown in the figure below was obtained through a campaign of one thousand replications of one million simulations each.

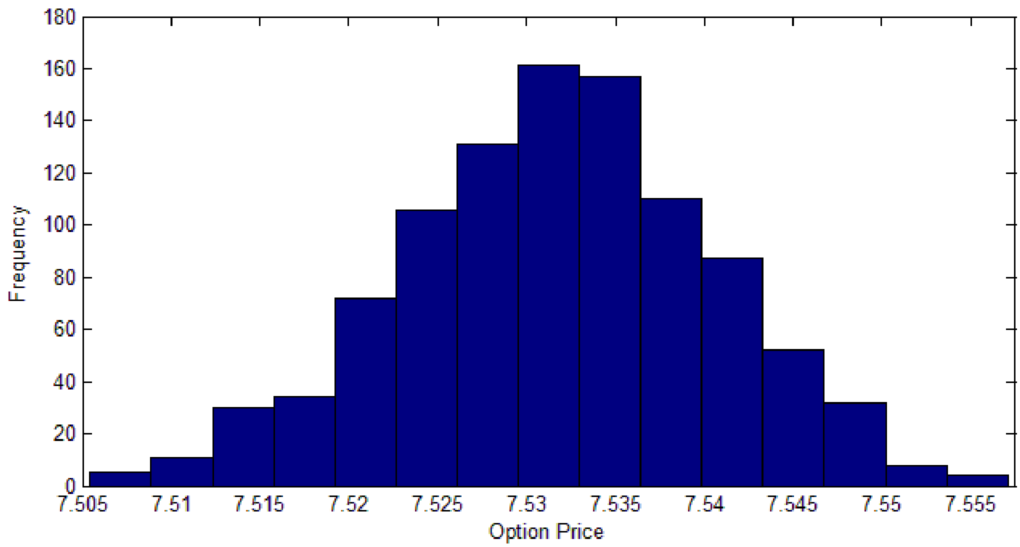

Figure III.125 European arithmetic average spread option

The theoretical fair value for this financial instrument is equal to the average of the prices obtained by the Monte Carlo method.

The price of the European arithmetic average spread option considered is 7.531744, with a standard deviation of 0.008887.

In the case of three underlyings S_1, S_2 and S_3, the formulas become:

$$S_1 + \Delta S_1 = S_1 \exp\left[\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\Delta t + \sigma_1 \alpha_{1,t}\sqrt{\Delta t}\right] \quad \textit{(Eq. III.222)}$$
$$S_2 + \Delta S_2 = S_2 \exp\left[\left(\mu_2 - \frac{1}{2}\sigma_2^2\right)\Delta t + \sigma_2 \alpha_{2,t}\sqrt{\Delta t}\right] \quad \textit{(Eq. III.223)}$$
$$S_3 + \Delta S_3 = S_3 \exp\left[\left(\mu_3 - \frac{1}{2}\sigma_3^2\right)\Delta t + \sigma_3 \alpha_{3,t}\sqrt{\Delta t}\right] \quad \textit{(Eq. III.224)}$$

The correlation between the three assets is considered by setting:

$$\alpha_{1,t} = \epsilon_{1,t} \quad \textit{(Eq. III.225)}$$

$$\alpha_{2,t} = \rho_{1,2}\epsilon_{1,t} + \epsilon_{2,t}\sqrt{1 - \rho_{1,2}^2} \quad \textit{(Eq. III.226)}$$

$$\alpha_{3,t} = \frac{\epsilon_{3,t}}{g} + \left(\rho_{2,3} - \rho_{1,3}\rho_{1,2}\right)\epsilon_{2,t} + \rho_{1,3}\epsilon_{1,t}\sqrt{\frac{1}{1-\rho_{1,2}^2}} \quad \textit{(Eq. III.227)}$$

$$g = \sqrt{\frac{1-\rho_{1,3}^2}{1-\rho_{1,2}^2-\rho_{2,3}^2-\rho_{1,3}^2+2\rho_{1,2}\rho_{1,3}\rho_{2,3}}} \quad \textit{(Eq. III.228)}$$

Let us now present an example with three underlyings, considering an exotic option written on three assets, called Option on Maximum of two spread options. Its pay-off at maturity is defined as $\max[z \cdot (S_1 - S_2 - K),\ z \cdot (S_3 - S_2 - K), 0]$ where $z = +1$ if call and $z = -1$ if put. The financial characteristics on which the valuation has been made using the Monte Carlo technique are the following:

$z = -1$

$\mathrm{S}_1 = 100,\ \mathrm{S}_2 = 55,\ \mathrm{S}_3 = 107$ are the spot prices for the underlyings.

$K = 50$ is the strike price.

$T = 0.5$ is the time to maturity expressed in year fractions.

$r = 5\%$ is the risk-free rate.

$b_1 = 1\%,\ b_2 = 3\%,\ b_3 = 2.5\%$ are the cost-of-carry of the underlyings.

$\sigma_1 = 30\%,\ \sigma_2 = 20\%,\ \sigma_3 = 15\%$ are the volatilities of the underlyings.

$\rho_{1,2} = 0.25, \rho_{1,3} = -0.1, \rho_{2,3} = 0.2$ are the correlations between the three underlyings.

$nSimulations = 10^6$ is the number of simulations (or paths) for computing one price.

The price histogram shown below was obtained through a campaign of one thousand replications of one million simulations each. The theoretical fair value for this financial instrument is equal to the average of the prices obtained by the Monte Carlo method.

The price of the Put Option on Maximum of two spread options is 13.18749 with a standard deviation of 0.011635.

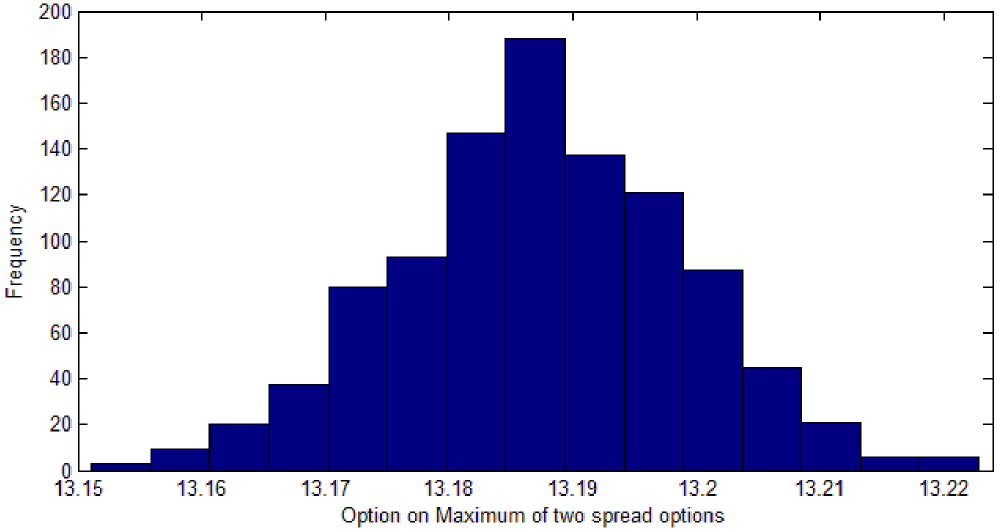

Figure III.126 Option on Maximum of two spread options

In order to build a correlation matrix to deal with N generic assets in the Monte Carlo simulation, the **Cholesky decomposition** is employed. If we have ϵ_1, ϵ_2, ϵ_3, …, ϵ_N multiple uncorrelated random numbers, the Cholesky decomposition is used to transform them into the correlated variables $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_N$. If we then define ϵ and α as column vectors having the values ϵ_i and α_i respectively in the rows, we can use the transformation: $\alpha = M\epsilon$ with M satisfying the condition: $MM^T = R$, where R is a positive definite symmetric correlation matrix. The objective therefore is to decompose the correlation matrix into a product of two matrices: M multiplied by transposed M must return R. The Cholesky decomposition is the most popular methodology for accomplishing this task. For example, in the case of two assets, the matrices assume the values:

$$R = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \; M = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \quad \textit{(Eq. III.229)}$$

The most common method for calculating Greeks with the Monte Carlo method is to use the approximation of finite differences.

To this aim, the first-order partial derivatives Δ and ϑ can be estimated using the **two-sided finite difference method:**

$$\Delta = \frac{f(S+\Delta S)-f(S-\Delta S)}{2\cdot\Delta S} \quad \textit{(Eq. III.230)} \qquad \vartheta = \frac{f(\sigma+\Delta\sigma)-f(\sigma-\Delta\sigma)}{2\cdot\Delta\sigma} \quad \textit{(Eq. III.231)}$$

The first-order partial derivative θ can be approximated through the **one-sided finite difference method:**

$$\theta = \frac{f(\theta)-f(T-\Delta T)}{\Delta T} \quad \textit{(Eq. III.232)}$$

Lastly, the second-order partial derivative Γ can be approximated with the **central finite difference method:**

$$\Gamma = \frac{f(S+\Delta S)-2\cdot f(S)+f(S-\Delta S)}{\Delta S^2} \quad \textit{(Eq. III.233)}$$

Where f is the pricing function coded for the derivative.

This way of proceeding is generally valid and can be considered efficient for all numerical methodologies. In the case of a stochastic technique, such as the Monte Carlo technique, if there is no optimal management of the error on the output, it may not be satisfactory. The reduction and control of the uncertainty that characterizes the Monte Carlo pricing method becomes a crucial point. In this context, among the different variance reduction methodologies present in the literature we mention: Antithetic Variates, Control Variates, Stratified Sampling, Latin Hypercube Sampling, Moment Matching and Importance Sampling.
Among the variance control methods present in the literature we mention the Mean Square Error (MSE) and the Mean Square Pure Error (MSPE).
The choice of the correct number of simulations to be conducted for estimating the fair value with a pre-fixed level of error on the output becomes an essential problem in order to correctly determine the price of the financial instrument and, consequently, its sensitivity measures. In order to consider the possible early exercise using the Monte Carlo methodology, the Longstaff-Schwartz 2001 method based on least squares regression is often implemented. When dealing with American options, it is not always easy to decide on the convenience of exercising the acquired right immediately or waiting till the expiry date. This difficulty, which in fact exists in the real world, is reflected in the integration engine. In other words, it is necessary to implement the correct logic for early exercise convenience in the algorithm. A good strategy could be represented by the following rule: if the immediate payment is greater than the expected future one, the holder will exercise the right, otherwise not. Assuming that the price of the underlying is $S(t)$ and the pay-off can be expressed in function of $\tilde{P}(\mathrm{S}(t),t)$, the most convenient exercise strategy at time $t = t_1$ can be expressed in mathematical terms in the form:

$$Exercise \leftarrow Yes, \tilde{P}(\mathrm{S}(t_1),t_1) > E_{t_1}\left[\tilde{P}(\mathrm{S}(t_2),t_2)\middle|\varphi_{t_1}\right] \quad \textit{(Eq. III.234)}$$

$$Exercise \leftarrow No, \tilde{P}(\mathrm{S}(t_1),t_1) < E_{t_1}\left[\tilde{P}(\mathrm{S}(t_2),t_2)\middle|\varphi_{t_1}\right] \quad \textit{(Eq. III.235)}$$

$E_{t_1}\left[\tilde{P}(\mathrm{S}(t_2),t_2)\middle|\varphi_{t_1}\right]$ is the expected value of future payments from time $t_2 > t_1$ to current time t_1 and φ_{t_1} represents the information available to the holder for $t = t_1$.
It is therefore necessary to "know" the expected value of future payments, to determine the correct fair value of the American option. For this purpose, the scientific literature proposes the usage of least squares regression. The starting point is a procedure generating M paths from the standard integration engine to simulate the future dynamics of asset S. Thus, for each path it is necessary to regress the future pay-offs on a normally polynomial

basis function F_i, which depends on the price assumed by S. Let us define Y as the pay-off vector for the M paths and $\mathbf{1}$, $F_1(S)$, $F_2(S)$ as the basis functions. By regressing Y on these basis functions we obtain the expression:

$$E[Y|S] = \alpha + \beta F_1(S) + \gamma F_2(S) \quad \textit{(Eq. III.236)}$$

This represents an estimation for the expected future value of the pay-offs as a function of the underlying S. Starting from this relationship, we can decide whether it is preferable to exercise the option immediately or to hold it until the following period. Such procedure must be recalled iteratively starting from the expiry date of the derivative T, up to the valuation time, $t = 0$ (which constitutes the so called Longstaff-Schwartz backwardation).

To better illustrate the method, let us consider the valuation of a Bermuda put option. Let us assume that $S(0) = 1$, the strike price is equal to $K = 1.1$, and the maturity of the derivative is 3 years. The option can be exercised every year, i.e. for $T = 1,\ 2,\ 3$ years. If $S(0) \neq 1$, it is advisable to normalize it to $S(0) = 1$ and adjust the strike price $K/S(0)$, to have a smaller error in the regression.

Let us now assume a risk-free rate of $r = 6\%$ and 8 simulated paths, for simplicity. The below table shows, on the left, the simulations obtained with the Monte Carlo method for asset S.

Path	t = 0	t = 1	t = 2	t = 3
1	1	1.07	1.53	1.95
2	1	0.76	0.78	0.71
3	1	0.85	0.69	0.76
4	1	0.96	1.01	0.97
5	1	0.95	1.06	1.28
6	1	1.59	1.26	1.07
7	1	1.28	1.23	0.97
8	1	1.11	1.57	1.89

Path	t = 0	t = 1	t = 2	t = 3
1	1	0	0	0.00
2	1	0	0	0.39
3	1	0	0	0.34
4	1	0	0	0.13
5	1	0	0	0.00
6	1	0	0	0.03
7	1	0	0	0.13
8	1	0	0	0.00

Table III.60 Simulated paths and Pay-off at time t=3

To use the Longstaff-Schwartz least squares method, the procedure is initialized at the end, and the discretization time steps are retraced back up to $t = 0$. For $t = 3$, the holder of the option exercises it only if it is convenient to do so: the pay-off matrix is shown on the right on Table III.60.

Now we want to determine for which paths it is preferable to exercise the option at time $t = 2$.

For $t = 2$, only the 4 in-the-money paths need to be taken into consideration since the other 4 have zero as pay-off. Among the paths $t = 2{,}3{,}4{,}5$, we determine the expected discounted payoffs Y, as shown below:

Path	S(t=2)	pay off(t=2)	In the money	pay off(t=3)	DF	Y
1	1.53	0	NO	0.00	0.941765	0
2	0.78	0.32	YES	0.39	0.941765	0.36738168
3	0.69	0.41	YES	0.34	0.941765	0.320199941
4	1.01	0.09	YES	0.13	0.941765	0.122429389
5	1.06	0.04	YES	0.00	0.941765	0
6	1.26	0	NO	0.00	0.941765	0
7	1.23	0	NO	0.00	0.941765	0
8	1.57	0	NO	0.00	0.941765	0
	[A]	[B]		[C]	[D]	[E]

[A] see Table III.60, [B] max[K-S(t=2),0], [C] max[K-S(t=3),0] , [D] exp(-rΔt)=exp(-0.06*1) , [E]=[C]x[D]

Table III.61 Determination of the variables for the first regression

We use the following as basis functions: $\mathbf{1}$, $F_1(S) = S$, $F_2(S) = S^2$.

We now need to run a regression of Y on these functions:

$E[Y|S] = \alpha + \beta F_1(S) + \gamma F_2(S) = \alpha + \beta S + \gamma S^2$ *(Eq. III.237)*

The regression coefficients are obtained by calculating the following matrix product:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T Y = \begin{pmatrix} -2.7077 \\ +7.8084 \\ -4.9566 \end{pmatrix} \quad \textit{(Eq. III.238)}$$

Where:

$$\boldsymbol{X} = (1 \quad F_1(S) = S \quad F_2(S) = S^2) = \begin{pmatrix} 1 & 0.78 & 0.78^2 \\ 1 & 0.69 & 0.69^2 \\ 1 & 1.01 & 1.01^2 \\ 1 & 1.06 & 1.06^2 \end{pmatrix} = \begin{pmatrix} 1 & 0.78 & 0.6084 \\ 1 & 0.69 & 0.4761 \\ 1 & 1.01 & 1.0201 \\ 1 & 1.06 & 1.1236 \end{pmatrix}, Y = \begin{pmatrix} 0.3672882 \\ 0.3201999 \\ 0.1224294 \\ 0 \end{pmatrix}$$

From the equation of the regression, we evaluate the function $E[Y|S]$ for significant values of S at $t = 2$:

$E[Y|S] = -2.7077 + 7.8084 \cdot S - 4.9566 \cdot S^2$
$E[Y|S = 0.78] = +0.367257$ $\quad$ $E[Y|S = 0.69] = +0.320259$
$E[Y|S = 1.01] = +0.122556$ $\quad$ $E[Y|S = 1.06] = -0.000032$

We then compare these values with the payoffs resulting from an immediate exercise of the option:

Path	Exercise	Continuation	Early exercise convenience
1	0	0	-
2	0.32	0.367257	0.32 > 0.367257 → NO
3	0.41	0.320259	0.41 > 0.320259 → YES
4	0.09	0.122556	0.09 > 0.122556 → NO
5	0.04	0	0.04 > 0 → YES
6	0	0	-
7	0	0	-
8	0	0	-

Table III.62 Decision of exercise at time $t = 2$

It should be noted that it is preferable to exercise the option at time $t = 2$ for paths $= j = 3,\ 5$.

For the simulations $j = 2,\ 4$, the expected pay-off values are higher when the option is not exercised.

Considering the payoffs of paths 3 and 5 at time 2, Table III.63 of the payoffs is updated, obtaining:

Path	t = 1	t = 2	t = 3
1	0	0	0.00
2	0	0	0.39
3	0	0.41	0.00
4	0	0	0.13
5	0	0.04	0.00
6	0	0	0.03
7	0	0	0.13
8	0	0	0.00

Table III.63 Payoff at time $t = 2$

It is necessary to implement the same logical steps to obtain the payoffs at time $t = 1$.

For $t = 1$, only three paths provide a value of S greater than the strike price ($j = 6{,}7{,}8$).

Then, we implement the regression for the remaining 5 paths.

Path	S(t=1)	pay off(t=1)	In the money	last pay off	DF	Y
1	1.07	0.03	YES	0.00	0.88692	0
2	0.76	0.34	YES	0.39	0.88692	0.345899
3	0.85	0.25	YES	0.41	0.941765	0.386123
4	0.96	0.14	YES	0.13	0.88692	0.1153
5	0.95	0.15	YES	0.04	0.941765	0.037671
6	1.59	0	NO	0	0.88692	0
7	1.28	0	NO	0	0.88692	0
8	1.11	0	NO	0	0.88692	0
	[A]	[B]		[C]	[D]	[E]

[A] see Table III.60, [B] max[K-S(t=1),0], [C] max[K-S(t=last),0] , [D] Disc. Factors, [E]=[C]x[D]

Table III.64 Determination of the variables for the second regression

We now perform the second regression:

$$E[Y|S] = \alpha + \beta F_1(S) + \gamma F_2(S) = \alpha + \beta S + \gamma S^2$$

The regression coefficients are obtained by calculating the following matrix product:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = (\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T Y =$$

$$\left[\begin{pmatrix} 1 & 1.07 & 1.07^2 \\ 1 & 0.76 & 0.76^2 \\ 1 & 0.85 & 0.85^2 \\ 1 & 0.96 & 0.96^2 \\ 1 & 0.95 & 0.95^2 \end{pmatrix}^T \begin{pmatrix} 1 & 1.07 & 1.07^2 \\ 1 & 0.76 & 0.76^2 \\ 1 & 0.85 & 0.85^2 \\ 1 & 0.96 & 0.96^2 \\ 1 & 0.95 & 0.95^2 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 1.07 & 1.07^2 \\ 1 & 0.76 & 0.76^2 \\ 1 & 0.85 & 0.85^2 \\ 1 & 0.96 & 0.96^2 \\ 1 & 0.95 & 0.95^2 \end{pmatrix}^T \begin{pmatrix} 0 \\ 0.345899 \\ 0.386123 \\ 0.1153 \\ 0.037671 \end{pmatrix} =$$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} +1.3847 \\ -1.2804 \\ -0.0378 \end{pmatrix}$$

$$E[Y|S] = 1.3847 - 1.2804 \cdot S - 0.0378 \cdot S^2$$

From the equation of the regression, we evaluate the function $E[Y|S]$ for significant values of S at $t = 1$:

$$E[Y|S] = 1.3847 - 1.2804 \cdot S - 0.0378 \cdot S^2$$

$E[Y|S = 1.07] = -0.028605$ $\qquad$ $E[Y|S = 0.76] = +0.3897627$

$E[Y|S = 0.85] = 0.2690495,\ E[Y|S = 0.96] = 0.1206795,\ E[Y|S = 0.95] = 0.134205$

We then compare these values with the payoffs resulting from an immediate exercise of the option:

Path	Exercise	Continuation	Early exercise convenience
1	0.03	-0.02860522	0.03 > -0.02860522 → **YES**
2	0.34	0.38976272	0.34 > 0.38976272→ NO
3	0.25	0.2690495	0.25 > 0.2690495 → NO
4	0.14	0.12067952	0.14 >0.12067952 → **YES**
5	0.15	0.1342055	0.15 > 0.1342055 → **YES**
6	0	0	-
7	0	0	-
8	0	0	-

Table III.65 Decision of exercise at time $t = 1$

We conlude that it is preferable to exercise the option at time $t = 1$ for paths $j = 1,4,5$.

We thus proceed to update Table III.63 of the payoffs, obtaining:

Path	t = 1	t = 2	t = 3
1	0.03	0	0
2	0	0	0.39
3	0	0.41	0
4	0.14	0	0
5	0.15	0	0
6	0	0	0.03
7	0	0	0.13
8	0	0	0

DF(0,t)	0.941765	0.886920437	0.83527
Path	t=1	t=2	t=3
1	0.028253	0	0
2	0	0	0.325755
3	0	0.363637379	0
4	0.131847	0	0
5	0.141265	0	0
6	0	0	0.025058
7	0	0	0.108585
8	0	0	0

Table III.66 Payoff at time $t = 2$ and Discounted Pay-off matrix

The price of the Bermuda put option is obtained by calculating the average value of all the discounted pay-offs and it is equal to 0.14055. If the option had been a European type, its price would have been 0.1065. The early-exercise feature is therefore worth 0.14055-0.1065=0.034053.

We have illustrated above the operation of the Longstaff-Schwartz method for considering the possibility of exercising the option right before expiry, let us now consider the valuation of an American put option having the following financial characteristics: S=50, K=55, T=0.5, r=5%, σ=30%.

The price frequency distribution was obtained for a million launches, replicated 1,000 times. The fair value we obtain is equal to 6.678385±0.02003.

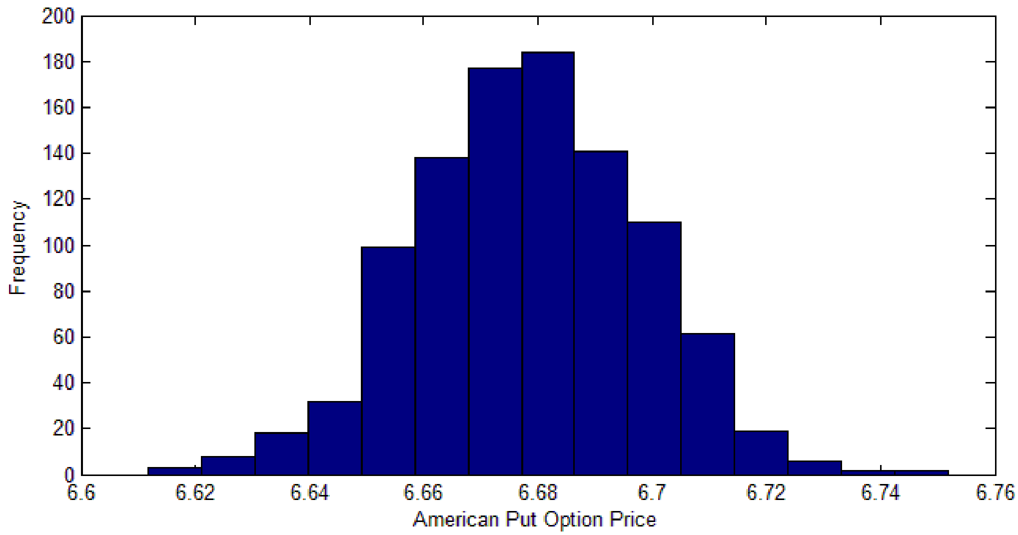

Figure III.127 American Put Option price using the Longstaff-Schwartz technique

For pricing purposes, simulating the behavior of a financial asset in the future means solving a stochastic differential equation (SDE). Not all financial underlyings follow a Geometric Brownian motion though. For example, it is experimentally verified that interest rates are characterized by a mean-reversion, or return to the mean. The SDE that regulates the stochastic process assumes the following form:

$$dS_t = \kappa[\theta - S_t]dt + \sigma S_t dW_t \quad \text{(Eq. III.239)}$$

Where:

θ is the mean reversion level.
κ is the mean reversion speed.
σ is the volatility.

The parameters are generally calibrated on the financial market typically starting from the zero-coupon bonds constituting the term structure of the forward rates or from the options linked to the actively negotiated rates (cap, floor, swaption). These SDEs are generally integrated using numerical schemes.

FURTHER READINGS

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Giribone P. G. – "Mathematical modeling in Quantitative Finance and Computational Economics – Part II: Deterministic and Stochastic Optimal Control" – PhD Thesis in Economics – XXXIII Cycle – Supervisor: prof. Marco Guerrazzi (Dept. of Economics – DIEC) and prof. Ottavio Caligaris (Dept. of Engineering – DIME) (2021).

Glasserman P. – "Monte Carlo Methods in Financial Engineering" – Springer (2003).

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According to the Black-Scholes-Merton framework, volatility is a fundamental input parameter for calculating the fair value of an option. There are different ways of estimating volatility starting from the rates of change in prices of a historical series. It is worth to note that the reference sample is $N+1$ prices, thus N returns observations and that price sampling must be conducted through fixed and regular time intervals i. Furthermore, the historical volatility estimated by the methodologies presented here measures the dispersion of the data for the time interval with which the data were sampled.

We now introduce the following notations:

$Close_i$ is the closing price of the asset at the end of interval i.
$High_i$ is the highest price level recorded by the asset in interval i.
Low_i is the lowest price level recorded by the asset in interval i.

The most used method to calculate volatility is the so-called **close-to-close Volatility**, which in mathematical terms means:

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N} \ln\left(\frac{Close_i}{Close_{i-1}}\right)^2 - \frac{1}{N(N-1)}\left[\sum_{i=1}^{N} \ln\left(\frac{Close_i}{Close_{i-1}}\right)\right]^2} \quad \textit{(Eq. III.240)}$$

Parkinson in 1980 suggested to estimate the standard deviation using the **High-Low Volatility** method:

$$\sigma = \frac{1}{2N\sqrt{\ln(2)}}\sum_{i=1}^{N+1} \ln\left(\frac{High_i}{Low_i}\right) \quad \textit{(Eq. III.241)}$$

Finally, in the same year, Garman and Klass proposed to use the following estimator:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{2}\left[\ln\left(\frac{High_i}{Low_i}\right)\right]^2 - \frac{1}{N}\sum_{i=1}^{N}[2\ln(2)-1]\cdot\left[\ln\left(\frac{Close_i}{Close_{i-1}}\right)\right]^2} \quad \textit{(Eq. III.242)}$$

This last approach is called **High-Low-Close Volatility**

Let us illustrate a practical example and estimate historical volatility with the **close-to-close** method on the time series shown in the table below. The contributions to derive the standard deviation on the rates of change are:

$$u_i = \ln\left(\frac{Close(i)}{Close(i-1)}\right); \sum_{i=1}^{N=20} u_i^2 = 0.006929; \left(\sum_{i=1}^{N=20} u_i\right)^2 = 0.016542$$

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N} u_i^2 - \frac{1}{N(N-1)}\left(\sum_{i=1}^{N} u_i\right)^2} = \sqrt{\frac{1}{19}\cdot 0.006929 - \frac{1}{19\cdot 20}\cdot 0.016542} = 0.017921$$

Trading Day	Close	u(i)	$u(i)^2$
1	127.5	-	-
2	128.5	0.00781254	6.10358E-05
3	130	0.011605546	0.000134689
4	128	-0.015504187	0.00024038
5	128	0	0
6	132	0.030771659	0.000946895
7	130	-0.015267472	0.000233096
8	130	0	0
9	137.5	0.056089467	0.003146028
10	138	0.003629768	1.31752E-05
11	139.5	0.010810916	0.000116876
12	140	0.003577821	1.28008E-05
13	141	0.007117468	5.06583E-05
14	144	0.021053409	0.000443246
15	143	-0.006968669	4.85624E-05
16	142	-0.007017573	4.92463E-05
17	142	0	0
18	142	0	0
19	140	-0.014184635	0.000201204
20	140	0	0
21	145	0.03509132	0.001231401

Table III.67 Close to close volatility

The formula that allows the conversion of a volatility expressed on a daily basis to a volatility expressed on an annual basis is:

$$\sigma_{annual} = \sigma_{daily} \cdot \sqrt{Ndays} \quad \textit{(Eq. III.243)}$$

Where $Ndays$ denotes the number of trading days in a year.

This conversion is necessary because in all pricing models this measure of dispersion is expressed on an annual basis: $\sigma_{annual} = 1.7921\% \cdot \sqrt{252} = 28.4489\%$.

We now present an example using the **high-low** Volatility:

Trading Day	High(i)	Low(i)	ln(High/Low)
1	127.5	126	0.011834458
2	129	126	0.023530497
3	131	129	0.015384919
4	132	128	0.030771659
5	131	128	0.023167059
6	132	128	0.030771659
7	131.5	130	0.011472401
8	131	130	0.007662873
9	138.5	132	0.048068403
10	140	137	0.021661497
11	142	137	0.035846132
12	142.5	140	0.017699577
13	142	138	0.028573372
14	145	143	0.013889112
15	144	141.5	0.017513582
16	144.5	142	0.01745245
17	142.5	141	0.010582109
18	144	141.5	0.017513582
19	142.5	139.5	0.021277398
20	140	139	0.007168489
21	145	138.5	0.045863417

Table III.68 High-Low volatility

The dataset is the same, as shown in the table, but we reach a different result:

$$\sum_{i=1}^{N+1} \ln\left(\frac{High_i}{Low_i}\right) = 0.4577$$

$$\sigma = \frac{1}{2(N+1)\sqrt{\ln 2}} 0.4577 = 0.0130895$$

$$\sigma_{annual} = 1.30895\% \cdot \sqrt{252} = 20.7789\%$$

Lastly, an example using the **high-low-close** Volatility still on the same dataset:

Day	Close	High	Low	hl(i)	hl(i)²	u(i)	u(i)²
1	127.5	127.5	126				
2	128.5	129	126	0.023530	0.000554	0.007813	0.000061
3	130	131	129	0.015385	0.000237	0.011606	0.000135
4	128	132	128	0.030772	0.000947	-0.015504	0.000240
5	128	131	128	0.023167	0.000537	0.000000	0.000000
6	132	132	128	0.030772	0.000947	0.030772	0.000947
7	130	131.5	130	0.011472	0.000132	-0.015267	0.000233
8	130	131	130	0.007663	0.000059	0.000000	0.000000
9	137.5	138.5	132	0.048068	0.002311	0.056089	0.003146
10	138	140	137	0.021661	0.000469	0.003630	0.000013
11	139.5	142	137	0.035846	0.001285	0.010811	0.000117
12	140	142.5	140	0.017700	0.000313	0.003578	0.000013
13	141	142	138	0.028573	0.000816	0.007117	0.000051
14	144	145	143	0.013889	0.000193	0.021053	0.000443
15	143	144	141.5	0.017514	0.000307	-0.006969	0.000049
16	142	144.5	142	0.017452	0.000305	-0.007018	0.000049
17	142	142.5	141	0.010582	0.000112	0.000000	0.000000
18	142	144	141.5	0.017514	0.000307	0.000000	0.000000
19	140	142.5	139.5	0.021277	0.000453	-0.014185	0.000201
20	140	140	139	0.007168	0.000051	0.000000	0.000000
21	145	145	138,5	0.045863	0.002103	0.035091	0.001231

Table III.69 High-Low-Close volatility

$$u_i = \ln\left(\frac{Close(i)}{Close(i-1)}\right);\ hl_i = \ln\left(\frac{High(i)}{Low(i)}\right);\ [2\ln 2 - 1]\sum_{i=1}^{N} u_i^2 = 0.002677;\ \frac{1}{2}\sum_{i=1}^{N=20} hl_i^2 = 0.006218$$

$$\sigma = \sqrt{\frac{0.006218}{20} - \frac{0.0026771}{20}} = 0.013307;\ \sigma_{annual} = 1.3307\% \cdot \sqrt{252} = 21.124\%$$

Choosing an appropriate number of statistical observations to determine historical volatility is not an easy task. It is in fact true that the more data we use, the greater the accuracy, but it is also true that σ changes over time and too old data can be irrelevant in its impact on the future. A good trade-off suggested by the literature is to use the daily closing prices of the last 90-180 days, and a rule of thumb, which is often adopted by practictioners, is to match the time period in which volatility is measured, with the time period to which it is applied. Therefore, for pricing a two-year option, two years of historical data of the underlying S_i are required.

In the previous analyzes, we generally assumed that the stock does not pay any dividend, but the adaptation to the case in which this phenomenon occurs is not complicated. The rate of return $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$ has to be adjusted to consider the paid dividend D: $u_i = \ln\left(\frac{S_i + D}{S_{i-1}}\right)$.

Another type of volatility is constituted by the **Exponential Weighted Historical Volatility**, also called Exponentially Weighted Moving Average (EWMA) Volatility, which gives more importance to more recent observations. An **EWMA** Volatility can be calculated using the recursive formula:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda)\left[\ln\left(\frac{S_t}{S_{t-1}}\right)\right]^2 \quad \textit{(Eq. III.244)}$$

Where σ_t is the current volatility and σ_{t-1} is the measure estimated in the previous observation. If daily data is used, volatility is annualized by multiplying it by the square root of the number of trading days in a year:

$$\sigma_{annual}(t) = \sqrt{\sigma_t^2 \cdot Ndays} \quad \textit{(Eq. III.245)}$$

λ is defined as the "smoothing factor" and empirical research has shown that a reasonable value for this parameter is $0.75 \leq \lambda \leq 0.98$. In fact, the popular software developed by J.P. Morgan, RiskMetrics, sets it at 0.94.

Here is a practical example using the **EWMA Volatility**, calculated from the closing prices time series: $\lambda = 0.94$

For the first variance, we have $t = 3$, then $\sigma_3^2 = \lambda u(t-1)^2 + (1-\lambda)u(t)^2$.

For $t > 3$, the recursive formula is used and we obtain $\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda)u(t)^2$.

From the last calculated variance value, we find the volatility expressed on a daily basis and therefore on an annualized basis ($Ndays = 252$):

$$\sigma = \sqrt{0.000257147} = 0.0160358;\ \sigma_{annual} = 25.45613\%$$

Trading Day	Close	u(t)	u(t)^2	σ(t)^2
1	127.5			
2	128.5	0.00781254	6.10358E-05	
3	130	0.011605546	0.000134689	6.5455E-05
4	128	-0.015504187	0.00024038	7.59504E-05
5	128	0	0	7.13934E-05
6	132	0.030771659	0.000946895	0.000123924
7	130	-0.015267472	0.000233096	0.000130474
8	130	0	0	0.000122645
9	137.5	0.056089467	0.003146028	0.000304048
10	138	0.003629768	1.31752E-05	0.000286596
11	139.5	0.010810916	0.000116876	0.000276413
12	140	0.003577821	1.28008E-05	0.000260596
13	141	0.007117468	5.06583E-05	0.000248
14	144	0.021053409	0.000443246	0.000259715
15	143	-0.006968669	4.85624E-05	0.000247045
16	142	-0.007017573	4.92463E-05	0.000235177
17	142	0	0	0.000221067
18	142	0	0	0.000207803
19	140	-0.014184635	0.000201204	0.000207407
20	140	0	0	0.000194962
21	145	0.03509132	0.001231401	0.000257149

Table III.70 EWMA volatility

Under the assumption that the percentage change in asset prices is normally distributed, the following formula calculates confidence intervals for estimated historical volatility:

$$P\left[\hat{\sigma}\sqrt{\frac{n-1}{\chi^2_{\left(n-1,\frac{\alpha}{2}\right)}}} \leq \sigma \leq \hat{\sigma}\sqrt{\frac{n-1}{\chi^2_{\left(n-1,1-\frac{\alpha}{2}\right)}}}\right] = 1-\alpha \quad \textit{(Eq. III.246)}$$

Where:

n is the number of observations.
$\hat{\sigma}$ is the estimated historical volatility.
$\chi^2_{\left(n-1,\frac{\alpha}{2}\right)}$ is the value of the chi square distribution with $(n-1)$ degrees of freedom for a confidence level equal to $1-\alpha$.

We now present an example, considering the annualized historical volatility estimation calculated with the close-to-close method of the previous example: $\hat{\sigma}=28.45\%$. This estimation is based on 21 closing prices and, therefore, 20 price changes ($n=20$). The 95% confidence interval ($\alpha=0.05$) of this statistical inference is:

$$P\left[\hat{\sigma}\sqrt{\frac{n-1}{\chi^2_{\left(n-1,\frac{\alpha}{2}\right)}}}\leq\sigma\leq\hat{\sigma}\sqrt{\frac{n-1}{\chi^2_{\left(n-1,1-\frac{\alpha}{2}\right)}}}\ \right]=1-\alpha$$

$$P\left[0.2845\cdot\sqrt{\frac{20-1}{\chi^2_{\left(20-1,\frac{0.05}{2}\right)}}}\leq\sigma\leq 0.2845\cdot\sqrt{\frac{20-1}{\chi^2_{\left(20-1,1-\frac{0.05}{2}\right)}}}\ \right]=1-0.05$$

$$P\left[0.2845\cdot\sqrt{\frac{19}{8.9065}}\leq\sigma\leq 0.2845\cdot\sqrt{\frac{19}{32.8523}}\ \right]=0.95$$

$$P[0.2164\leq\sigma\leq 0.4155\ ]=0.95$$

With 20 observations and an estimated volatility of 28.45%, there is a 95% probability that the true volatility lies between 21.64% and 41.55%. Now let σ_n be the volatility of a market variable at day n, as estimated at the end of day $n-1$. The square of the volatility σ_n^2 is called the rate of variance. Let S_i be the value of the market variable at the end of day i. The variable u_i, which represents the continuously compounded rate of return during day i (more precisely between the end of day $i-1$ and the end of day i) is defined as follows:

$u_i=\ln\left(\frac{S_i}{S_{i-1}}\right)$ *(Eq. III.247)*

A correct estimate of the daily rate of variance, σ_n^2, obtained based on the most recent m observations of u_i, is:

$\sigma_n^2=\frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i}-\bar{u})^2$ *(Eq. III.248)*

Where $\bar{u}$ is the average of the u_i:

$\bar{u}=\frac{1}{m}\sum_{i=1}^{m}u_{n-i}$ *(Eq. III.249)*

In financial forecasting mathematical models, it is often customary to make a few changes to this formula, in particular:

- u_i is defined as the proportional rate of change of the market variable between the end of day $i-1$ and the end of day i, so that: $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$. Such definition is consistent with the way in which volatility is estimated for the purposes of the Var (Value-at-Risk).

- It is assumed that $\bar{u}$ is zero. This assumption generally has little effect on the estimate of the variance, since the expected value of the daily change of a variable is very small compared to the standard deviation.

- it is also common to substitute $m-1$ for m. Through this substitution, the analyst passes from an unbiased estimate of the variance to a maximum likelihood one.

These three changes result in minimal changes in the variance estimates and the formula for the rate of variance simplifies to:

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i}-\bar{u})^2 \rightarrow \sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m}u_{n-i}^2 \quad \textit{(Eq. III.250)}$$

where:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \rightarrow u_i = \frac{S_i - S_{i-1}}{S_{i-1}}. \quad \textit{(Eq. III.251)}$$

Let us present an example, considering the time series of closing prices in the example shown above, thus we have:

S_i are the Closing prices.

n is the rate of variance referring to the nth day and it is equal to 21.

m is the number of observations or number of price changes, which are 20.

The mean of u_i^2 is equal to the rate of variance expressed on a daily basis:

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m}u_{n-i}^2 = 0.000359397$$

We now express the rate of variance on an annualized basis, assuming that there are 252 trading days in a year:

$$\sigma_n^{annual} = \sqrt{\sigma_n^2 \cdot Ndays} = 30.095\%$$

This estimation method associates a weight equal to $u_{n-1}^2, u_{n-2}^2,.., u_{n-m}^2$.

Since the goal is to estimate the current volatility level σ_n, it makes sense to give more weight to the most recent data.

Trading Day	S(i)	u(i)=[S(i)-S(i-1)]/S(i-1)	u(i)^2
1	127.5		
2	128.5	0.007843137	6.15148E-05
3	130	0.011673152	0.000136262
4	128	-0.015384615	0.000236686
5	128	0	0
6	132	0.03125	0.000976563
7	130	-0.015151515	0.000229568
8	130	0	0
9	137.5	0.057692308	0.003328402
10	138	0.003636364	1.32231E-05
11	139.5	0.010869565	0.000118147
12	140	0.003584229	1.28467E-05
13	141	0.007142857	5.10204E-05
14	144	0.021276596	0.000452694
15	143	-0.006944444	4.82253E-05
16	142	-0.006993007	4.89021E-05
17	142	0	0
18	142	0	0
19	140	-0.014084507	0.000198373
20	140	0	0
21	145	0.035714286	0.00127551

Table III.71 Traditional model

The model is thus extended by adding the weight α_i:

$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i \cdot u_{n-i}^2$ *(Eq. III.252)*

Variable α_i represents the weight assigned to the observation of i past days and, since we want to give greater prominence to the most recent observations, the following condition must be respected: $\alpha_i < \alpha_j,\ i > j$. The sum of the weights must obviously be equal to unity: $\sum_{i=1}^{m} \alpha_i = 1$. For a sufficiently large m, the EWMA is a particular case of the model just presented, in which the weights α_i decrease exponentially as we go further back in time.

To prove this statement, let us reconsider the formula for the EWMA: $\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$.

The volatility estimate σ_n for day n (obtained at the end of day $n-1$) is computed based on σ_{n-1} (the volatility estimate for day $n-1$ obtained at the end of day $n-2$) and on u_{n-1} (the observed rate of change of the asset between day $n-2$ and $n-1$).

To understand the reason why the formula $\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$ involves exponentially decreasing weights, we need to replace, recursively, the expression of σ_{n-1}^2 inside it, so as to obtain:

$\sigma_n^2 = \lambda\{\lambda\sigma_{n-2}^2 + (1-\lambda)u_{n-2}^2\} + (1-\lambda)u_{n-1}^2$ *(Eq. III.253)*

$\sigma_n^2 = (1-\lambda)\{u_{n-1}^2 + \lambda u_{n-2}^2\} + \lambda^2\sigma_{n-2}^2$ *(Eq. III.254)*

Analogously, substituting σ_{n-2}^2, we obtain:

$\sigma_n^2 = (1-\lambda)\{u_{n-1}^2 + \lambda u_{n-2}^2 + \lambda^2 u_{n-3}^2\} + \lambda^3\sigma_{n-3}^2$ *(Eq. III.255)*

Continuing with progressive steps, the sequence becomes:

$\sigma_n^2 = (1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}u_{n-i}^2 + \lambda^m\sigma_{n-m}^2$ *(Eq. III.256)*

For m sufficiently large - and so they must be for the volatility estimates to be significant - the term $\lambda^m\sigma_{n-m}^2$ is negligible.

$\sigma_n^2 = (1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}u_{n-i}^2 + \lambda^m\sigma_0^2 \rightarrow \sigma_n^2 = (1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}u_{n-i}^2$ *(Eq. III.257)*

The latter expression thus obtained is equivalent to the formulation:

$\sigma_n^2 = \sum_{i=1}^{m}\alpha_i \cdot u_{n-i}^2$ *(Eq. III.258)*

With $\alpha_i = (1-\lambda)\cdot\lambda^{i-1}$.

The weights assigned to the u_i decrease at the rate λ as we go further back in time, in particular, it is observed that each weight is equal to λ times the previous weight.

Another interesting feature of the EWMA approach is that there is relatively little data to store in the memory. On any given day, only the current estimate of the rate of variance and the most recent value of the market variable need be used.
Furthermore, this type of approach considers changes in volatility: let us suppose, for example, that on day $n-1$, the market variable undergoes a sharp increase, so that u_{n-1}^2 assumes a high value.
This results in an increase of σ_n, i.e., the daily volatility estimates for day n. In fact, the value of λ determines the extent to which the estimate of daily volatility reacts to recent observations of u_i^2.
In particular, a low value of λ causes a consistent weight to be assigned to u_{n-1}^2, so the estimates of volatility in the following days will be very high. On the other hand, at a high value of this parameter, then the daily volatility estimates will react little in relation to the new information provided by u_i^2.

A very popular model for variance estimation is the approach proposed by Engle in 1982 and called the Autoregressive Conditional Heteroskedasticity – ARCH. The basic idea is to assume the existence of a long-term average volatility, to which a certain weight must be assigned, and consequently a model is obtained in the following form:

$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i \cdot u_{n-i}^2$ *(Eq. III.259)*

Where V_L is the long-term variance, and γ is the weight associated to V_L. Furthermore, given that the sum of the weights must be equal to one, the following constraint must be respected:

$\gamma + \sum_{i=1}^{m} \alpha_i = 1$ *(Eq. III.260)*

The model just described is known as $ARCH(m)$, where m represents the number of observations: the further the observation is, the lower the weight assigned to it. If we set $\omega = V_L \gamma$, the model can be rewritten as follows:

$\sigma_n^2 = \omega + \sum_{i=1}^{m} \alpha_i \cdot u_{n-i}^2$ *(Eq. III.261)*

Consequently, in 1986 Bollerslev presented the **Generalized Autoregressive Conditional Heteroskedasticity** (GARCH) model. The actual difference between $GARCH(1,1)$ and $EWMA$ is analogous to the difference between $ARCH$ and the variance estimation on weighting schemes.
Indeed in $GARCH(1,1)$, σ_n^2 is calculated based on the average long-term variance rate V_L, σ_{n-1}^2 and u_{n-1}^2.
The equation representing $GARCH(1,1)$ is: $\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$, and the sum of the weights must be equal to one: $\alpha + \beta + \gamma = 1$.
We notice that $EWMA$ turns out to be a particular case of $GARCH(1,1)$ with $\gamma = 0$, $\alpha = 1 - \lambda$ and $\beta = \lambda$.
The parameters $(1,1)$ of the $GARCH$ indicate that the calculation of σ_n^2 is based on the most recent observation of u^2 and on the most recent estimate of the rate of variance. Since the more general expression of the model $GARCH(p, q)$ computes σ_n^2 from the most recent p observations on u^2 and the most recent q estimates of the variance rate, then the recursive formula of the model generalizes as follows:

$\sigma_n^2 = V_L \gamma + \sum_{i=1}^{q} \alpha_i \cdot u_{n-i}^2 + \sum_{i=1}^{p} \beta_i \cdot \sigma_{n-i}^2$ *(Eq. III.262)*

We note that setting $p = 0$, we obtain the $ARCH$ model. However, it should also be noted that the most popular model of $GARCH$ is the model with $p = 1$ and $q = 1$, i.e., the $GARCH(1,1)$. Setting then $\omega = \gamma V_L$, we can rewrite the estimation model:

$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$ *(Eq. III.263)*

This form of the model is generally used to estimate parameters, and once ω, α and β have been calibrated, γ can be computed as $1 - \alpha - \beta$. Besides, the long-term variance V_L is equal to the ratio: ω/γ.

For the $GARCH(1,1)$ to be stable, it is required that $\alpha + \beta < 1$, otherwise the weight assigned to the long-term variance becomes negative. As we have seen above, the structure of the weights is very similar to that

given in the $EWMA$ model, except that, in addition to assigning exponentially decreasing weights to the previous term u^2, it also assigns a weight to the **long-run average volatility**.
To justify this statement, similarly to what was done previously, we iteratively replace the expression of σ_{n-1}^2 in $\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2$.

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta[\omega + \alpha u_{n-2}^2 + \beta\sigma_{n-2}^2] \rightarrow \sigma_n^2 = \omega + \beta\omega + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \beta\sigma_{n-2}^2$$

Substituting σ_{n-2}^2, we have:

$$\sigma_n^2 = \omega + \beta\omega + \beta^2\omega + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \alpha\beta^2 u_{n-3}^2 + \beta^3\sigma_{n-3}^2 \quad \textit{(Eq. III.264)}$$

Continuing like this, we observe that the weight applied to u_{n-i}^2 is $\alpha\beta^{i-1}$. Thus, the weights decline exponentially at the rate β, which can be interpreted as a **decay rate**. This is similar to the factor λ in the $EWMA$ model, it defines the relative importance of observations on u in determining the current state of variance. For example, if we set $\beta = 0.9$, then u_{n-2}^2 is as important as 90% of u_{n-1}^2; u_{n-3}^2 is as important as 81% (0.9^2) of u_{n-1}^2 and so on…

The $GARCH(1,1)$ model recognizes that the variance, although moving randomly, tends to converge towards a long-term average level, V_L over time, with an associated weight equal to $\gamma = 1 - \alpha - \beta$. This estimation method is equivalent to a model in which the variance V follows a stochastic process of the type: $dV = a(V_L - V)dt + \xi V dW_t$.
Where:
$a = 1 - \alpha - \beta$.
$\xi = \alpha\sqrt{2}$.
dW_t is a stochastic Wiener process, and time is measured in days.

The model in fact incorporates the "**mean-reverting**" effect, in which the variance has a drift (trend) which brings it back towards V_L at the speed of a. In particular, when $V > V_L$, the variance has a negative drift, otherwise, when $V < V_L$ the slope becomes positive.

In practical terms, a tendency of the rate of variance to be mean-reverting is observed in the markets: the $GARCH(1,1)$ favors this behavior observed experimentally on the markets, which makes the $GARCH(1,1)$ theoretically more attractive than $EWMA$. However, if the estimation of the parameters of $GARCH(1,1)$ leads to a negative ω, it can no longer be considered stable and it is therefore more reasonable to use the $EWMA$ approach.

Forecasting models for volatility envisage free parameters which must be estimated to obtain volatility values that are consistent with observed historical data. The most common method to find the parameters of $GARCH(1,1)$ is to implement the maximum likelihood method.
Let us define the estimated variance for day i as $v_i = \sigma_i^2$ and assume that the conditional probability distribution on the variance of u_i is normal.

In this context, the maximum likelihood function, L, to be maximized with respect to the model parameters is given by:

$$\boldsymbol{L} = \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi v_i}} \exp\left(-\frac{u_i^2}{2v_i}\right) \quad \textit{(Eq. III.265)}$$

By applying the logarithm, the maximum points of the previous function do not vary, so it is equivalent to maximizing the following expression:

$$\boldsymbol{L} = \sum_{i=1}^{m} \left[-\ln(v_i) - \frac{u_i^2}{v_i}\right] = \sum_{i=1}^{m} L_i \quad \textit{(Eq. III.266)}$$

We implement a practical example and take as a reference the time series of the S&P 500 index from 18th July 2005 to 13th August 2010. We now want to estimate ω, α and β of $GARCH(1,1)$.

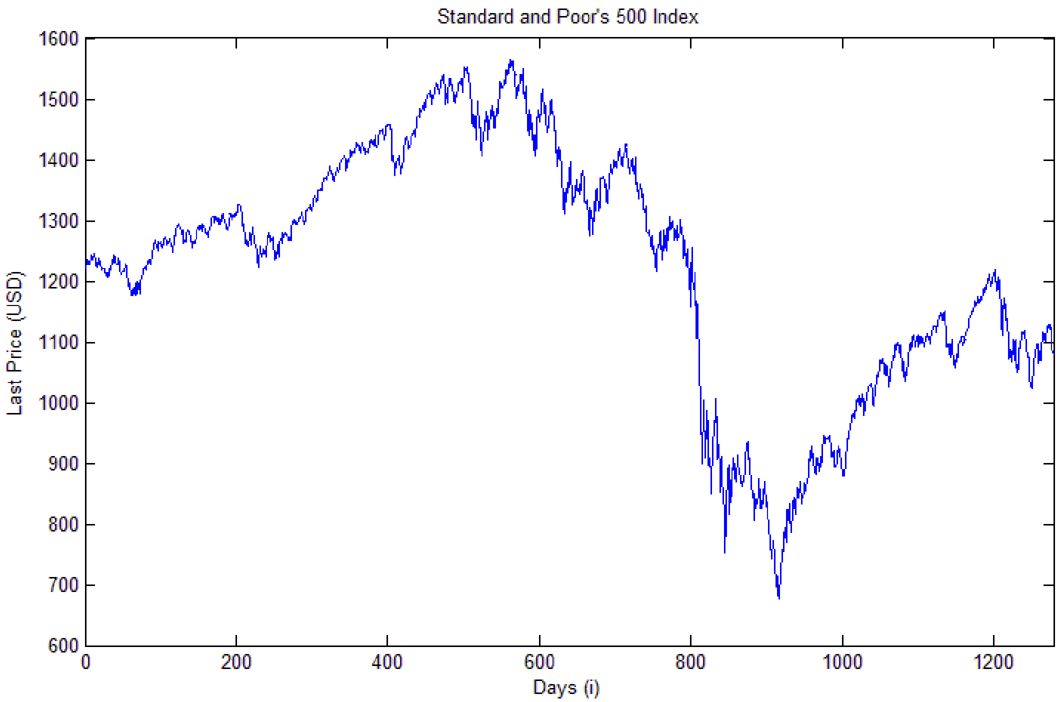


Figure III.128 Standard and Poor's 500 index – last prices

The iterative optimization procedure shows that the parameters which maximize $\boldsymbol{L}$ are: $\omega = 0.0000013439$, $\alpha = 0.083246$ and $\beta = 0.91025$.

Date	PX_LAST	i	S(i)	u(i) [A]	v(i) [B]	L(i) [C]
07/18/2005	1221.13	1	1221.13			
07/19/2005	1229.35	2	1229.35	0.00673147	[Binit]	
07/20/2005	1235.2	3	1235.2	0.004758612	4.53127E-05	9.502187262
07/21/2005	1227.04	4	1227.04	-0.006606218	4.44744E-05	9.039310703
07/22/2005	1233.68	5	1233.68	0.005411397	4.54625E-05	9.354505097
07/25/2005	1229.03	6	1229.03	-0.003769211	4.51643E-05	9.6906419
…	…	…	…	…	…	…
08/06/2010	1121.64	1274	1121.64	-0.003704	0.000143829	8.751497048
08/09/2010	1127.79	1275	1127.79	0.005483043	0.00013339	8.696851826
08/10/2010	1121.06	1276	1121.06	-0.005967423	0.000125252	8.700874195
08/11/2010	1089.47	1277	1089.47	-0.028178688	0.000118308	2.330621776
08/12/2010	1083.61	1278	1083.61	-0.005378762	0.000175234	8.484286609
08/13/2010	1079.25	1279	1079.25	-0.004023588	0.00016324	8.621112193

omega	alpha	beta
1.3439E-06	0.083246	0.91025

$\sum_{i=0}^{m} L(i)$ =10236.637987

[A] $u_i = \frac{S_i - S_{i-1}}{S_{i-1}}$; [Binit] $v_2 = u_2^2$; [B] $v(i) = \omega + \alpha u_{i-1}^2 + \beta v_{i-1}$ and [C] $L(i) = -\ln(v_i) - \frac{u_i^2}{v_i}$

Table III.72 GARCH volatility

In the example, the long-term rate of variance, V_L is $V_L = \frac{\omega}{1-\alpha-\beta} = 0.000206$. The long-term volatility is therefore equal to $\sqrt{0.0002066} = 1.437\%$ calculated daily.

An alternative and sometimes more robust approach to the above presented parameter estimation in $GARCH(1,1)$ is the **variance targeting method**. This method sets the long-term mean rate of variance, V_L , equal to the sample variance estimated from historical data. The value of ω is therefore equal to: $\omega = V_L(1-\alpha-\beta)$ and only two parameters of $GARCH(1,1)$ have to be estimated (α and β).

In the case of the previous example, we obtain a daily volatility of 1.55249% which corresponds to a variance rate equal to 0.000241023. If we then set $V_L =$ 0.000241023, we can express ω as a function of α and β allowing a simplification of the optimization problem: in fact, the maximum likelihood function $\boldsymbol{L}$ must only be a function of two independent variables, rather than three. Through a numerical optimization routine, the calibrated parameters are obtained: $\alpha =$ 0.08435278, β=0.9101804986 and ω=0.0000013176034891.

The value of the objective function $\boldsymbol{L}$ obtained in correspondence with the maximum point thus found is equal to 10236.59467801490.

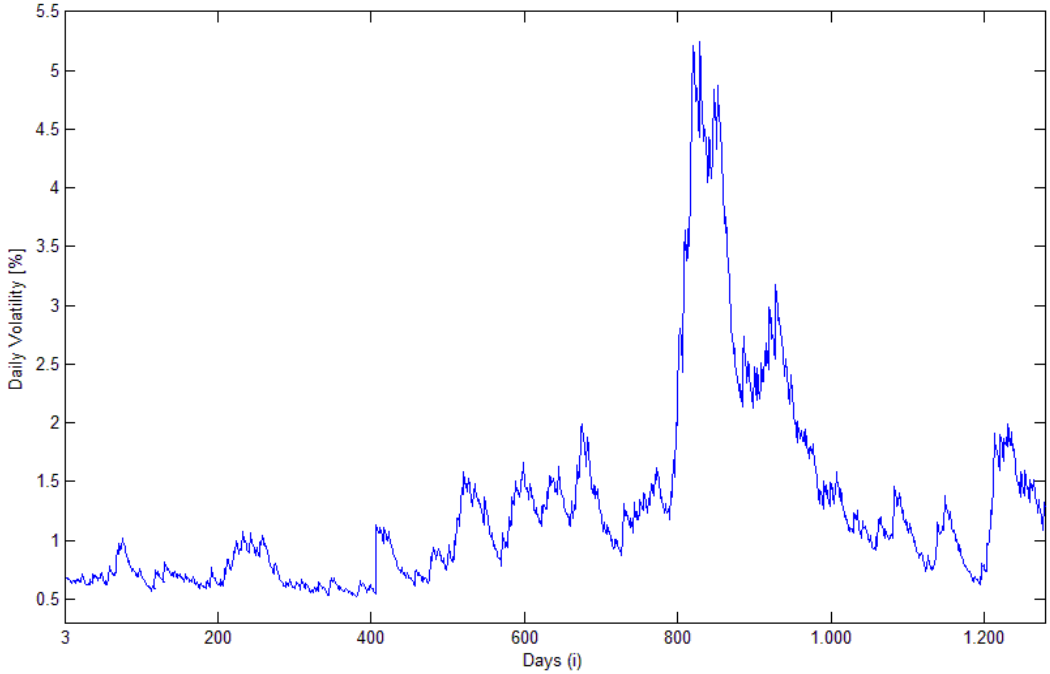


Figure III.129 Standard and Poor's 500 index – daily volatility

Similarly, the same maximum likelihood function can be used to estimate the other discussed forecast volatility models. In particular, in the $ARCH$ model, we set $\beta = 0$ and we maximize $\boldsymbol{L}$ with respect to the independent variables α and ω.

Using the data of the previous example, the following parameters are obtained: ω = 0.00015649237 and $\alpha =$ 0.4389432682.

In the $EWMA$ model, on the other hand, we set the parameters ω = 0, $\alpha = 1 - \lambda$ and $\beta = \lambda$, and the optimization becomes one-dimensional and $\boldsymbol{L}$ must be maximized with respect to the independent variable λ.

Using the data of the previous example, we obtain an optimal value of λ equal to 0.9375629785, corresponding to a value of the objective function equal to 10200.92827125.

Since the presence of a single maximum for the maximum likelihood function $\boldsymbol{L}$ cannot be demonstrated a priori, it is advisable to use different starting points of the solver to be confident of converging to the optimal solution for the calibration problem.

Let us develop the concept in mathematical terms. The estimated rate of variance at the end of day $n-1$ for day n, using the generalized model $GARCH(1,1)$ is:

$\sigma_n^2 = V_L(1-\alpha-\beta) + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2 \rightarrow \sigma_n^2 - V_L = \alpha(u_{n-1}^2 - V_L) + \beta(\sigma_{n-1}^2 - V_L)$ *(Eq. III.267)*

At a generic future time $n+t$, we have:

$\sigma_{n+t}^2 - V_L = \alpha(u_{n+t-1}^2 - V_L) + \beta(\sigma_{n+t-1}^2 - V_L)$ *(Eq. III.268)*

The expected value of u_{n+t-1}^2 is σ_{n+t-1}^2. So:

$E[\sigma_{n+t}^2 - V_L] = (\alpha+\beta)\cdot E[\sigma_{n+t-1}^2 - V_L]$ *(Eq. III.269)*

Where $E[\cdot]$ denotes the expected value. Using this equation repeatedly leads to:

$E[\sigma_{n+t}^2 - V_L] = (\alpha+\beta)^t \cdot E[\sigma_n^2 - V_L] \rightarrow E[\sigma_{n+t}^2] = V_L + (\alpha+\beta)^t \cdot E[\sigma_n^2 - V_L]$ *(Eq. III.270)*

This equation is able to predict the volatility at day $n+t$, using the information available at the end of day $n+1$.

In the $EWMA$ model, $\alpha+\beta=1$ and therefore we obtain that the expected future value of the variance rate is equal to the current variance rate.

In the case of $\alpha+\beta<1$, then the final term in the equation becomes progressively smaller as t increases. As discussed above, the rate of variance shows a mean reversion V_L at a speed rate of $1-\alpha-\beta$.
On the other hand, when $\alpha+\beta>1$, then the weight assigned to the long-term average variance is negative, and $GARCH(1,1)$ is no longer a stable process: in these critical cases we say that we have a mean fleeing process.

Considering the data from the S&P example above, $GARCH(1,1)$, and using the recursive equation above, we can simulate the rate of variance in the future.
The dataset is: $\alpha+\beta$ =0.083246+0.91025=0.993496 and V_L =0.0002066, the current level of the daily rate of variance is 0.00016324.

We compute:

$$E[\sigma_{n+t}^2] = V_L + (\alpha+\beta)^t \cdot E[\sigma_n^2 - V_L]$$

$$E[\sigma_{n+t}^2] = 0.0002066 + 0.993496^t \cdot (0.00016324 - 0.0002066)$$

For $t=10$ days, $E[\sigma_{n+10}^2] = 0.000165979$, and for $t=1000$ days, $E[\sigma_{n+1000}^2] = 0.00020654$.

Lastly, for $t\rightarrow+\infty$, $E[\sigma_{t\rightarrow+\infty}^2] = V_L$.

The graph in the Figure III.130 shows the asymptotic limit of the model $GARCH(1,1)$ at the value V_L.

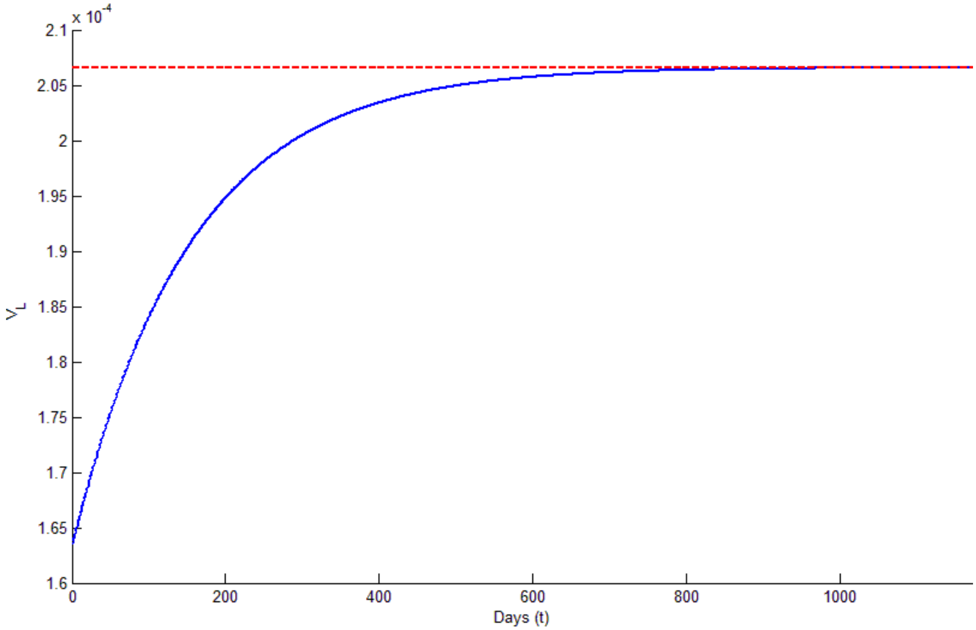


Figure III.130 Asymptotic limit of the model $GARCH(1,1)$ at the value V_L

Let us further analyze and suppose it is day n. We define the quantities: $V(t) = E(\sigma_{n+t}^2)$ and $a = \ln\left(\frac{1}{\alpha+\beta}\right)$, such that:

$$E[\sigma_{n+t}^2] = V_L + (\alpha+\beta)^t \cdot E[\sigma_n^2 - V_L] \rightarrow V(t) = V_L + \exp(-a\,t)\,[V(0) - V_L] \quad \textit{(Eq. III.271)}$$

Where $V(t)$ is an estimation of the instantaneous rate of variance on day t.

The mean rate of variance expressed daily between today and time to maturity T is given by:

$$\bar{V}(t) = \frac{1}{T}\int_0^T V(t)dt = V_L + \frac{1-\exp(-a\,T)}{a\,T}[V(0) - V_L] \quad \textit{(Eq. III.272)}$$

The bigger T, the more $\bar{V}(t)$ tends to V_L.

We then define $\sigma(T)$ as the annualized volatility which should be used, for example, to value an option with a T day expiration with the model $GARCH(1,1)$.

Assuming a year consisting of 252 working days, the following relationship is valid:

$$\sigma(T)^2 = 252\left(V_L + \frac{1-\exp(-a\,T)}{a\,T}[V(0) - V_L]\right) \quad \textit{(Eq. III.273)}$$

The last equation allows to derive a term structure of volatility.

In the case of the S&P 500 index example we illustrated above, the volatility term-structure is given by:

$a = \ln\frac{1}{\alpha+\beta} = \ln\frac{1}{0.993496} = 0.006525243$; $V(0) = 0.00016324$ and $V_L = 0.0002066$.

$$\sigma(T)^2 = 252\left(0.0002066 + \frac{1-\exp(-0.006525243 \cdot T)}{0.006525243 \cdot T}[0.0001632 - 0.0002066]\right)$$

Where T is measured in days.

The following figure shows the chart of the historical volatility term structure of the S&P500.

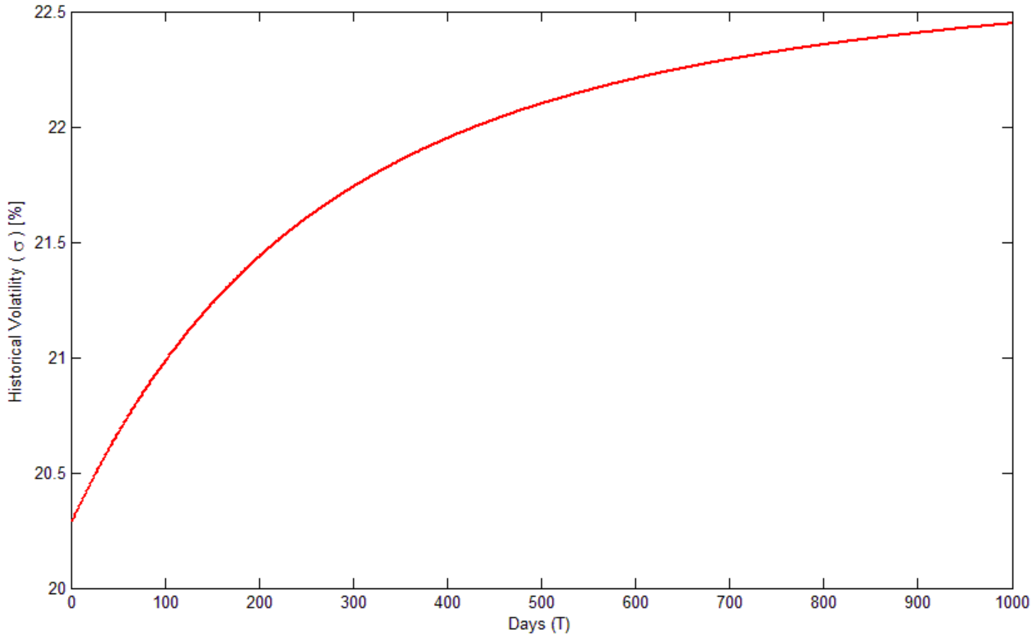


Figure III.131 S&P 500 Historical Volatility term structure

The volatility estimation techniques have the disadvantage to be based on the assumption that the future is strongly correlated to the past realizations. For this reason, it is appropriate to introduce another possible approach, based on the so-called **implied volatility**, σ_{IMPL}. The implied volatility is defined as that value of σ, which inserted in the Black-Scholes pricing formula, allows to obtain a theoretical value of the option equal to the market value. Unfortunately, this pricing formula is not analytically invertible, so it is not possible to derive a closed analytic expression for $\sigma = f(S_0, K, r, T)$. However, there are numerical tools that allow the value of volatility to be obtained, starting from a goal seeking algorithm. The following figure shows the surface of the implied volatilities obtained from the options on the S&P 500 index on the reference date of 29th September 2017 as a function of the time to maturity T and the strike price K.

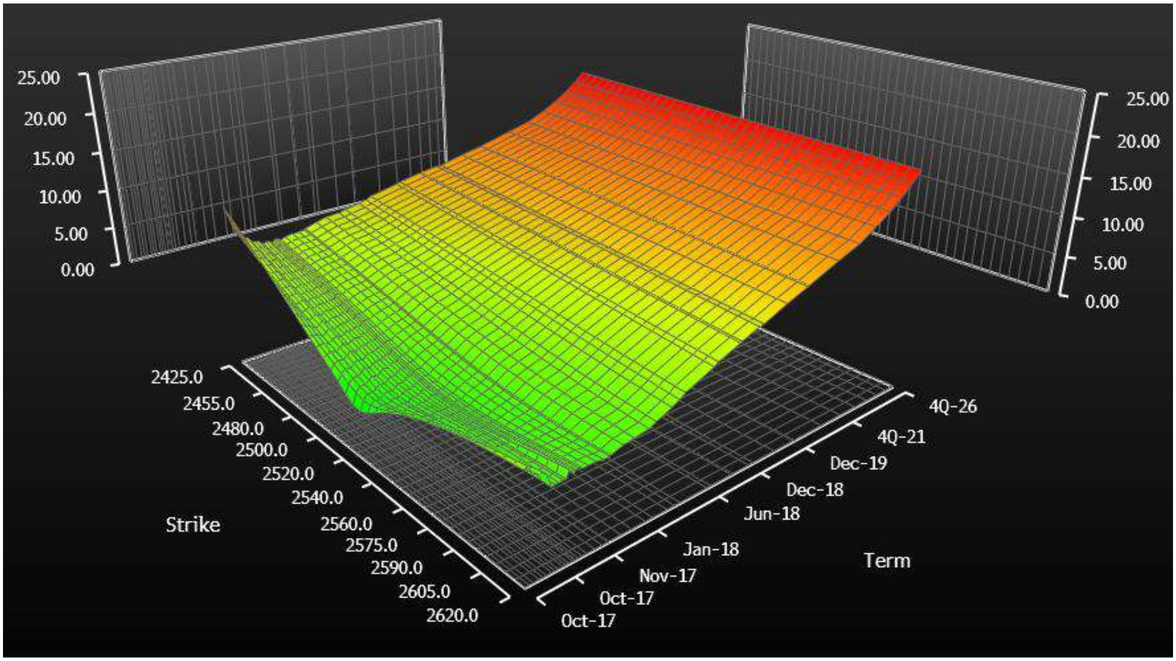


Figure III.132 Implied volatility surface. Source: Bloomberg®

The implied volatilities in option prices are used to ascertain current market opinions about the expected volatility associated with a particular security. In their work, Quantitative Analysts often calculate the implied volatilities in the prices of options written on a certain security that are more actively traded and then calculate the prices of other exotic options written on the same security but less actively traded.

It is important to underline though that the valuations of the options whose exercise price is much below the market price of the security, or much above (deep out of the money), are relatively insensitive to volatility.

Therefore, the implied volatilities calculated based on these options tend to be unreliable and it is preferable to resort to a methodology based on historical series.

Another important aspect to consider when analyzing this type of volatility is the different value it can assume in the presence of different strike prices. When we use in financial jargon the term "**volatility smiles**", we mean the graphs that represent the σ_{IMPL} as a function of the strike price (K).

Figure III.133 represents a section of the surface of implied volatilities, maintaining T constant.

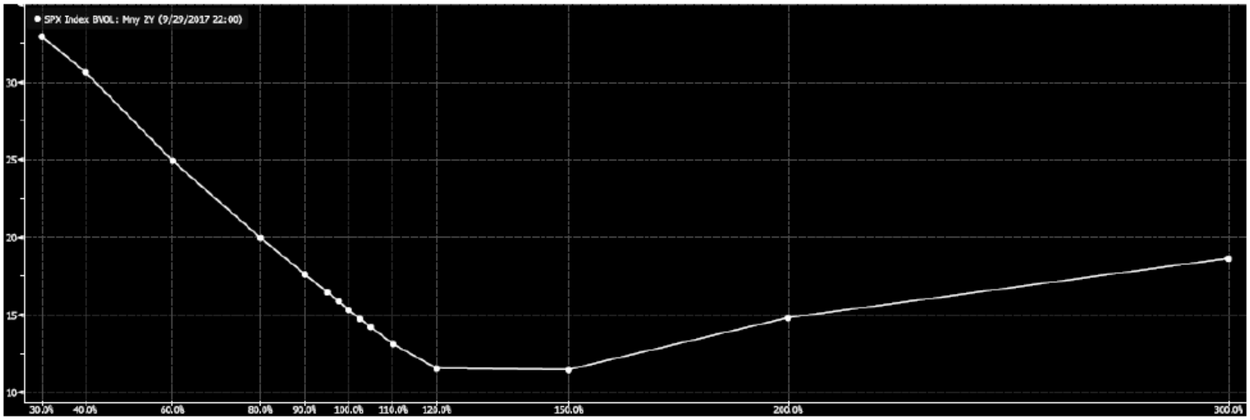


Figure III.133 Implied volatility section. Source: Bloomberg®

An empirical phenomenon observed on the markets is that the implied volatility associated with in/out of the money options tends to be higher than that calculated at-the-money, i.e., the strike price is equal to or close to the current price of the security. The previous figure highlights the described effect on volatility: the abscissa axis expresses the option's **moneyness** as a percentage. We notice that both for very low strike levels (for example for a strike level equal to 60% of the spot level of the underlying: $K = 60\% \cdot S_0$), and for very high strike levels, the volatility value is higher than those recorded for values around the at-the-money strike. Consequently, to correctly estimate the **fair value** of an OTC derivative, in order to infer a correct volatility value it is necessary that the starting listed contracts have similar characteristics (with particular reference to the duration and the strike price). Trading must also be characterized by significant liquidity.

We now introduce the concept of **correlation** between two variables X and Y, which can be defined as:

$\text{corr}(X, Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$ *(Eq. III.274)*

Where:

σ_X and σ_Y are the standard deviations of X and Y .

$\text{cov}(X, Y)$ is the covariance between X and Y .

The covariance between the two variables X and Y is defined as:

$\text{cov}(X, Y) = E[(X - \mu_X) \cdot (Y - \mu_Y)]$ *(Eq. III.275)*

Where: μ_X and μ_Y are the means of X and Y and $E[\cdot]$ denotes the expected value.

Now let x_i and y_i be the percentage changes of X and Y between the end of day $i - 1$ and i:

$x_i = \frac{X_i - X_{i-1}}{X_{i-1}}, y_i = \frac{Y_i - Y_{i-1}}{Y_{i-1}}$

Where: X_i and Y_i are the values of X and Y at the end of the i-th day.

We further define:

$\sigma_{x,n}$ as the daily volatility of variable X estimated for the n-th day.
$\sigma_{y,n}$ as the daily volatility of variable Y estimated for the n-th day.
cov_n as an estimation of the covariance between the daily variation in X and Y, calculated for the n-th day.

The estimated correlation between X and Y on day n is:

$$\text{corr}_n = \frac{\text{cov}_n}{\sigma_{x,n}\sigma_{y,n}} \quad \textit{(Eq. III.276)}$$

Similarly to what was done for the volatility estimation, using equal weights and assuming that the mean of x_i and y_i is zero, the rate of variance of X and Y can be estimated from the most recent m observations:

$$\sigma_{x,n}^2 = \frac{1}{m}\sum_{i=1}^{m} x_{n-i}^2\,, \sigma_{y,n}^2 = \frac{1}{m}\sum_{i=1}^{m} y_{n-i}^2 \quad \textit{(Eq. III.277)}$$

Following the same logic, a similar estimate for the covariance between X and Y is:

$$\text{cov}_n = \frac{1}{m}\sum_{i=1}^{m} x_{n-i} \cdot y_{n-i} \quad \textit{(Eq. III.278)}$$

An alternative for updating the covariances is provided by a model similar to the equation for $EWMA$. In this case the formula is given by:

$$\text{cov}_n = \lambda \cdot \text{cov}_{n-1} + (1-\lambda) \cdot x_{n-1} \cdot y_{n-1} \quad \textit{(Eq. III.279)}$$

Let us present an example involving the concept of correlation, assuming that $\lambda = 0.95$ and the estimation of the correlation between the two variables X and Y at day $n-1$ is equal to $\text{corr}(X,Y) = 0.6$. We also suppose that the estimate of the volatilities for X and Y at day $n-1$ is respectively equal to $\sigma_X = 1\%$ and $\sigma_Y = 2\%$. From the relationship between correlation and covariance, it follows that:

$$\text{Cov}_{n-1}(\text{X}, \text{Y}) = \text{corr}(\text{X}, \text{Y}) \cdot \sigma_X \cdot \sigma_Y = 0.6 \cdot 0.01 \cdot 0.02 = 0.00012.$$

Let us assume that the percentage change between X and Y at day $n-1$ is $x_{n-1} = 0.5\%$, and $y_{n-1} = 2.5\%$ y, respectively. According to the $EWMA$ method, the variance and covariance for day n must be updated as:

$$\sigma_{x,n}^2 = \lambda \cdot \sigma_{X,n-1}^2 + (1-\lambda) \cdot x_{n-1} = 0.95 \cdot 0.01^2 + 0.05 \cdot 0.005^2 = 0.00009625$$

$$\sigma_{y,n}^2 = \lambda \cdot \sigma_{Y,n-1}^2 + (1-\lambda) \cdot y_{n-1} = 0.95 \cdot 0.02^2 + 0.05 \cdot 0.025^2 = 0.00041125$$

$$\text{Cov}_n = \lambda \cdot \text{cov}_{n-1} + (1-\lambda) \cdot x_{n-1} \cdot y_{n-1} = 0.95 \cdot 0.00012 + 0.05 \cdot 0.05 \cdot 0.025 = 0.00012025.$$

The new volatility of X is $\sqrt{0.00009625} = 0.981\%$.

The new volatility of Y is $\sqrt{0.00041125} = 2.028\%$.

The new correlation between X and Y is: $\text{corr}_n = \frac{0.00012025}{0.00981 \cdot 0.02028} = 0.6044$.

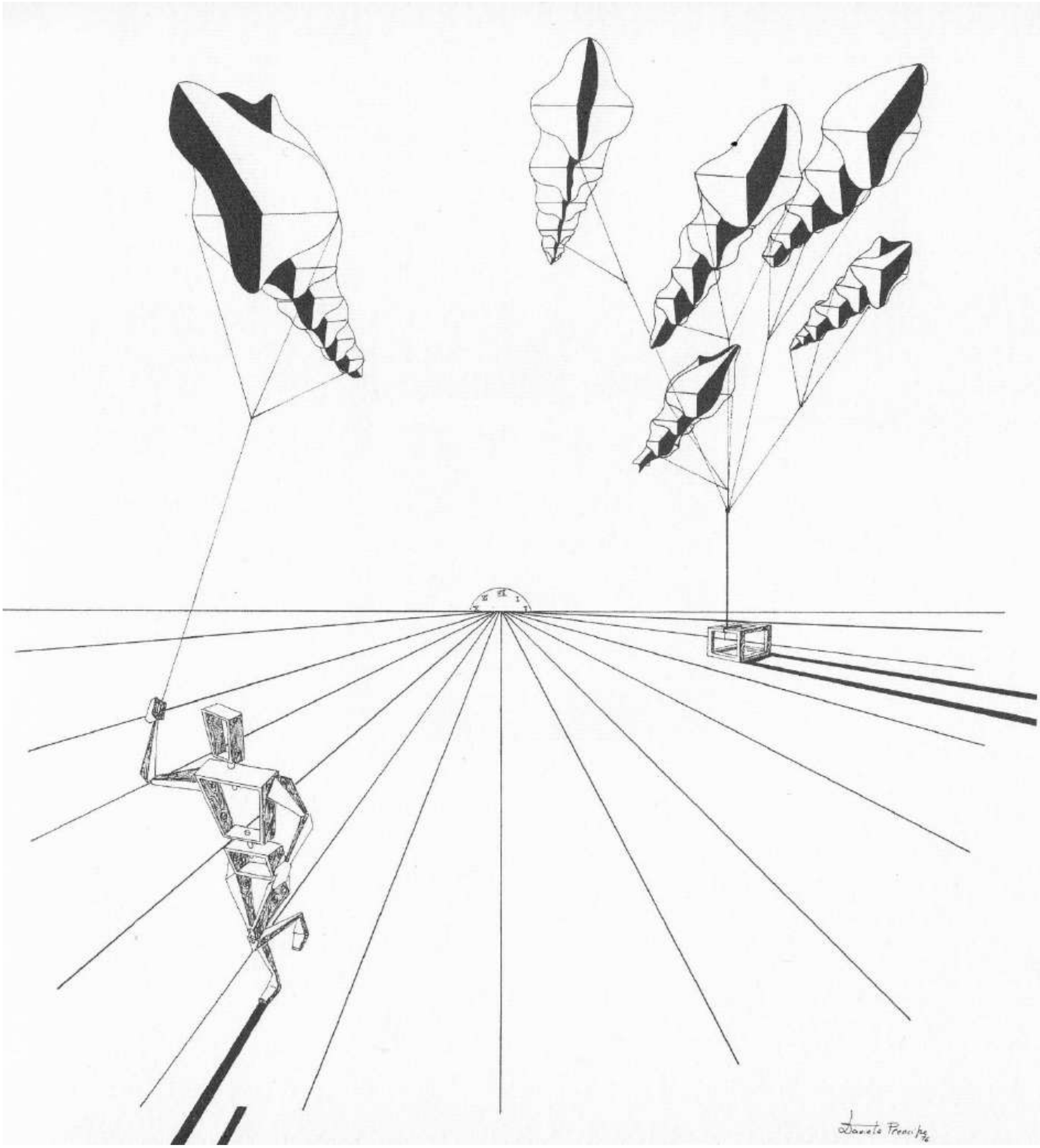
Some researchers have proposed a method for updating $GARCH(1,1)$ as a basis for predicting future levels of covariance:

$\text{Cov}_n = \omega + \alpha x_{n-1} y_{n-1} + \beta \text{cov}_{n-1}$.

Once all variances and covariances have been calculated, a variance-covariance matrix can be constructed. The element (i, j) of this matrix reports the value of the covariance between variable i and j, when $i \neq j$. On the other hand, when $i = j$, the value represents the variance of variable i.
Not all variance-covariance matrices are internally consistent though, as they must necessarily be positive semi-definite and this condition is satisfied if all the eigenvalues of the matrix are positive. If this does not occur, it is necessary to use mathematical methodologies for generating valid matrices. Among such methods which aim to create a positive semi-definite matrix as close as possible to the original one, the following can be mentioned: the Shrinkage Approach, the Hypersphere Decomposition, the Spectral Decomposition and the SDP, or Semidefinite Programming.

FURTHER READINGS

Caligaris O., Giribone P. G., Neffelli M. – "Ricostruzione di superfici di volatilità mediante l'utilizzo di reti neurali auto-associative: un caso studio basato sull'analisi non lineare delle componenti principali" – Risk Management Magazine Vol. 12, N. 3 (2017).
Giribone P. G. – "Implementazione della tecnica AFO per la stima dei parametri di un modello GARCH (1,1). Analisi di robustezza e confronto prestazionale con i solver tradizionali" – Risk Management Magazine Vol. 13, N. 2 (2018).
Haug E. G. – "The Complete Guide to Option Pricing Formulas" – McGraw Hill (2007).
Hull J. – "Options, Futures and other derivatives" – Pearson (2014).



PART IV: SWAPS

Chapter 1 – Fundamentals

Introduction and definitions
Swap mechanics
Interest rate swaps (IRS)
Swap quotes
Closing a swap contract
Other IRS typologies
Currency swaps
Other currency swap typologies
Trading and hedging strategies
Total return, commodity and variance swaps
Swaptions

Chapter 2 – Quantitative Analysis

Swap pricing as a bond portfolio
Swap pricing as a forward portfolio
IRS pricing and sensitivity
Currency swap pricing and sensitivity
Swap curve stripping

IV.1 FUNDAMENTALS

Among all the financial innovations introduced in the 1980s none can compete with the swap market. The growth of swaps over the last forty years has been phenomenal and today they have established themselves as one of the most important classes of derivatives in the capital market. These financial instruments have become so widespread that they can be considered a complementary risk management tool to futures and options.
A swap is a contract in which two counterparties agree to make payments to each other or to exchange a stream of cash flows in the future, according to an agreed formula for a predetermined period of time. The start date of the swap is called the effective date, while the end date is called the termination date (or maturity date).

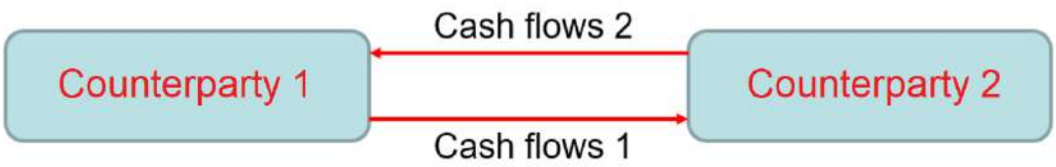


Figure IV.1 Swap between two counterparties

In the early days of the swap market, transactions were conducted by banks and financial institutions which were compensated for the offered service through a fee. This implied that the contract was signed directly between the two counterparties. Nowadays this is no longer feasible due to the very high market volumes, and the problem was easily solved by introducing a financial intermediary: in exchange for a commission the **swap brokers** were entrusted with the task of looking for the suitable counterparty that met the client's needs. The brokers take no risk connected to the contract, as they do not take any position in the swap. On the other hand, **swap dealers** (or market makers) are willing to act as counterparty in a swap contract. When the dealers do not want to assume the risks associated with the operation, they will find an interested counterparty to replace them or hedge their own book. The swap dealer's profit consists of the bid-ask spread he applies to swap quotes.

For operators on the swap market, working with a swap dealer is beneficial for two reasons: first of all, the use of an intermediary reduces the time searching for a suitable counterparty, since the swap dealer is ready to enter into the swap at any time. Secondly, an intermediary can reduce credit assessment costs. Such assessment is necessary, as each of the two counterparties can potentially incur a default that prevents from fulfilling its contractual duties.

In conclusion, the swap with a dealer saves the costs of a detailed creditworthiness analysis.

Figure IV.2 Swap dealer

The cash flows typically associated with a swap occur at the initial and at the final stage of the contract, but also during the life of the swap, and they can be summarized as follows:

- Initial notional exchange, which is optional as it is not required for all types of swaps:

Figure IV.3 Initial notional exchange

- Periodic payments between counterparties, which are mandatory for all swaps:

Figure IV.4 Periodic payments

- Final notional exchange, which is also optional as it is not required for all types of swaps:

Figure IV.5 Final notional exchange

Unlike the main financial markets, the swap market did not include a centralized clearing mechanism until a few years ago.
The terms of the contract for an "over the counter", or OTC swap are determined between the counterparties who follow the best practices proposed by the main professional categories among which we mention ISDA, the International Swap Dealer Association and the British Banker's Association. Following this set of rules and guidelines for defining the contractual characteristics leads to greater standardization and consequently greater speed in closing the deal. Although the instrument is not subjected to strict regulation, the market participants on the other hand, obviously are.

An **Interest Rate Swap** (IRS) is a derivative in which two counterparties agree to make periodic payments calculated based on contractually agreed interest rates on a reference notional. Therefore, to define an IRS, the following characteristics must be specified in the contract:

- the swap start date.

- The time to maturity of the swap or expiration time.

- The interest rates to be exchanged, fixed rates or floating rates.

- The frequency of cash flow payments.

- The notional to be considered on which interest will be calculated and paid.

The most common type of interest rate swap is the **fixed-for-floating rate swap**, also called **plain vanilla swap** or generic swap, in which payments are calculated for one counterparty using the variable interest rate, while for the other they are determined based on a fixed interest rate. Payments are usually made in arrears on an annual or semi-annual basis. The **effective date** is the date from which interest begins to accrue and the **payment date** is the date on which the interest is paid. As an example, the mechanics of an IRS can be illustrated as follows:

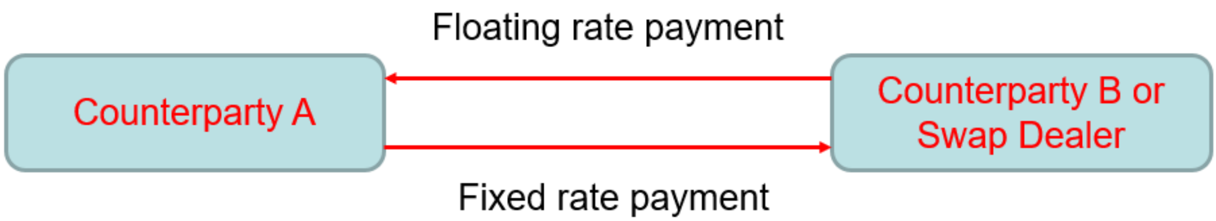


Figure IV.6 Interest rate swap

The floating interest rate is periodically redefined (**floating rate reset**), i.e., it is fixed at a specific spot market rate (reference rate) observed on the market at the contractual date (**reset dates**). The most widespread floating rate is LIBOR.

The two series of payments are called **swap legs** or swap sides. The fixed rate is called **coupon swap**. In an IRS, the notional is usually not exchanged, only the interest is paid.

Let us examine a practical example: on 12/28/YYYY-1, Company ALPHA enters into a swap transaction with Company BETA. Both agree to exchange starting from 01/01/YYYY, cash flows on a notional amount of CHF 100 million until 12/31/YYYY+4 (5 years). Company ALPHA will pay counterparty BETA a fixed rate of 3% annually on the notional amount of CHF 100 million, while Company BETA will pay a variable interest rate indexed to 6-month LIBOR every six months on the notional amount of CHF 100 million. Payment dates are 01/01 and 07/01 as shown in the table below.

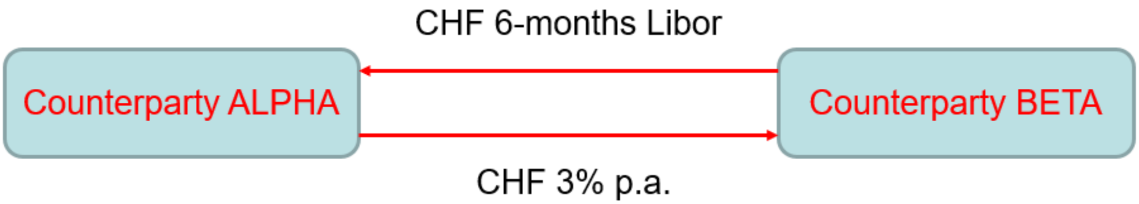


Figure IV.7 Interest rate swap example

From the perspective of Company BETA, on the swap starting date, the cash flows are as follows:

Payment Date	Receive Fix [CHF]	Pay Floating [CHF]	LIBOR setting	LIBOR
			01/01/YYYY	3.25%
07/01/YYYY		-1,625,000	01/07/YYYY	?
01/01/YYYY+1	+3,000,000	- 6 month LIBOR	01/01/YYYY+1	?
07/01/YYYY+1		- 6 month LIBOR	01/07/YYYY+1	?
01/01/YYYY+2	+3,000,000	- 6 month LIBOR	01/01/YYYY+2	?
07/01/YYYY+2		- 6 month LIBOR	01/07/YYYY+2	?
01/01/YYYY+3	+3,000,000	- 6 month LIBOR	01/01/YYYY+3	?
07/01/YYYY+3		- 6 month LIBOR	01/07/YYYY+3	?
01/01/YYYY+4	+3,000,000	- 6 month LIBOR	01/01/YYYY+4	?
07/01/YYYY+4		- 6 month LIBOR	01/07/YYYY+4	?
12/31/YYYY+4	+3,000,000	- 6 month LIBOR		

Table IV.1 Interest rate swap example: Cash flows

Figure IV.8 Interest rate swap. Main characteristics. Source: Bloomberg®

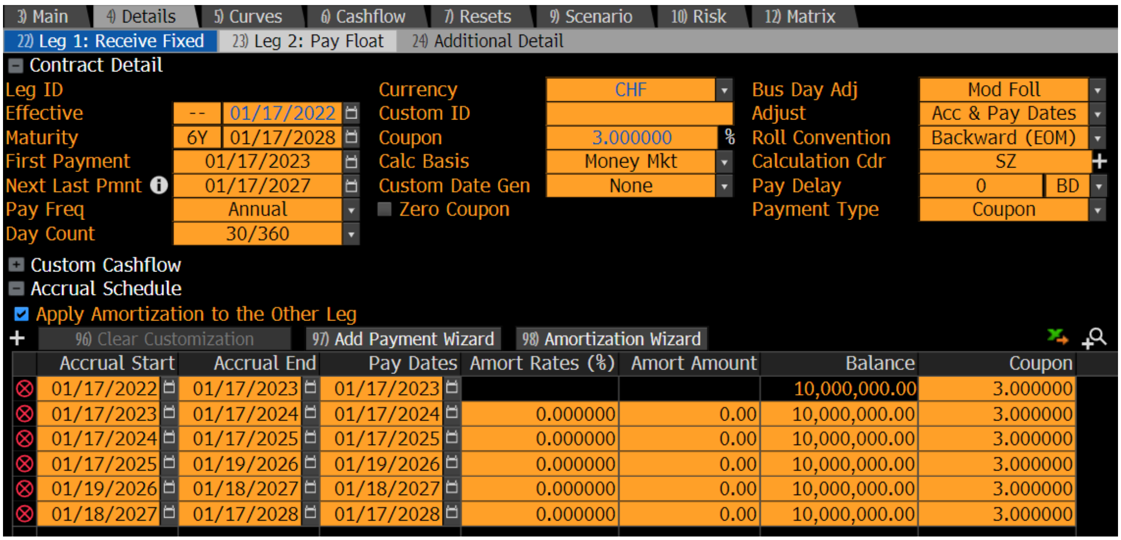


Figure IV.9 Interest rate swap. Details of the Leg 1: Receive Fixed. Source: Bloomberg®

Figure IV.10 Interest rate swap. Details of the Leg 2: Pay Float. Source: Bloomberg®

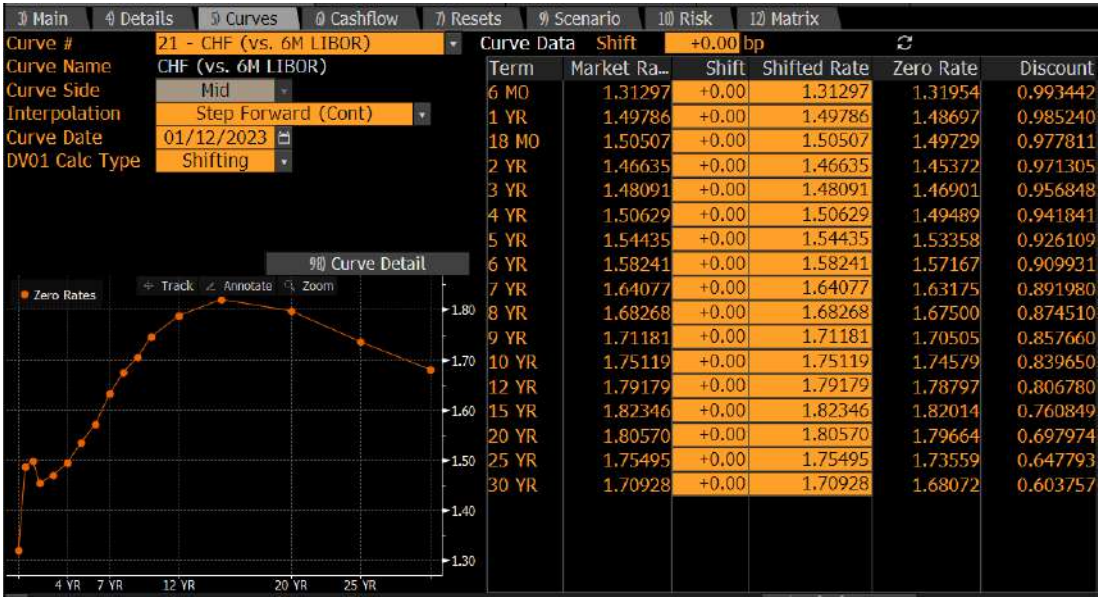


Figure IV.11 Interest rate swap. Curves. Source: Bloomberg®

3) Main | 4) Details | 5) Curves | 6) Cashflow | 7) Resets | 9) Scenario | 10) Risk | 12) Matrix

21) Cashflow Table | 22) Cashflow Graph

Cashflow: Leg 1: Receive Fixed — Historical Cashflows — Accrued 295,833.33
Currency: CHF — Zero Rate — NPV 1,734,118.34
Equiv. Coupon

Pay Date	Accrual Start	Accrual End	Da...	Notional	Principal	Payment	Discount	Zero Rate	PV
01/17/2023	01/17/2022	01/17/2023	360	10,000,000.00	0.00	300,000.00	0.999819	1.319543	299,945.78
01/17/2024	01/17/2023	01/17/2024	360	10,000,000.00	0.00	300,000.00	0.985057	1.485242	295,517.07
01/17/2025	01/17/2024	01/17/2025	360	10,000,000.00	0.00	300,000.00	0.971125	1.453055	291,337.54
01/19/2026	01/17/2025	01/19/2026	362	10,000,000.00	0.00	301,666.67	0.956586	1.468752	288,570.14
01/18/2027	01/19/2026	01/18/2027	359	10,000,000.00	0.00	299,166.67	0.941705	1.494408	281,726.83
01/17/2028	01/18/2027	01/17/2028	359	10,000,000.00	0.00	299,166.67	0.925975	1.533111	277,020.99

21) Cashflow Table | 22) Cashflow Graph

Cashflow: Leg 2: Pay Float — Historical Cashflows — Accrued -16,227.67
Currency: CHF — Zero Rate — NPV -754,746.92
Equiv. Coupon

Pay Date	Accrual Start	Accrual End	Da...	Notional	Principal	Payment	Discount	Zero Rate	PV
01/17/2023	07/18/2022	01/17/2023	183	-10,000,000.00	0.00	-17,257.92	0.999819	1.319543	-17,254.80
07/17/2023	01/17/2023	07/17/2023	181	-10,000,000.00	0.00	-64,516.40	0.993298	1.319543	-64,084.03
01/17/2024	07/17/2023	01/17/2024	184	-10,000,000.00	0.00	-84,107.27	0.985057	1.485242	-82,850.45
07/17/2024	01/17/2024	07/17/2024	182	-10,000,000.00	0.00	-75,791.69	0.977634	1.495691	-74,096.55
01/17/2025	07/17/2024	01/17/2025	184	-10,000,000.00	0.00	-67,531.39	0.971125	1.453055	-65,581.43
07/17/2025	01/17/2025	07/17/2025	181	-10,000,000.00	0.00	-74,440.93	0.963930	1.462250	-71,755.85
01/19/2026	07/17/2025	01/19/2026	186	-10,000,000.00	0.00	-76,739.95	0.956586	1.468752	-73,408.37
07/17/2026	01/19/2026	07/17/2026	179	-10,000,000.00	0.00	-77,273.70	0.949239	1.483189	-73,351.22
01/18/2027	07/17/2026	01/18/2027	185	-10,000,000.00	0.00	-79,969.17	0.941705	1.494408	-75,307.39
07/19/2027	01/18/2027	07/19/2027	182	-10,000,000.00	0.00	-84,349.34	0.933807	1.515895	-78,766.02
01/17/2028	07/19/2027	01/17/2028	182	-10,000,000.00	0.00	-84,549.56	0.925975	1.533111	-78,290.81

Figure IV.12 Interest rate swap. Cashflow Tables. Source: Bloomberg®

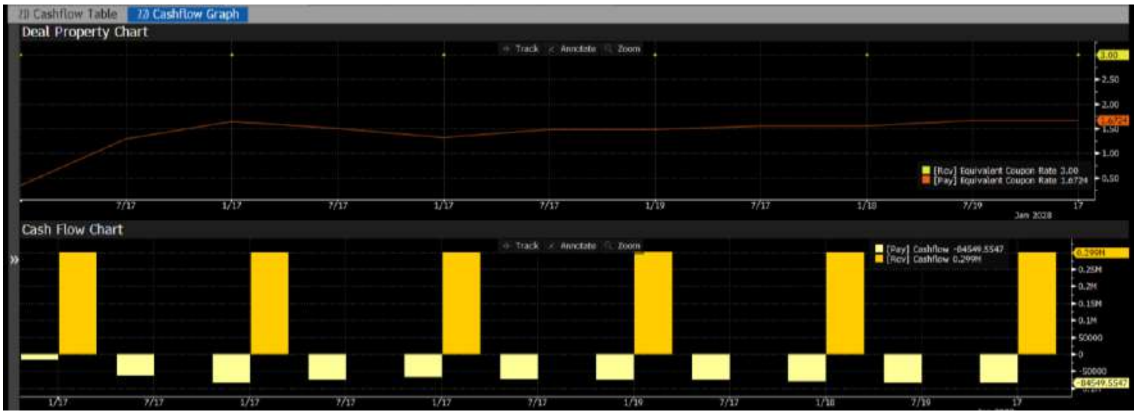


Figure IV.13 Interest rate swap. Cashflow Graph. Source: Bloomberg®

Unlike futures and options, most IRS contracts are traded over the counter, therefore there is currently no location where trades take place and most of the time there is no clearing house to ensure that commitments can be honored.
Swap dealers therefore regularly prepare indicative schedules of quotations to be applied.
The reference floating rate is generally LIBOR, while Swap prices are frequently quoted as a spread over government issues.

Maturity	Spread over Treasury yield if dealer pays fixed	Spread over Treasury yield if dealer receives fixed	Current Treasury yield
2 years	38 basis points	44 basis points	1.55%
3 years	46 basis points	52 basis points	1.72%
4 years	50 basis points	58 basis points	1.85%
5 years	52 basis points	60 basis points	1.92%
6 years	58 basis points	66 basis points	1.96%
7 years	62 basis points	70 basis points	2.00%
10 years	74 basis points	84 basis points	2.08%

Table IV.2 Swap quotes

The table assumes half-yearly rates and bullet transactions (i.e. without amortization plan) made by companies characterized by the best level of creditworthiness. All rates are quoted against 6-months LIBOR flat. For each maturity, in order to obtain the swap price, it is necessary to add the quoted spread to the current yield of the government bond.

For example, if the dealer pays fixed on a 5-year swap, the fixed rate on the swap should equal the yield on the 5-year bond plus 52 basis points. If the dealer pays floating and receives fixed, the fixed swap rate should be equal to the yield on the government bond plus 60 basis points. The mid-rate can simply be calculated by estimating the average between the pay and receive rates.

For example, the 5-year mid-rate swap is equal to:

$$[(0.0192 + 0.0052) + (0.0192 + 0.0060)] / 2 = 2.48\%.$$

Generally, fixed rates are expressed on an annual or half-yearly basis according to the conventions adopted by the single countries.

Euro Zone | 99) Settings | Interest Rate Swap Rates

40) Annual Swaps | 41) Sprs to Gov. | 42) 3M Euribor | 43) Ann Swaps vs ESTR | 44) ESTR/EURIBOR Basis | 45) Combined

Euro Swap Annual Fixed Annual v.s. Float Semi-annual					Euro Annual Swap Spreads to Government				
Tenor	Bid	Ask	Mid	Change	Tenor	Bid	Ask	Mid	Change
1) 1 YR	3.213 /	3.246	3.229	-0.027	51) 1 YR	56.549 /	63.997	60.273	-0.719
2) 18 MO	3.223 /	3.263	3.243	-0.018	52) 18 MO	/			
3) 2 YR	3.153 /	3.186	3.170	-0.027	53) 2 YR	58.726 /	64.550	61.638	-0.338
4) 3 YR	2.957 /	2.979	2.968	-0.034	54) 3 YR	60.298 /	64.983	62.641	-1.129
5) 4 YR	2.837 /	2.851	2.844	-0.046	55) 4 YR	56.809 /	60.067	58.438	-0.224
6) 5 YR	2.769 /	2.778	2.773	-0.056	56) 5 YR	60.030 /	62.184	61.107	-0.095
7) 6 YR	2.729 /	2.740	2.734	-0.062	57) 6 YR	63.031 /	65.545	64.288	-0.608
8) 7 YR	2.704 /	2.715	2.710	-0.065	58) 7 YR	57.681 /	60.385	59.033	-0.721
9) 8 YR	2.696 /	2.707	2.701	-0.068	59) 8 YR	63.095 /	65.184	64.139	-0.066
10) 9 YR	2.698 /	2.709	2.704	-0.070	60) 9 YR	61.178 /	63.073	62.126	-0.118
11) 10 YR	2.705 /	2.715	2.710	-0.073	61) 10 YR	56.947 /	58.963	57.955	-0.361
12) 11 YR	2.709 /	2.723	2.716	-0.073		/			
13) 12 YR	2.714 /	2.723	2.718	-0.072		/			
14) 13 YR	2.709 /	2.723	2.716	-0.070		/			
15) 14 YR	2.700 /	2.714	2.707	-0.067		/			
16) 15 YR	2.684 /	2.696	2.690	-0.065	66) 15 YR	44.748 /	47.610	46.179	0.755
17) 16 YR	2.658 /	2.673	2.665	-0.068		/			
18) 17 YR	2.628 /	2.642	2.635	-0.066		/			
19) 18 YR	2.591 /	2.606	2.599	-0.066		/			
20) 19 YR	2.553 /	2.568	2.560	-0.063		/			
21) 20 YR	2.515 /	2.525	2.520	-0.068	71) 20 YR	34.657 /	37.477	36.067	0.214
22) 21 YR	2.476 /	2.488	2.482	-0.062		/			

Figure IV.14 Swap quotes. Source: Bloomberg®

Normal market bid-ask spreads are around 4-12 basis points. Obviously, if the dealer deals in swaps different from the plain vanilla ones, adjustments to the base list prices should be made.

A popular market benchmark is the **swap curve**, which is a curve that reports the fixed rate of a plain vanilla swap (fixed vs. six-month LIBOR) for different maturities. Although the interest rates expressed by the curve are not very uniform in fact due, for example, to overlapping maturities between the instruments and different interpolations, the swap curve remains a fundamental element in valuing fixed-income products and in measuring the interest rate expectations. In constructing interest rates term structures, the swap curve is often used to determine the zero rate and the related discount factors for the long end (typically for maturities of more than two years).

Figure IV.15 below shows the standard S-45 Bloomberg® Curve. The short term has been stripped using Cash Rates, the middle-term using Serial Forward Rates Agreements (FRA) and the long term using Swaps.

Figure IV.15 The Swap curve. Source: Bloomberg®

Before their maturity, swap positions may be closed by selling the derivative to the swap dealer or to another counterparty. Alternatively, the swap holder could hedge his position by taking an opposite position to the current one in another swap or by hedging his position for the remaining time with futures and bonds.

Let us consider, as an example the holder of a swap who pays a fixed rate and believes that interest rates are going to fall sharply. Consequently, he wishes to change his exposure and his alternatives can be either to sell the swap to the dealer and close his position or to enter into a new swap contract where he pays a floating interest rate: the new position should require a LIBOR payment which cancels out what was received from the first swap. The spread between the two positions should therefore be equal to the difference between the higher fixed interest rate paid in the first swap and the lower fixed interest rate received in the second swap. A third alternative consists in using futures contracts if the time to cover is not excessively long.

Up to now, we have seen the most common and standard Interest Rate Swaps category, i.e., the plain vanilla swap. With these derivatives, one counterparty (the payer) pays a fixed interest rate to another counterparty (the receiver) against a variable interest rate (fixed rate versus floating rate). There are many variations of the standard IRS in the financial market and in this context, we will discuss the most widespread types of swaps.
A counterparty may decide to enter a swap that exchanges a floating interest rate against another floating interest rate, such as LIBOR against another prime interest rate. This allows financial institutions to hedge an exposure that arises from assets and liabilities subject to different interest rates. Swaps that have the characteristic of having both legs linked to a floating rate are called **basis swaps**. Within this category, **Yield curve swaps** are

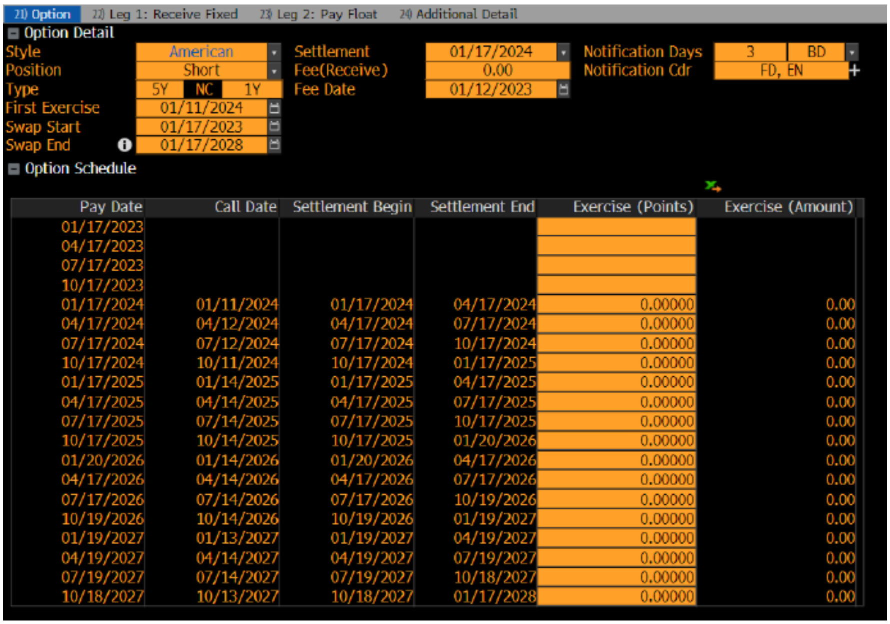
basis swaps in which the counterparties agree to exchange payments based on the difference between two rates of the same content at two different maturities on a given yield curve.

Figure IV.16 A basis swap. Source: Bloomberg®

Some swaps are characterized by having an amortization schedule, they are therefore called **amortising swaps**. Then, we have **Accreting swaps,** which have an increasing notional over time. On the other hand, swaps that have a period in which the notional increases and subsequently decreases are called **Roller coaster swaps**.

Zero-coupon swaps are fixed-for-floating, where the fixed rate is a zero-coupon bond, in this case the fixed leg does not pay until maturity, but a large payment is made on that occasion. **Rate-capped and Rate-floored swaps** are swaps that have a maximum or a minimum rate on the floating leg.
Callable Swaps and **Putable Swaps** are swaps for which the payer (or the receiver) has the option to terminate the swap early. They are also called **Cancellable Swaps**. **Extendible swaps** are swaps in which one of the two counterparties has the option to extend the maturity of the derivative, and **Reversible swaps** are swaps in which the payer and the receiver swap roles one or more times during the life of the derivative.

Figure IV.17 A cancellable swap. Main Characteristics. Source: Bloomberg®



Pay Date	Call Date	Settlement Begin	Settlement End	Exercise (Points)	Exercise (Amount)
01/17/2023					
04/17/2023					
07/17/2023					
10/17/2023					
01/17/2024	01/11/2024	01/17/2024	04/17/2024	0.00000	0.00
04/17/2024	04/12/2024	04/17/2024	07/17/2024	0.00000	0.00
07/17/2024	07/12/2024	07/17/2024	10/17/2024	0.00000	0.00
10/17/2024	10/11/2024	10/17/2024	01/17/2025	0.00000	0.00
01/17/2025	01/14/2025	01/17/2025	04/17/2025	0.00000	0.00
04/17/2025	04/14/2025	04/17/2025	07/17/2025	0.00000	0.00
07/17/2025	07/14/2025	07/17/2025	10/17/2025	0.00000	0.00
10/17/2025	10/14/2025	10/17/2025	01/20/2026	0.00000	0.00
01/20/2026	01/14/2026	01/20/2026	04/17/2026	0.00000	0.00
04/17/2026	04/14/2026	04/17/2026	07/17/2026	0.00000	0.00
07/17/2026	07/14/2026	07/17/2026	10/19/2026	0.00000	0.00
10/19/2026	10/14/2026	10/19/2026	01/19/2027	0.00000	0.00
01/19/2027	01/13/2027	01/19/2027	04/19/2027	0.00000	0.00
04/19/2027	04/14/2027	04/19/2027	07/19/2027	0.00000	0.00
07/19/2027	07/14/2027	07/19/2027	10/18/2027	0.00000	0.00
10/18/2027	10/13/2027	10/18/2027	01/17/2028	0.00000	0.00

Figure IV.18 A cancellable swap. Embedded option. Source: Bloomberg®

A **currency swap** is a transaction in which two counterparties agree to make periodic payments to each other based on agreed interest rates on a notional amount denominated in two different currencies. Actually, a currency swap is very similar to an IRS, except for the fact that there are two currencies involved therefore the notional amounts and the cash flows exchanged are expressed in two different currencies.
In this case, the counterparties exchange interest payment flows of different types: fixed against fixed, floating against floating, fixed against floating. Usually in a currency swap the initial and the final exchange of the principal amount occur.
Therefore, a standard currency swap has three distinct classes of cash flows: the first is the initial exchange of notional amounts denominated in different currencies, which set the current exchange rate. Then there are the interest payments made by each counterparty to the other.
Lastly, the final exchange of notional amounts based on the exchange rate registered at the conclusion of the swap.

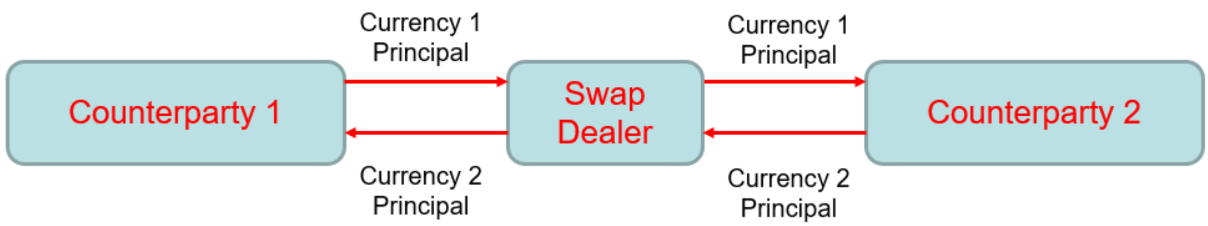


Figure IV.19 Exchange of principals

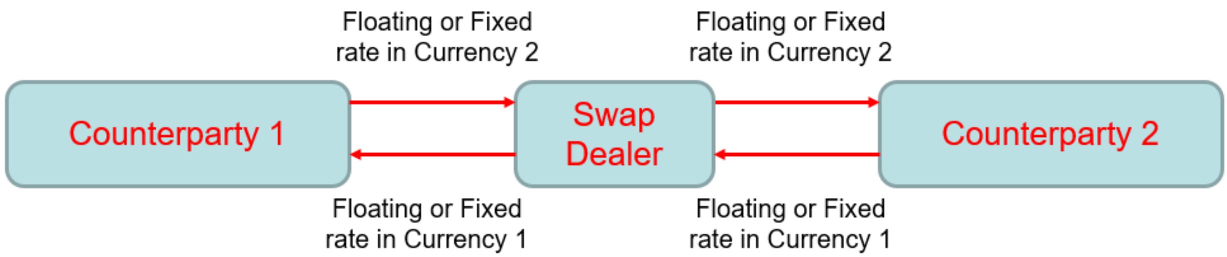


Figure IV.20 Periodic service payments

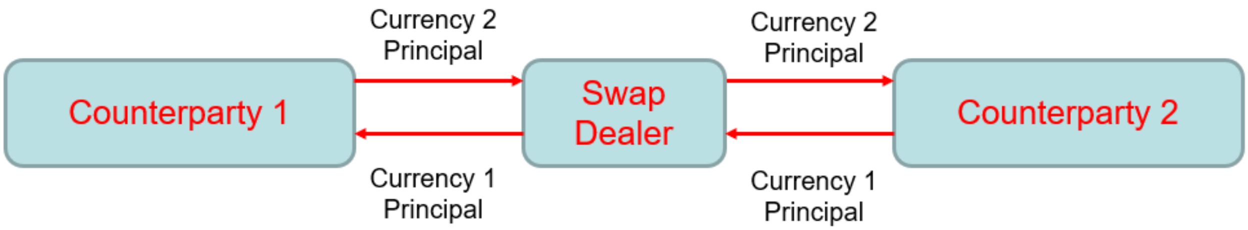


Figure IV.21 Re-Exchange of principals

Let us analyze a practical example of currency swap. On 12/28/YYYY-1, Company ALPHA enters a swap transaction with Company BETA.

Both agree to exchange cash flows on a notional of CHF 100 million starting from 01/01/YYYY until 12/31/YYYY+4 (5 years) at a fixed rate of 3% and cash flows on a notional of USD 80 million at 6-month USD LIBOR.

Company ALPHA will pay counterparty BETA a fixed rate of 3% annually on the notional amount of CHF 100 million and Company BETA will pay a floating interest rate indexed to 6-month LIBOR on the USD 80 million notional every six months.

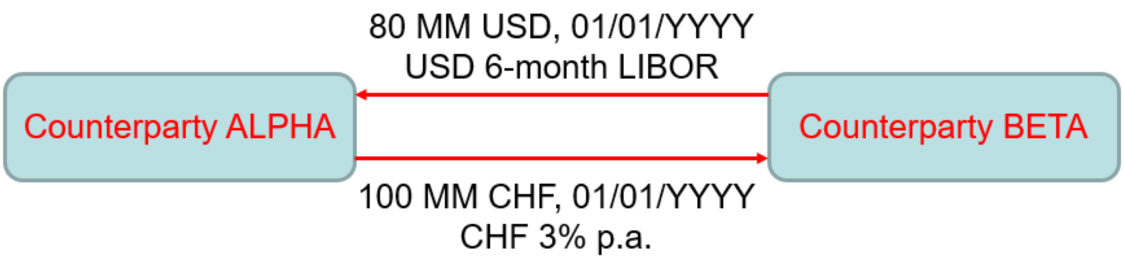


Figure IV.22 Currency Swap example

In this example, the USD/CHF spot exchange rate would be 1.25, so on 12/31/YYYY+4 Company ALPHA will repay BETA CHF 100 million and receive USD 80 million from BETA, regardless of the market spot exchange rate recorded at the expiration date.

From the perspective of Company BETA, on the swap starting date, the cash flows are:

Payment Date	Receive Fix [CHF]	Pay Floating [USD]	LIBOR setting	LIBOR
01/01/YYYY	-100,000,000	+80,000,000	01/01/YYYY	3.25%
07/01/YYYY		-1,300,000	07/01/YYYY	?
01/01/YYYY+1	+3,000,000	- 6 month LIBOR	01/01/YYYY+1	?
07/01/YYYY+1		- 6 month LIBOR	07/01/YYYY+1	?
01/01/YYYY+2	+3,000,000	- 6 month LIBOR	01/01/YYYY+2	?
07/01/YYYY+2		- 6 month LIBOR	07/01/YYYY+2	?
01/01/YYYY+3	+3,000,000	- 6 month LIBOR	01/01/YYYY+3	?
07/01/YYYY+3		- 6 month LIBOR	07/01/YYYY+3	?
01/01/YYYY+4	+3,000,000	- 6 month LIBOR	01/01/YYYY+4	?
07/01/YYYY+4		- 6 month LIBOR	07/01/YYYY+4	?
12/31/YYYY+4	+3,000,000	- 6 month LIBOR		
12/31/YYYY+4	+100,000,000	-80,000,000		

Table IV.3 Currency Swap example

Figure IV.23 Currency Swap. Main Characteristics. Source: Bloomberg®

Figure IV.24 Currency Swap. Details of the Leg 1: Receive Fixed. Source: Bloomberg®

Figure IV.25 Currency Swap. Details of the Leg 2: Pay Float. Source: Bloomberg®

Figure IV.26 Currency Swap. Cashflow Graph. Source: Bloomberg®

3) Main | 4) Details | 5) Curves | 6) Cashflow | 7) Resets | 9) Scenario | 10) Risk | 11) CVA | 12) Matrix

21) Cashflow Table | 22) Cashflow Graph

Cashflow: Leg 1: Receive Fixed — Historical Cashflows — Accrued 2,950,000.00
Currency: CHF — Zero Rate — NPV 113,214,763.91
Equiv. Coupon

Pay Date	Accrual Start	Accrual End	Da...	Notional	Principal	Payment	Discount	Zero Rate	PV
01/17/2023	01/18/2022	01/17/2023	359	100,000,000.00	0.00	2,991,666.67	0.999909	0.664335	2,991,394.42
01/17/2024	01/17/2023	01/17/2024	360	100,000,000.00	0.00	3,000,000.00	0.989910	1.000449	2,969,729.14
01/17/2025	01/17/2024	01/17/2025	360	100,000,000.00	0.00	3,000,000.00	0.981064	0.948094	2,943,191.56
01/20/2026	01/17/2025	01/20/2026	363	100,000,000.00	0.00	3,025,000.00	0.972423	0.924563	2,941,578.18
01/19/2027	01/20/2026	01/19/2027	359	100,000,000.00	0.00	2,991,666.67	0.964381	0.901770	2,885,107.61
01/18/2028	01/19/2027	01/18/2028	359	100,000,000.00	100,000,000.00	102,991,666.67	0.956230	0.891707	98,483,763.00

Cashflow: Leg 2: Pay Float — Historical Cashflows — Accrued -1,671,860.06
Currency: USD — Zero Rate — NPV -111,497,125.02
Equiv. Coupon

Pay Date	Accrual Start	Accrual End	Da...	Notional	Principal	Reset Date	Reset Rate	Payment	Discount	Zero Rate	PV
01/17/2023	07/18/2022	01/17/2023	183	-107,631,040.79	0.00	07/14/2022	3.38129	-1,849,986.46	0.999401	4.372514	-1,848,878.69
07/17/2023	01/17/2023	07/17/2023	181	-107,631,040.79	0.00	01/13/2023	5.14186	-2,782,491.60	0.975924	4.782496	-2,715,498.95
01/17/2024	07/17/2023	01/17/2024	184	-107,631,040.79	0.00	07/13/2023	5.14229	-2,828,848.07	0.952918	4.757434	-2,695,661.27
07/17/2024	01/17/2024	07/17/2024	182	-107,631,040.79	0.00	01/15/2024	4.34234	-2,362,819.89	0.934540	4.476566	-2,208,150.70
01/17/2025	07/17/2024	01/17/2025	184	-107,631,040.79	0.00	07/15/2024	3.46467	-1,905,962.02	0.920182	4.125280	-1,753,832.09
07/17/2025	01/17/2025	07/17/2025	181	-107,631,040.79	0.00	01/15/2025	3.17640	-1,718,895.03	0.907616	3.858333	-1,560,096.48
01/20/2026	07/17/2025	01/20/2026	187	-107,631,040.79	0.00	07/15/2025	3.17724	-1,776,342.65	0.894851	3.673076	-1,589,562.14
07/17/2026	01/20/2026	07/17/2026	178	-107,631,040.79	0.00	01/16/2026	3.15871	-1,680,987.86	0.882974	3.543506	-1,484,268.49
01/19/2027	07/17/2026	01/19/2027	186	-107,631,040.79	0.00	07/15/2026	3.15981	-1,757,151.32	0.870733	3.441649	-1,530,009.03
07/19/2027	01/19/2027	07/19/2027	181	-107,631,040.79	0.00	01/15/2027	3.20243	-1,732,980.92	0.858737	3.370936	-1,488,175.07
01/18/2028	07/19/2027	01/18/2028	183	-107,631,040.79	-107,631,040.79	07/15/2027	3.20272	-109,383,325.71	0.846774	3.313711	-92,622,992.11

Figure IV.27 Currency Swap. Cashflow Table. Source: Bloomberg®

Since plain vanilla currency swaps (fixed versus floating) have not always proved suitable to meet the needs of investors, swap dealers have created numerous variants of the derivative.

The most common modifications are the following swaps:

Fixed-for-fixed rate currency swaps are swaps in which both parties pay a fixed interest rate. It should be noted that such a product can be created using a single swap contract or it can be synthesized by means of a combination of two fixed-for-floating swaps featuring the same floating leg. Such swaps are also called circus swaps.

Floating-for-floating currency swaps (also called basis swaps in the case of LIBOR-LIBOR swaps) are swaps in which both counterparties pay a variable interest rate. Again, such a product can be created using a single swap contract or it can be synthesized by means of a combination of two fixed-for-floating swaps featuring the same fixed leg.

Amortising currency swaps are swaps in which the notional amounts are gradually swapped over the life of the derivative.

Figure IV.28 Fixed-for-fixed currency swap. Source: Bloomberg®

Figure IV.29 Floating-for-floating currency swap. Source: Bloomberg®

As we have seen, generally speaking, swaps are instruments that allow to create fixed-rate and floating-rate liabilities. For example, if we consider a company where the treasurer expects interest rates to rise: he may use a swap to convert existing floating rate debt into a synthetic fixed rate liability. Under the swap, the firm will pay a fixed rate and receive a floating rate, locking in the cost of its short-term debt.

Figure IV.30 Synthetic fixed rate liability

Similarly, if a treasurer expects a decrease in short-term rates, he will wish to convert an existing fixed-rate debt into a synthetic floating-rate liability and a swap would be the right instrument to implement this type of debt transformation.
The firm will pay a floating rate and receive a fixed rate, which may allow the firm to lower the cost of interest if the short-term interest rate falls.

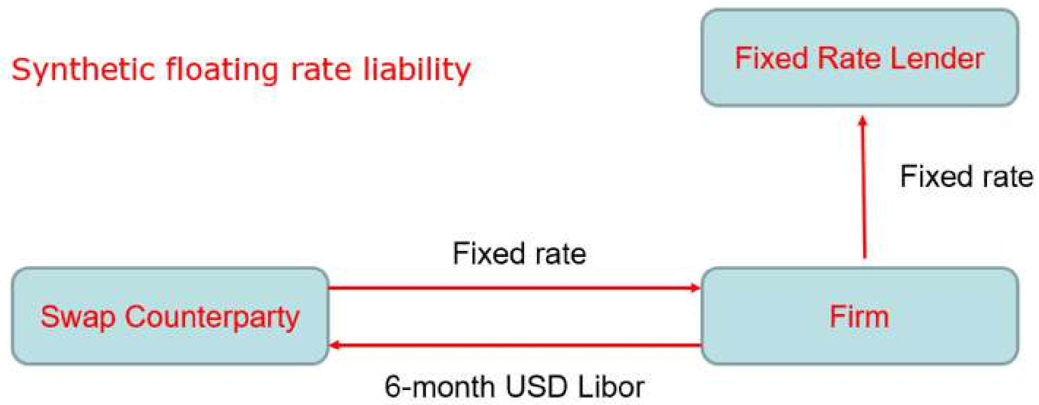


Figure IV.31 Synthetic floating rate liability

Currency swaps can also be used to convert the currency denomination of certain liabilities to a more appropriate one or to hedge against a currency exposure.
As an example, an obligor of an existing liability denominated in a currency that is expected to strengthen can hedge its exposure by entering into a currency swap.

Let us analyze an example contemplating the above-described situation. On October 1, a Dutch company issued a 5-year fixed-rate bond for a total amount of USD 10 million.
The EUR/USD exchange rate was 1.17 at the time of issue, and after a year, the EUR/USD exchange rate has risen to 1.25. The company is worried about a sharp increase in the next 4 years and decides to enter a currency swap in order to hedge against the risk.
To set up the hedging, the Dutch company stipulates a currency swap with the bank through which it will pay interest in EUR on an amount of EUR 8 million and it will receive the interest in USD paid on a notional amount of USD 10 million.
Clearly, the firm should choose to enter a swap in which it receives USD interest from the bank as close as possible to the USD interest paid on its loan.
In short, using a swap structured in the way we described, the company is protected against a possible appreciation of the USD, as shown in the figure below.

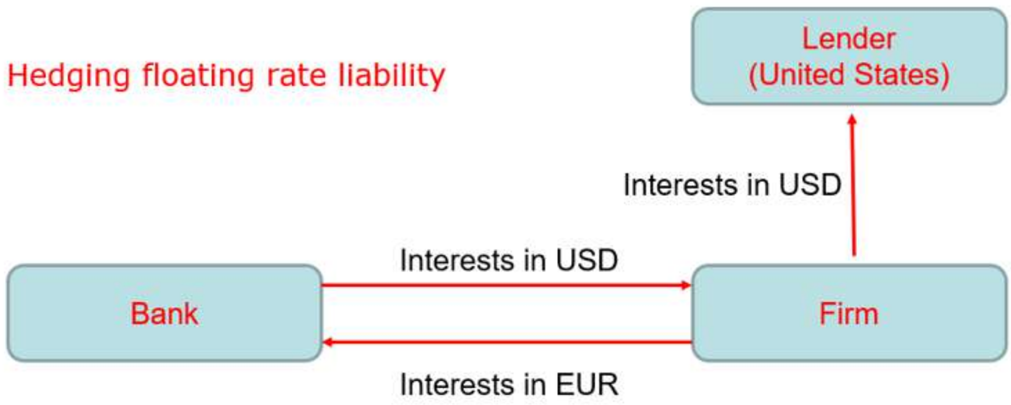


Figure IV.32 Hedging floating rate liability

Besides what we have seen so far, swaps can also be used to profitably manage a portfolio. For example, in an environment where the yield curve slopes upward, a portfolio manager might enter a swap transaction in order to convert fixed rate investments to floating rate investments.

Another useful feature of swaps is that they can be used to lock in gains or losses on fixed rate investments. If we consider, as an example, a fixed-rate portfolio manager, after a significant decrease in interest rates, he can lock in his capital gain by "swapping" his investments in floating rates.
Finally, swaps can also be used to enhance portfolio performance on both fixed and floating rate investments. We will illustrate this last characteristic through the following example. An investor has a portfolio of floating-rate securities (USD 6M LIBOR).
In order to increase the overall return, the investor could simultaneously enter two swaps: in the first, called original swap, he will exchange his floating rate (USD 6M LIBOR) against a fixed rate (e.g. 3%); and in the second, called reverse swap, he will swap a lower fixed rate (e.g. 2.5%) against the same floating rate (USD 6M LIBOR).
As a result of the two transactions, the new rate of return on the overall portfolio is USD 6M LIBOR + 50 basis points.

Another category is constituted by **Equity index swaps,** in which at least one leg is linked to the total return of an equity index.
Let us explain this type of contract through an example, considering a well-diversified portfolio, i.e., highly correlated to a benchmark stock index, such as the S&P 500 index.
The portfolio manager who wishes to hedge against the risk of a stock market crash for a certain period of time could enter into an equity swap, that strategy is fined as downside market risk hedging.
More specifically, the manager may contact a swap dealer in order to exchange the total return of the index, including capital appreciation/depreciation and dividends, against a floating interest rate.

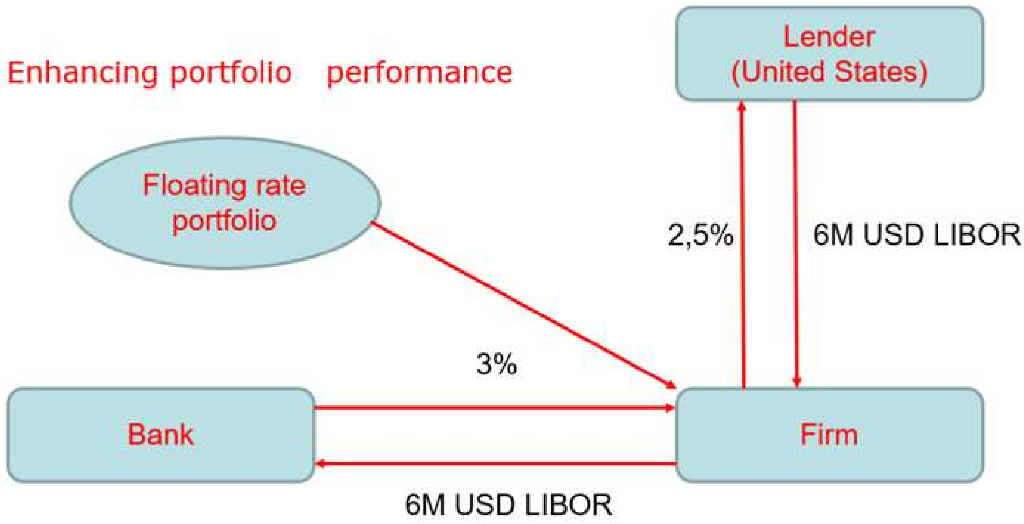


Figure IV.33 Enhancing portfolio performance

Figure IV.34 Equity index swap

Commodity swaps is another type of swap, introduced in 1986 by Chase Manhattan Bank for hedging purposes.
As the name suggests, the simplest case is a contract with which a commodity producer wants to hedge against changes in the price of the commodity itself.
The producer can thus enter into a commodity swap, in which he pays the swap dealer a variable price per unit of the commodity (for example, based on the average of the spot prices recorded in the market) and he receives a fixed price.
Since the hedger is also a producer, he can resell the commodity unit on the spot market and receive the corresponding spot price per unit, which he will use to pay the swap dealer. By entering a swap with such characteristics, the producer is assured of receiving a fixed price for his goods.

In this case, the notional amount is not exchanged and all transactions are cash-settled. Clearly, a commodity swap can be used not only for hedging purposes, but also for speculation without the transactions going to the spot market.

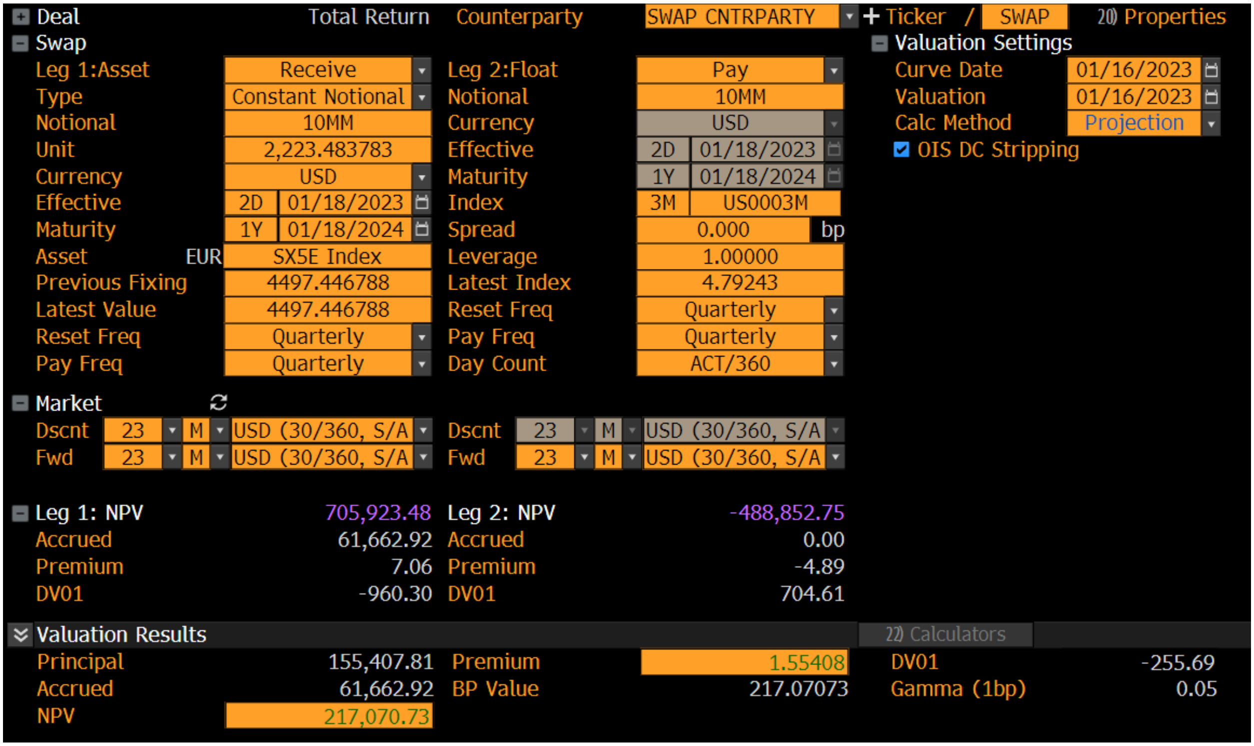


Figure IV.35 Total Return Swap. Source: Bloomberg®

Figure IV.36 Commodity swap

Figure IV.37 Commodity swap. Source: Bloomberg®

A **volatility swap** is a forward contract on realized volatility. Its pay-off at maturity is equal to:

$$Payoff = Notional\ Amount \cdot (Realized\ Volatility\ - Volatility\ Strike\ Price)\ (Eq.\ IV.1)$$

As a standard, all volatilities are annualized and quoted in percentage points. Similarly, a **variance swap** is a forward contract on the realized variance, which is the square root of the realized volatility. Therefore, the related pay-off is:

$$Payoff = Notional\ Amount \cdot (Realized\ Variance\ - Variance\ Strike\ Price)\ (Eq.\ IV.2)$$

Figure IV.38 Variance swap. Current Market. Source: Bloomberg®

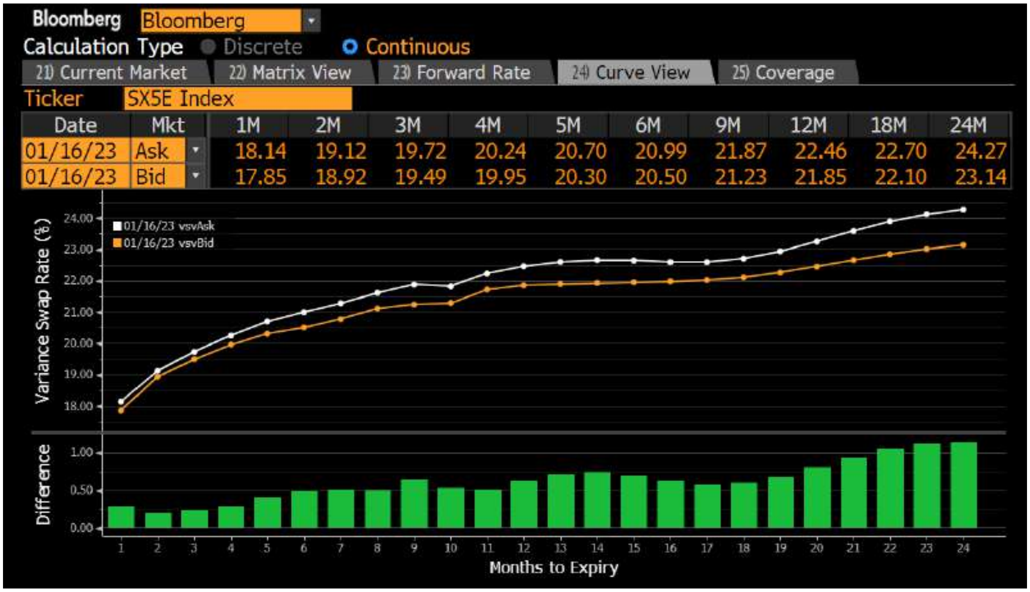


Figure IV.39 Variance swap. Curve view. Source: Bloomberg®

Lastly, a **swaption** is an option that gives the right to enter a swap. In a swaption written on an interest rate swap, a **swaption payer** gives the option holder the right to enter a swap that pays a fixed interest rate. A **swaption receiver** gives the holder the right to enter a swap that receives a fixed rate. Swaptions are normally European options created ad-hoc to meet the specific needs of a trader and therefore they are mainly traded on OTC markets.

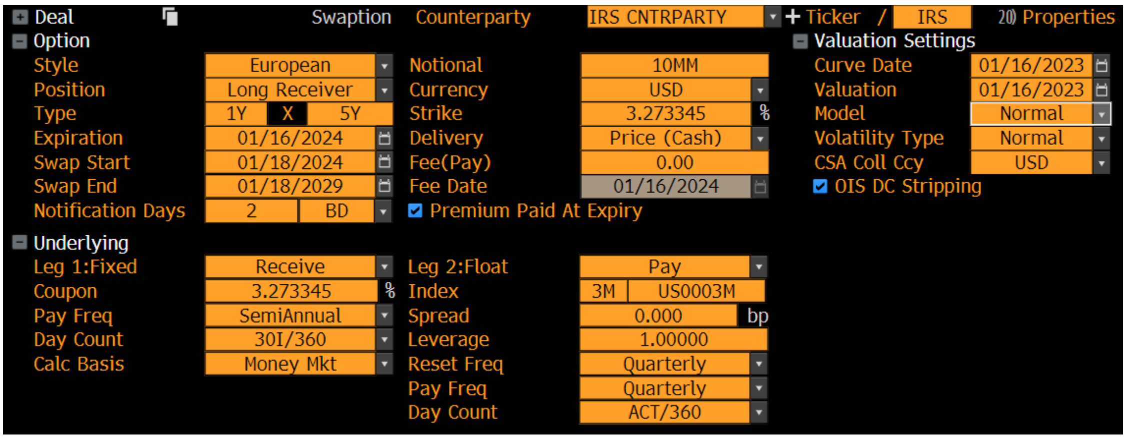


Figure IV.40 Swaption. Source: Bloomberg®

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IV.2 QUANTITATIVE ANALYSIS

The analysis starts from the observation that a plain vanilla swap can be valued either as a long position in one bond combined with a short position in another bond or as a portfolio of forward contracts. Therefore, the pricing techniques differ depending on if we consider the former or the latter. Let us examine both techniques.

Swap pricing as a bond portfolio

From a cash flow perspective, in a generic interest rate swap, the fixed-rate payer is in the situation where he has sold a fixed-coupon bond and bought a floating-rate note. Similarly, the fixed-rate receiver is in the position where he has bought a fixed-rate bond and sold a floating-rate bond. Thus, the simplest approach to valuing a generic interest rate swap is to consider it as a portfolio of two fixed-income securities. Swap pricing is therefore equivalent to valuing the difference between the theoretical value of the two securities, which can be done using traditional bond valuation techniques. However, two factors must be taken into account that complicate this pricing methodology. First of all, an IRS generally requires neither an initial nor a matured investment since only the interests are exchanged.
This fact is not actually a problem if both counterparties simultaneously buy and sell the same amount of fixed and floating rate securities since the cash flows equalize the same amount of money both at inception and at maturity. Secondly, the expected value of the cash flows of the floating leg represented by the floating-rate bond are uncertain, since they depend on the future level of interest rates.

In mathematical terms, we assume that, in accordance with the swap statement, a financial institution receives a fixed rate payment and makes a floating rate payment at the same time. We denote by V the value of the swap, B_1 the value of the fixed-rate bond underlying the swap, B_2 the value of the floating-rate bond underlying the swap, Q the notional on which interest is calculated (swap principal).

$R_{0,t}$ is the effective discount rate to be applied on date t_i and K is the fixed payment which corresponds to the interest to be paid at time t_i. At each time, the value of the swap can be represented by: $V = B_1 - B_2$. Since B_1 is the discounted value of the cash flows of a fixed rate bond, we have:

$$B_1 = \sum_{i=1}^{n} \frac{K}{\left(1+R_{0,t_i}\right)^{t_i}} + \frac{Q}{\left(1+R_{0,t_n}\right)^{t_n}} \quad \text{(Eq. IV.3)}$$

For bond B_2, the cash flows are uncertain, since they are dependent on the future level of interest rates. But in practical terms, bond B_2 will have a value close to par, given its re-pricing characteristics. Also, when entering a swap and immediately after a coupon reset date, the value of B_2 is equal to the notional Q. Expressing this concept in the usual mathematical notation, we obtain:

$$B_2 = \frac{K^*}{\left(1+R_{0,t_1}\right)^{t_1}} + \frac{Q}{\left(1+R_{0,t_1}\right)^{t_1}} \quad \text{(Eq. IV.4)}$$

Where K^* is the initially known variable amount used for payment on date t_1. The previous mathematical expressions can be expressed using the continuous composition of rates:

$$B_1 = \left[\sum_{i=1}^{n} K \cdot \exp\left(-r_{0,t_i} \cdot t_i\right)\right] + Q \cdot \exp\left(-r_{0,t_n} \cdot t_n\right) \text{ (Eq. IV.5)}$$

$$B_2 = K^{*}\exp\left(-r_{0,t_1} \cdot t_1\right) + Q \cdot \exp\left(-r_{0,t_1} \cdot t_1\right) \text{ (Eq. IV.6)}$$

At the inception date of the swap, its value should be zero, as the transaction is commonly conducted at current market rates such that the net discounted value of the payments is zero, or rather, the fixed rate on a typical IRS is set such that the market value of the fixed leg equals the floating leg. In fact, if the swap is not concluded at zero market value, one counterparty has to pay the difference to the other. At maturity, the value of the swap is zero, while during its life the NPV of the derivative varies according to the value of B_1 and B_2.

We now present as an example a company that has entered an IRS, in which it pays 6-month LIBOR and receives the fixed rate of 4% (compounded semi-annually) on a notional of USD 100 million. The swap has a remaining life of 1.25 years. The continuously compounded discount rates over 3, 9 and 15 months are 5%, 5.5% and 6%, respectively.
The last LIBOR set at the last payment is 5.10% (composed on a semi-annual basis).

The value of the derivative is: $V = B_1 - B_2$. $K = \text{USD 2 million}$. $\text{K}^{*} = \text{USD 2.55 million}$. We can thus calculate:

$$B_1 = \left[\sum_{i=1}^{n} K \cdot \exp\left(-r_{0,t_i} \cdot t_i\right)\right] + Q \cdot \exp\left(-r_{0,t_n} \cdot t_n\right) = 2 \cdot \exp(-0.25 \cdot 0.05) + 2 \cdot \exp(-0.75 \cdot 0.055) + 2 \cdot \exp(-1.25 \cdot 0.06) + 100 \cdot \exp(-1.25 \cdot 0.06) = 98.52$$

$$B_2 = K^{*}\exp\left(-r_{0,t_1} \cdot t_1\right) + Q \cdot \exp\left(-r_{0,t_1} \cdot t_1\right) = 2.55 \cdot \exp(-0.25 \cdot 0.05) + 100 \cdot \exp(-0.25 \cdot 0.05) = 101.28$$

Thus, the value of a swap to the counterparty paying the floating rate and receiving the fixed rate is:
$V = 98.52 - 101.28 = -USD\ 2.75 \text{ million}$.

For the other counterparty who pays fixed and receives variable, the value is + USD 2.75 million.

Swap pricing as a forward contract using the zero-coupon yield curve

On the other hand, in the absence of default risk, an IRS can also be decomposed into a portfolio of forward contracts. Using the same notation, let us consider the case of a company that enters a swap with a notional Q, paying a floating rate and receiving a fixed rate. Payments are made every six months. The payment to the company at each payment date (every 6 months) is equal to:

$$\frac{Q}{2} \cdot (Floating\ Rate\ - Fixed\ Rate) \text{ (Eq. IV.7)}$$

Where the floating rate is the reference rate recorded on the payment date. Such formula is the pay-off of a standard forward contract written on the reference floating rate, with the exception that it is always settled 6 months in arrears. Let us now assume that F_i is the forward interest rate for the six-month period prior to payment date i ($i > 1$). The forward contract value for the i-th payment ($i > 1$) for the counterparty receiving the fixed rate payment (which amounts to K) and paying the floating rate (i.e. $0.5 \cdot F_i \cdot Q$) is as follows:

$$\left(K-\frac{1}{2}\cdot F_i\cdot Q\right)\cdot\frac{1}{\left(1+R_{0,t_i}\right)^{t_i}}\ (Eq.\ IV.8)$$

On the first payment date, the party makes a payment of K^* and receives a payment of K. The discounted value of the first payment is therefore:

$$(K-K^*)\cdot\frac{1}{\left(1+R_{0,t_1}\right)^{t_1}}\ (Eq.\ IV.9)$$

The total value of the swap for the counterparty receiving fixed and paying floating is:

$$V=(K-K^*)\cdot\frac{1}{\left(1+R_{0,t_1}\right)^{t_1}}+\sum_{i=2}^{n}\left(K-\frac{1}{2}\cdot F_i\cdot Q\right)\cdot\frac{1}{\left(1+R_{0,t_i}\right)^{t_i}}\quad (Eq.\ IV.10)$$

Similarly, the value of the swap for the counterparty receiving floating and paying fixed is:

$$V'=(K^*-K)\cdot\frac{1}{\left(1+R_{0,t_1}\right)^{t_1}}+\sum_{i=2}^{n}\left(\frac{1}{2}\cdot F_i\cdot Q-K\right)\cdot\frac{1}{\left(1+R_{0,t_i}\right)^{t_i}}\quad (Eq.\ IV.11)$$

The previous mathematical expressions can be rewritten using the continuous composition of rates:

$$V=(K-K^*)\cdot\exp\left(-r_{0,1}\cdot t_1\right)+\sum_{i=2}^{n}\left(K-\frac{1}{2}\cdot F_i\cdot Q\right)\cdot\exp\left(-r_{0,i}\cdot t_i\right)\ (Eq.\ IV.12)$$

$$V'=(K^*-K)\cdot\exp\left(-r_{0,1}\cdot t_1\right)+\sum_{i=2}^{n}\left(\frac{1}{2}\cdot F_i\cdot Q-K\right)\cdot\exp\left(-r_{0,i}\cdot t_i\right)\ (Eq.\ IV.13)$$

Let us now present an example, considering the same swap valuation previously discussed. This time it will be valued not as a portfolio of bonds, but as a portfolio of forward contracts. The first payment is USD 2 million (fixed leg) versus USD 2.55 million (floating leg based on the last LIBOR reset that occurred on the last payment date). For the next two payments, it is necessary to estimate the forward rate which is implicit in the zero rates: ($r_{0,0.25}=5\%$, $r_{0,0.75}=5.5\%$, $r_{0,1.25}=6\%$). Given that the rates are continuously compounded, the spot rate is an arithmetic average of forward rates:

$$f_{0.25,0.75}=\frac{r_{0,0.75}\cdot(0.75-0)-r_{0,0.25}\cdot(0.25-0)}{0.75-0.25}=\frac{5.5\%\cdot 0.75-5\%\cdot 0.25}{0.5}=5.75\%\ \text{p. a.}$$

$$f_{0.75,1.25}=\frac{r_{0,1.25}\cdot(1.25-0)-r_{0,0.75}\cdot(0.75-0)}{1.25-0.75}=\frac{6\%\cdot 1.25-5.5\%\cdot 0.75}{0.5}=6.75\%\ \text{p. a.}$$

We notice that all forward rates calculated are annualized and continuously compounded.

The second payment of the floating leg of the swap will therefore be equal to: $100\cdot[\exp(5.75\%\cdot 0.5)-1]=\text{USD } 2.917\text{ million}$.

The third payment of the floating leg of the swap is: $100\cdot[\exp(6.75\%\cdot 0.5)-1]=\text{USD } 3.433\text{ million}$.

The swap value for the counterparty paying the floating rate and receiving the fixed rate will be - USD 2.75 million:

$$V = (2.0 - 2.55) \cdot \exp(-0.05 \cdot 0.25) + (2.0 - 2.917) \cdot \exp(-0.055 \cdot 0.75) + (2.0 - 3.433) \cdot$$
$$\cdot \exp(-0.06 \cdot 1.25) = -\text{USD } 2.75 \text{ million.}$$

Using the same methodology, the fixed rate of a fix-to-floater swap can be set such that the initial value of the swap is zero. The pricing of a currency swap is very similar to that of a vanilla IRS. In the absence of default risk, a currency swap can be decomposed into a short/long position on two bonds or a series of Forex Forward contracts. We now provide a market example of pricing.

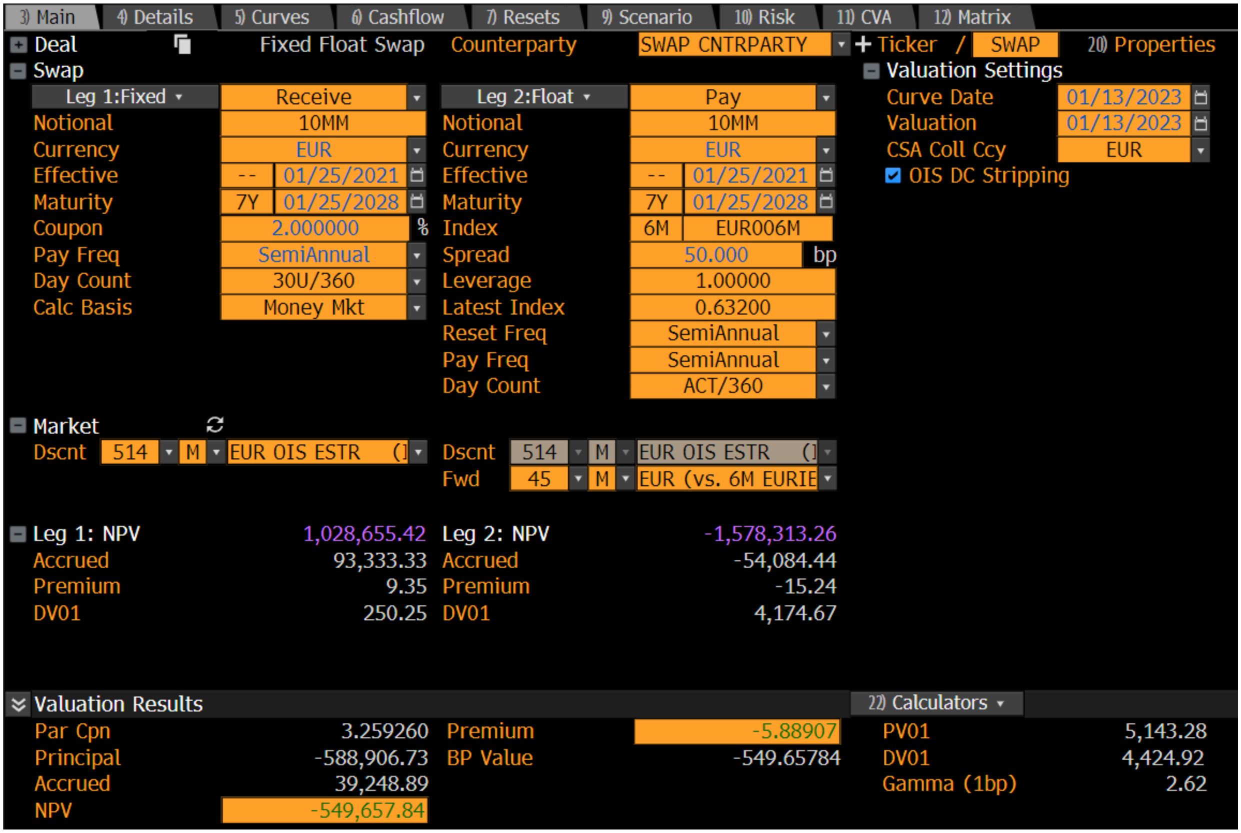


Figure IV.41 IRS pricing. Source: Bloomberg®

The Zero Rates we use for pricing come from the ESTER Curve, because of the underlying assumption that the swap is collateralized.

Term	Market Rates	Zero Rates	Discount Factors
1 WK	1.90225	1.92831	0.99963
2 WK	1.9024	1.92811	0.999261
1 MO	2.04425	2.07082	0.998243
2 MO	2.21025	2.2369	0.996391
3 MO	2.39775	2.42379	0.994041
4 MO	2.5226	2.54694	0.991661
5 MO	2.652	2.67379	0.988855
6 MO	2.7543	2.7734	0.986341
7 MO	2.8428	2.85842	0.983535
8 MO	2.916	2.92766	0.980619
9 MO	2.9675	2.97536	0.977992
10 MO	3.009	3.01268	0.97522
11 MO	3.04175	3.04116	0.972474
1 YR	3.066	3.06125	0.969851
18 MO	3.051	3.05566	0.95524
2 YR	2.947	2.94235	0.942775
3 YR	2.727	2.72014	0.921431
4 YR	2.6	2.5918	0.901393
5 YR	2.525	2.5161	0.881726
6 YR	2.4835	2.47452	0.861908
7 YR	2.455	2.44603	0.842521
8 YR	2.45	2.44223	0.822413
9 YR	2.458	2.4522	0.801743
10 YR	2.477	2.47414	0.780659
11 YR	2.48775	2.48675	0.760524
12 YR	2.506	2.50787	0.739966
15 YR	2.52	2.52429	0.684553

Table IV.4 Interest rates term structure. Tenor 1 day. Source: Bloomberg®

Term	Market Rates	Zero Rates	Discount Factors
6 MO	2.876	2.89506	0.985746
EUFR0AG	3.093	2.95582	0.982978
EUFR0BH	3.277	3.03846	0.979893
EUFR0CI	3.391	3.09135	0.977144
EUFR0DJ	3.457	3.13757	0.974206
EUFR0EK	3.526	3.1892	0.971154
EUFR0F1	3.54	3.22879	0.968228
EUFR0I1C	3.438	3.23754	0.96036
EUFR011F	3.263	3.24627	0.952515
2 YR	3.19425	3.14043	0.939042
3 YR	2.9855	2.93442	0.91551
4 YR	2.858	2.80808	0.893618
5 YR	2.7825	2.72895	0.872196
6 YR	2.745	2.69551	0.850545
7 YR	2.72075	2.67222	0.829276
8 YR	2.713	2.66572	0.80783
9 YR	2.71375	2.66788	0.786312
10 YR	2.7185	2.67331	0.765251
11 YR	2.72825	2.68499	0.744108
12 YR	2.73025	2.68778	0.724152
15 YR	2.70475	2.65936	0.670811
20 YR	2.5445	2.47233	0.609607
25 YR	2.3545	2.24652	0.570069
30 YR	2.183	2.04268	0.541588

Table IV.5 Interest rates term structure. Tenor 6 months. Source: Bloomberg®

The Receiving Leg NPV is equal to the sum of all discounted cash flows and in this case, it is EUR 1,028,655.42.

Zero Rate(4Y) =2.80808%

Zero Rate(5Y) =2.72895%

Date	T	Interp.Zero
21/01/2027	4.02	2.80611%
22/07/2027	4.52	2.76667%

$$F_{T_1.T_2} = \left(\frac{\left(1+R_{0.T_2}\right)^{T_2}}{\left(1+R_{0.T_1}\right)^{T_1}}\right)^{\frac{1}{T_2-T_1}} - 1 = \mathbf{2.4176\%}$$

Pay Date	Accrual Start	Accrual End	Days	Notional	Equiv. Coupon	Payment	Discount	Zero Rate	PV
07/26/2021	01/25/2021	07/26/2021	181	10,000,000	2	100555.56			
01/25/2022	07/26/2021	01/25/2022	179	10,000,000	2	99444.44			
07/25/2022	01/25/2022	07/25/2022	180	10,000,000	2	100000			
01/25/2023	07/25/2022	01/25/2023	180	10,000,000	2	100000	0.999366	1.92828	99936.62
07/25/2023	01/25/2023	07/25/2023	180	10,000,000	2	100000	0.985408	2.779983	98540.79
01/25/2024	07/25/2023	01/25/2024	180	10,000,000	2	100000	0.969	3.048869	96899.96
07/25/2024	01/25/2024	07/25/2024	180	10,000,000	2	100000	0.954482	3.041903	95448.16
01/27/2025	07/25/2024	01/27/2025	182	10,000,000	2	101111.11	0.941963	2.929267	95242.94
07/25/2025	01/27/2025	07/25/2025	178	10,000,000	2	98888.89	0.931513	2.802492	92116.27
01/26/2026	07/25/2025	01/26/2026	181	10,000,000	2	100555.56	0.920845	2.714092	92596.06
07/27/2026	01/26/2026	07/27/2026	181	10,000,000	2	100555.56	0.910778	2.642249	91583.8
01/25/2027	07/27/2026	01/25/2027	178	10,000,000	2	98888.89	0.90082	2.58819	89081.12
07/26/2027	01/25/2027	07/26/2027	181	10,000,000	2	100555.56	0.890939	2.546825	89588.87
01/25/2028	07/26/2027	01/25/2028	179	10,000,000	2	99444.44	0.881103	2.513703	87620.83

Table IV.6 Receiving Leg

The NPV of the Paying Leg is equal to the sum of all discounted cash flows and it is EUR 1,578,313.27.

Swap NPV = NPV Receive Leg – NPV Pay Leg = 1,028,655.42 - 1,578,313.27 = - EUR 549,657.85.

We now compute the Swap Accrued as the difference between the accrued interests of the two legs.

Leg 1 - Receive

Last Payment Date: 25th July 2022; Valuation Date: 13th January 2023; Day Basis: 30/360
Accrued Interest = Notional * (C/2) * (30*m+d)/180 =10,000,000 * (0.02/2) * (30 * 5+18)/180 = 93,333.33

Leg 2 - Pay

Last Payment Date: 25th July 2022. Last coupon = Reset Rate + Additive Margin = 0.632%+0.5%=1.132%
Valuation Date: 13th January 2023; Day Basis: ACT/360; Exact days 13th January 2023 - 25th July 2022 = 172 days.
Accrued Interest = Notional * (C/2) * (Exact days)/180 =10,000,000 * (0.01132/2) * (172)/180 = 54,084.44

Swap Accrued = Accrued Receive - Accrued Paying = 93,333.33 - 54,084.44 = EUR 39,248.89.

Swap Principal = Swap NPV - Swap Accrued = -549,657.85 – 39,248.89 = - EUR 588,906.73.

Pay Date	Accrual Start	Accrual End	Days	Notional	Reset Date	Reset	Coupon	Payment
07/26/2021	01/25/2021	07/26/2021	182	-10MM	01/21/2021	-0.527	-0.027	1365
01/25/2022	07/26/2021	01/25/2022	183	-10MM	07/22/2021	-0.517	-0.017	864.17
07/25/2022	01/25/2022	07/25/2022	181	-10MM	01/21/2022	-0.522	-0.022	1106.11
01/25/2023	07/25/2022	01/25/2023	184	-10MM	07/21/2022	0.632	1.132	-57857.78
07/25/2023	01/25/2023	07/25/2023	181	-10MM	01/23/2023	2.9342	3.4342	-172664.05
01/25/2024	07/25/2023	01/25/2024	184	-10MM	07/21/2023	3.53834	4.0383	-206404.26
07/25/2024	01/25/2024	07/25/2024	182	-10MM	01/23/2024	3.24331	3.7433	-189245.15
01/27/2025	07/25/2024	01/27/2025	186	-10MM	07/23/2024	2.791	3.291	-170034.88
07/25/2025	01/27/2025	07/25/2025	179	-10MM	01/23/2025	2.50499	3.005	-149414.56
01/26/2026	07/25/2025	01/26/2026	185	-10MM	07/23/2025	2.50183	3.0018	-154260.95
07/27/2026	01/26/2026	07/27/2026	182	-10MM	01/22/2026	2.40825	2.9083	-147028.23
01/25/2027	07/27/2026	01/25/2027	182	-10MM	07/23/2026	2.40848	2.9085	-147040.04
07/26/2027	01/25/2027	07/26/2027	182	-10MM	01/21/2027	2.41432	2.9143	-147335.27
01/25/2028	07/26/2027	01/25/2028	183	-10MM	07/22/2027	**2.41763**	2.9176	-148312.82

Pay Date	Days	Notional	Payment	Discount	Zero Rate	PV
07/25/2022	181	-10000000	1106.11			
01/25/2023	184	-10000000	-57857.78	0.999366	1.92828	-57821.11
07/25/2023	181	-10000000	-172664.05	0.985408	2.779983	-170144.52
01/25/2024	184	-10000000	-206404.26	0.969	3.048869	-200005.65
07/25/2024	182	-10000000	-189245.15	0.954482	3.041903	-180631.01
01/27/2025	186	-10000000	-170034.88	0.941963	2.929267	-160166.59
07/25/2025	179	-10000000	-149414.56	0.931513	2.802492	-139181.58
01/26/2026	185	-10000000	-154260.95	0.920845	2.714092	-142050.39
07/27/2026	182	-10000000	-147028.23	0.910778	2.642249	-133910.1
01/25/2027	182	-10000000	-147040.04	0.90082	2.58819	-132456.67
07/26/2027	182	-10000000	-147335.27	0.890939	2.546825	-131266.74
01/25/2028	183	-10000000	-148312.82	0.881103	2.513703	-130678.91

Table IV.7 Paying Leg

Valuation of the swap as a forwards-like portfolio

Pay Date	Payments (Rcv)	Payments (Pay)	Net Payments	Discount	Zero Rate	PV
07/26/2021	100555.56	1365	101920.56			
01/25/2022	99444.44	864.17	100308.61			
07/25/2022	100000	1106.11	101106.11			
01/25/2023	100000	-57857.78	42142.22	0.999366	1.92828	42115.51
07/25/2023	100000	-172664.05	-72664.05	0.985408	2.779983	-71603.73
01/25/2024	100000	-206404.26	-106404.26	0.969	3.048869	-103105.69
07/25/2024	100000	-189245.15	-89245.15	0.954482	3.041903	-85182.85
01/27/2025	101111.11	-170034.88	-68923.77	0.941963	2.929267	-64923.65
07/25/2025	98888.89	-149414.56	-50525.67	0.931513	2.802492	-47065.31
01/26/2026	100555.56	-154260.95	-53705.39	0.920845	2.714092	-49454.33
07/27/2026	100555.56	-147028.23	-46472.68	0.910778	2.642249	-42326.3
01/25/2027	98888.89	-147040.04	-48151.15	0.90082	2.58819	-43375.54
07/26/2027	100555.56	-147335.27	-46779.71	0.890939	2.546825	-41677.87
01/25/2028	99444.44	-148312.82	-48868.38	0.881103	2.513703	-43058.09

Table IV.8 Net Cash Flows

Pricing a Currency swap as a bond portfolio

We now assume that in a swap, a financial institution receives payments in foreign currency and makes payments in domestic currency. Let V be the swap value, B_F the value of the foreign currency bond underlying the swap, B_D the value of the domestic currency bond underlying the swap, and let S be the exchange rate. The value of the swap can be expressed as: $V = S \cdot B_F - B_D$, where B_F and B_D can be fixed or floating rate bonds.
Therefore, the fair value of a currency swap can be determined from the term structure of domestic currency interest rates, the term structure of foreign currency interest rates and the spot exchange rate.

Let us analyze a partial example. A financial institution has entered a swap in which it receives a fixed rate of 5% in JPY and pays a fixed rate of 8% in USD. All payments are made annually and the reference principals in the two currencies are USD 10 million and JPY 1,200 million. The time to maturity of the swap is 3 years. We assume a flat rate term structure with a continuously compounded Japanese interest rate of 4% p.a. and with a composed continuously US interest rate of 9% p.a.
The NPV of the swap is calculated assuming a USD/JPY exchange rate of 110. The domestic currency bond

is worth:

$B_D = 0.8 \cdot \exp(-0.09) + 0.8 \cdot \exp(-0.09 \cdot 2) + 10.8 \cdot \exp(-0.09 \cdot 3) = \text{USD } 9.64 \text{ million.}$

The foreign bond is worth:

$B_F = 60 \exp(-0.04) + 60 \exp(-0.04 \cdot 2) + 1260 \exp(-0.04 \cdot 3) = \text{JPY } 1230.55 \text{ million.}$

The swap fair value is $\frac{1230.55}{110} - 9.64 = \text{USD } 1.55 \text{ million.}$

For the counterparty paying JPY and receiving USD, the swap value is - USD 1.55 million.

Pricing a Currency swap as a forwards portfolio

The valuation of a currency swap can also be achieved by breaking down the swap into a series of Forex Exchange forward contracts. Recalling that a forward contract can be valued using the UIP (Uncovered Interest rate parity) formula:

$$F_{t.T} = S_t \cdot \frac{(1+R_D)^{T-t}}{(1+R_F)^{T-t}} \text{ (Eq. IV.14)}$$

Or employing the mathematical relationship in continuous time:

$$F_{t.T} = S_t \cdot \exp[(r_D - r_F) \cdot (T - t)] \text{ (Eq. IV.15)}$$

Where the subscript D indicates the reference to the domestic rate and the subscript F refers to the foreign rate. So, if we have the term structure of domestic currency interest rates, the term structure of foreign currency interest rates and the spot exchange rate, we can estimate the value of a currency swap.

Taking as a reference the case of the valuation of the previous currency swap, its price is calculated considering the derivative as a portfolio of FX forward contracts. The current exchange rate is 110 USD/JPY or 0.009091 JPY/USD. The USD-JPY interest rate differential is 5% p.a. We can estimate 1, 2 and 3-year future exchange rates as:

$F_{0,1} = 0.009091 \cdot \exp(0.05 \cdot 1) = 0.0096$

$F_{0,2} = 0.009091 \cdot \exp(0.05 \cdot 2) = 0.001$

$F_{0,3} = 0.009091 \cdot \exp(0.05 \cdot 3) = 0.0106$

Exchanging interest payments involves receiving JPY 60 million and paying USD 0.8 million. The risk-free rate in USD is 9%. The value of forward contracts expressed in million USD is:

$(60 \cdot 0.0096 - 0.8) \cdot \exp(-0.09 \cdot 1) = -0.21$

$(60 \cdot 0.001 - 0.8) \cdot \exp(-0.09 \cdot 2) = -0.16$

$(60 \cdot 0.0106 - 0.8) \cdot \exp(-0.09 \cdot 3) = -0.13$

The final exchange involves receiving JPY 1,200 million and paying USD 10 million.
The value of the corresponding forward contract, expressed in million USD is: $(1200 \cdot 0.0106 - 10) \cdot \exp(-0.09 \cdot 3) = 2.04$.

The fair value of the swap is: $2.04 - 0.13 - 0.16 - 0.21 = \text{USD } 1.54 \text{ million.}$

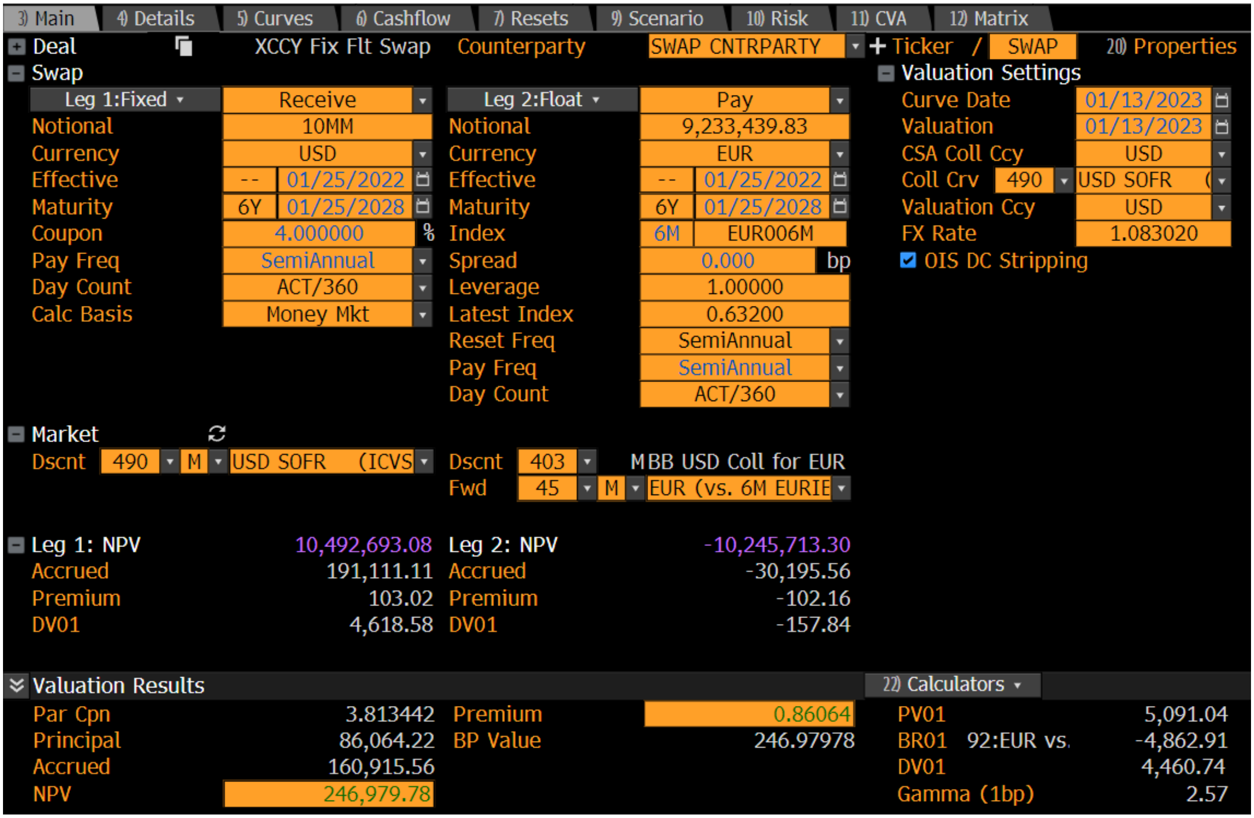


Figure IV.42 Currency Swap IRS pricing. Source: Bloomberg®

In this case, the notional exchanges occur on the effective date and on the maturity date (10 million /1.08302 = USD 9233439.83).

Other cash flows are calculated as usual through the formula: Notional * Reset Rate/100 * Days/360.

Discount factors are estimated using the continuous compounded formula starting from the zero rate: (exp (-Zero Rate * (Pay Date – Valuation Date)/365)).

PV = Payment * Discount.

Pay Date	Accrual Start	Accrual End	Days	Notional	Principal	Reset Date	Reset Rate
01/25/2022					9233439.83		
07/25/2022	01/25/2022	07/25/2022	181	-9233439.83	0	01/21/2022	-0.522
01/25/2023	07/25/2022	01/25/2023	184	-9233439.83	0	07/21/2022	0.632
07/25/2023	01/25/2023	07/25/2023	181	-9233439.83	0	01/23/2023	2.9342
01/25/2024	07/25/2023	01/25/2024	184	-9233439.83	0	07/21/2023	3.53834
07/25/2024	01/25/2024	07/25/2024	182	-9233439.83	0	01/23/2024	3.24331
01/27/2025	07/25/2024	01/27/2025	186	-9233439.83	0	07/23/2024	2.791
07/25/2025	01/27/2025	07/25/2025	179	-9233439.83	0	01/23/2025	2.50499
01/26/2026	07/25/2025	01/26/2026	185	-9233439.83	0	07/23/2025	2.50183
07/27/2026	01/26/2026	07/27/2026	182	-9233439.83	0	01/22/2026	2.40825
01/25/2027	07/27/2026	01/25/2027	182	-9233439.83	0	07/23/2026	2.40848
07/26/2027	01/25/2027	07/26/2027	182	-9233439.83	0	01/21/2027	2.41432
01/25/2028	07/26/2027	01/25/2028	183	-9233439.83	-9233439.83	07/22/2027	2.41763

Pay Date	Accrual Start	Accrual End	Payment	Discount	Zero Rate	PV
01/25/2022			9233439.83			
07/25/2022	01/25/2022	07/25/2022	24233.16			
01/25/2023	07/25/2022	01/25/2023	-29826.06	0.999252	2.276972	-29803.74
07/25/2023	01/25/2023	07/25/2023	-136216.47	0.986279	2.612778	-134347.51
01/25/2024	07/25/2023	01/25/2024	-166985.56	0.971435	2.805811	-162215.68
07/25/2024	01/25/2024	07/25/2024	-151398.29	0.959063	2.729245	-145200.49
01/27/2025	07/25/2024	01/27/2025	-133147.63	0.946637	2.686747	-126042.54
07/25/2025	01/27/2025	07/25/2025	-115005.67	0.937259	2.55958	-107790.07
01/26/2026	07/25/2025	01/26/2026	-118711.1	0.927671	2.471024	-110124.81
07/27/2026	01/26/2026	07/27/2026	-112417.55	0.918662	2.398581	-103273.69
01/25/2027	07/27/2026	01/25/2027	-112428.46	0.909739	2.344061	-102280.57
07/26/2027	01/25/2027	07/26/2027	-112701.05	0.900868	2.302395	-101528.8
01/25/2028	07/26/2027	01/25/2028	-9346915.26	0.892028	2.268993	-8337709.83

Table IV.9 Currency Swap pricing – Paying Leg

The NPV of the Receiving Leg is USD 10,492,693.10, The NPV of the Paying Leg is EUR 9,460,317.73, as a result the swap price is: 10,492,693.10 - 9,460,317.73 * 1.083020 = 10,492,693.10 - 10,245,713.31 = USD 246,979.79.

Pay Date	Accrual Start	Accrual End	Days	Notional	Principal	Coupon	Payment
01/25/2022					-10000000		-10000000
07/25/2022	01/25/2022	07/25/2022	181	10000000	0	4	201111.11
01/25/2023	07/25/2022	01/25/2023	184	10000000	0	4	204444.44
07/25/2023	01/25/2023	07/25/2023	181	10000000	0	4	201111.11
01/25/2024	07/25/2023	01/25/2024	184	10000000	0	4	204444.44
07/25/2024	01/25/2024	07/25/2024	182	10000000	0	4	202222.22
01/27/2025	07/25/2024	01/27/2025	186	10000000	0	4	206666.67
07/25/2025	01/27/2025	07/25/2025	179	10000000	0	4	198888.89
01/26/2026	07/25/2025	01/26/2026	185	10000000	0	4	205555.56
07/27/2026	01/26/2026	07/27/2026	182	10000000	0	4	202222.22
01/25/2027	07/27/2026	01/25/2027	182	10000000	0	4	202222.22
07/26/2027	01/25/2027	07/26/2027	182	10000000	0	4	202222.22
01/25/2028	07/26/2027	01/25/2028	183	10000000	10000000	4	10203333.33

Pay Date	Accrual Start	Accrual End	Days	Payment	Discount	Zero Rate	PV
01/25/2022				-10000000			
07/25/2022	01/25/2022	07/25/2022	181	201111.11			
01/25/2023	07/25/2022	01/25/2023	184	204444.44	0.998564	4.36955	204150.96
07/25/2023	01/25/2023	07/25/2023	181	201111.11	0.974954	4.796975	196074.11
01/25/2024	07/25/2023	01/25/2024	184	204444.44	0.951855	4.777226	194601.41
07/25/2024	01/25/2024	07/25/2024	182	202222.22	0.93316	4.517063	188705.59
01/27/2025	07/25/2024	01/27/2025	186	206666.67	0.918385	4.171213	189799.59
07/25/2025	01/27/2025	07/25/2025	179	198888.89	0.905923	3.902854	180178.01
01/26/2026	07/25/2025	01/26/2026	185	205555.56	0.893226	3.716326	183607.64
07/27/2026	01/26/2026	07/27/2026	182	202222.22	0.881157	3.577036	178189.56
01/25/2027	07/27/2026	01/25/2027	182	202222.22	0.869249	3.472235	175781.39
07/26/2027	01/25/2027	07/26/2027	182	202222.22	0.857395	3.393222	173384.23
01/25/2028	07/26/2027	01/25/2028	183	10203333.33	0.845628	3.329804	8628220.58

Table IV.10 Currency Swap pricing – Receiving Leg

Interest Rate Swap Curves are an essential tool for computing forward rates and discount factors used for pricing medium-long term financial instruments. We now illustrate how to create a zero curve from market IRS swap rates by using the bootstrapping methodology, and since market swap rates have three kinds according to their sources, i.e., cash (deposit), futures (Fra), swap, we present the bootstrapping for these three different cases.

Bootstrapping – Deposit (or Cash)

As the market swap rate for deposits is a quarterly compounding rate, the discount factor is estimated through this rate and the zero rate is calculated with the discount factor as follows:

$$DF(s,t_i) = \left(1 + R_{t_i}^{MKT} \times \frac{\tau(s,t_i)}{360}\right)^{-1} \quad \textit{(Eq. IV.16)}$$

$$R(s,t_i) = \frac{365}{\tau(s,t_i)} \times \ln\left(\frac{1}{DF(s,t_i)}\right) \quad \textit{(Eq. IV.17)}$$

Where $DF(s,t_i)$ is the discount factor from t_i to s, $R(s,t_i)$ is the zero or spot rate from t_i to s and $R_{t_i}^{MKT}$ is the market par rate at t_i and $\tau(s,t_i)$ is the day count. We use the Bloomberg® convention for this curve stripping.

Bootstrapping – Futures

Bloomberg provides the market rate for Euro dollar futures as a rate, not a price, i.e. yield = par value (100) – mkt price.

Since maturities of the contiguous futures are successive from 3M and non-overlapping, zero rates can be found in the following order:

1) Compute the discount factor from t_{i-1} to t_i:

$$DF(t_{i-1},t_i) = \left(1 + R_{t_i}^{MKT} \times \frac{\tau(t_{i-1},t_i)}{360}\right)^{-1} \quad \textit{(Eq. IV.18)}$$

2) Compute the discount factor from spot date to t_i:

$$DF(s,t_i) = DF(s,t_{i-1}) \times DF(t_{i-1},t_i) \quad \textit{(Eq. IV.19)}$$

3) Compute the zero rate from discount factor:

$$R(s,t_i) = \frac{365}{\tau(s,t_i)} \times \ln\left(\frac{1}{DF(s,t_i)}\right) \quad \textit{(Eq. IV.20)}$$

Particular attention must be paid to the first future contract. If its starting date exactly matches the maturity of the deposit (Case A) we can apply the previous general formula.

But if the starting date for the first futures contract does not exactly match the maturity of the first deposit (Case B) we have to consider a synthetic rate, known as **stub rate**, in order to perform the first zero rate computation from Futures/FRAs, then the general formula can be applied for the other contiguous futures.

1) Compute the discount factor from t_{STUB} to t_2:

$$DF(t_{STUB}, t_2) = \left(1 + R_2^{MKT} \times \frac{\tau(t_{STUB}, t_2)}{360}\right)^{-1} \quad \textit{(Eq. IV.21)}$$

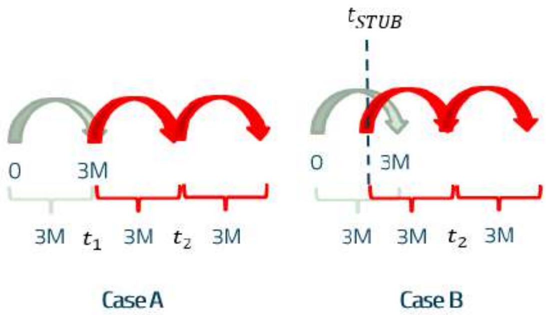
2) Interpolate the discount factor from spot date to t_{STUB}.

3) Compute the discount factor from spot date to t_i:

$$DF(s, t_2) = DF(s, t_{STUB}) \times DF(t_{STUB}, t_2) \quad \textit{(Eq. IV.22)}$$

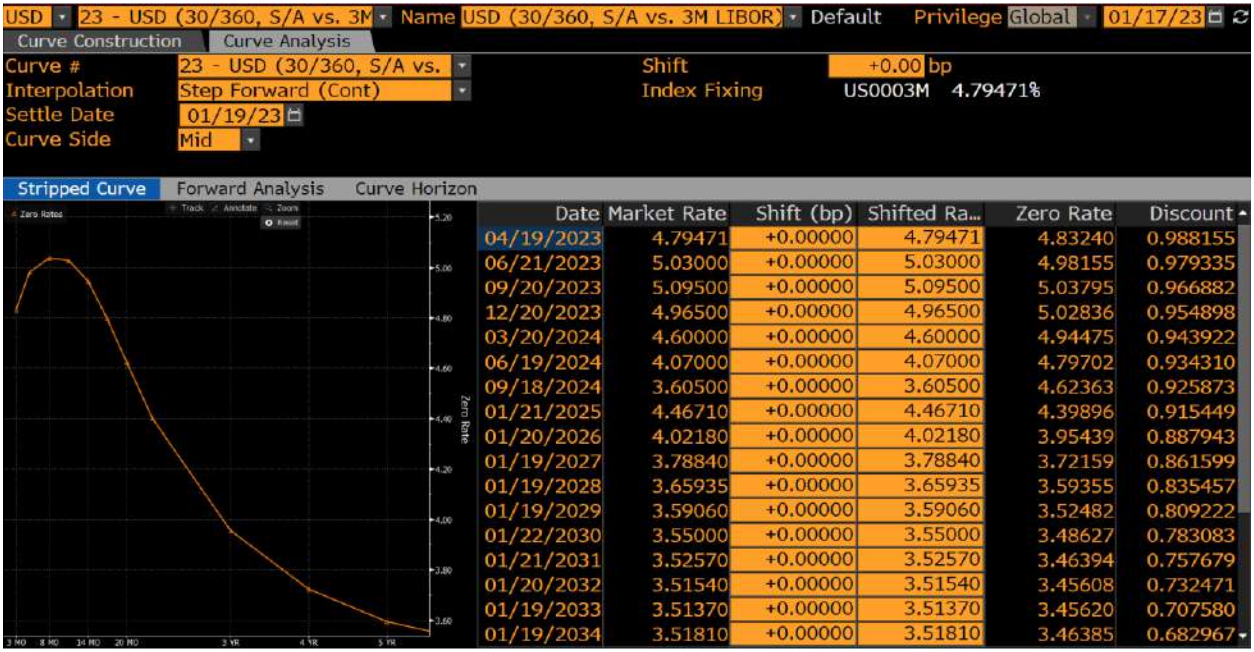
4) Compute the zero rate from discount factor:

$$R(s, t_2) = \frac{365}{\tau(s, t_2)} \times \ln\left(\frac{1}{DF(s, t_2)}\right) \quad \textit{(Eq. IV.23)}$$



We start by stripping the short and medium term of the US Swap curve (Bloomberg® id. 23), considering a Valuation Date of 17th January 2023 and a Settlement Date of 19th January 2023.

Figure IV.43 Swap Curve Builder. Curve Construction. Source: Bloomberg®



Date	Market Rate	Shift (bp)	Shifted Ra...	Zero Rate	Discount
04/19/2023	4.79471	+0.00000	4.79471	4.83240	0.988155
06/21/2023	5.03000	+0.00000	5.03000	4.98155	0.979335
09/20/2023	5.09500	+0.00000	5.09500	5.03795	0.966882
12/20/2023	4.96500	+0.00000	4.96500	5.02836	0.954898
03/20/2024	4.60000	+0.00000	4.60000	4.94475	0.943922
06/19/2024	4.07000	+0.00000	4.07000	4.79702	0.934310
09/18/2024	3.60500	+0.00000	3.60500	4.62363	0.925873
01/21/2025	4.46710	+0.00000	4.46710	4.39896	0.915449
01/20/2026	4.02180	+0.00000	4.02180	3.95439	0.887943
01/19/2027	3.78840	+0.00000	3.78840	3.72159	0.861599
01/19/2028	3.65935	+0.00000	3.65935	3.59355	0.835457
01/19/2029	3.59060	+0.00000	3.59060	3.52482	0.809222
01/22/2030	3.55000	+0.00000	3.55000	3.48627	0.783083
01/21/2031	3.52570	+0.00000	3.52570	3.46394	0.757679
01/20/2032	3.51540	+0.00000	3.51540	3.45608	0.732471
01/19/2033	3.51370	+0.00000	3.51370	3.45620	0.707580
01/19/2034	3.51810	+0.00000	3.51810	3.46385	0.682967

Figure IV.44 Swap Curve Builder. Curve analysis. Source: Bloomberg®

For the **Short-term Curve**, we consider a 3 months Deposit with maturity on 19th April 2023. The Market Rate is 4.79471%, thus:

$$DF(s,t_i) = \left(1 + R_1^{MKT} \times \frac{\tau(s,t_i)}{360}\right)^{-1} = \left(1 + 0.0479471 \times \frac{90}{360}\right)^{-1} = 0.9881552$$

$$R(s,t_i) = \frac{365}{\tau(s,t_i)} \times \ln\left(\frac{1}{DF(s,t_i)}\right) = \frac{365}{90} \times \ln\left(\frac{1}{0.9881552}\right) = 0.04832398$$

$$R_1^{MKT} = 0.0479471$$

$$\tau(s,t_1) = \tau(19 \text{ January } 2023,\ 19 \text{ April } 2023) = 90$$

For the **Medium-term Curve**, we have: the First Future in the medium-term section of the curve, which is the Eurodollar Futures (EDH3 contract), and the market rate yield which is (100% – 94.97%) 5.03%.

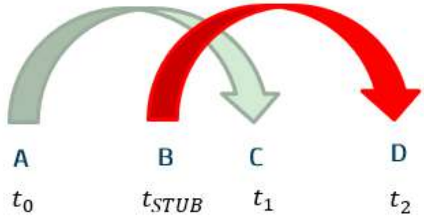
This interest starts to accrue on 13th March 2023 and ends on the futures maturity date, on 21st June 2023.

A is the IR curve spot date (19th January 2023), t_0.

B is the stub date of the first future (13th March 2023), t_{STUB}.

C is the maturity date of the spot (19th April 2023), t_1.

D is the maturity date of the first future (21st June 2023), t_2.



The first step is to compute the discount factor between B and D as follows:

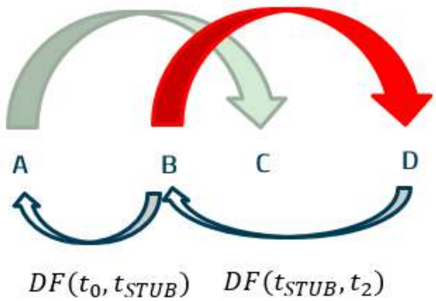
$$DF(t_{STUB}, t_2) = \left(1 + R_2^{MKT} \times \frac{\tau(t_{STUB}, t_2)}{360}\right)^{-1} =$$

$$= \left(1 + 0.0503 \times \frac{\tau(\text{13 March 2023,21 June 2023})}{360}\right)^{-1} = \left(1 + 0.0503 \times \frac{100}{360}\right)^{-1} = 0.98622$$

The second step is to interpolate the zero rate or the discount factor between A and B, $DF(t_0, t_{STUB})$.

We know that $DF(t_0, t_0) = 1$ and $DF(t_0, t_1) = 0.9881552$,

consequently the linear interpolated $DF(t_0, t_{STUB})$ is:



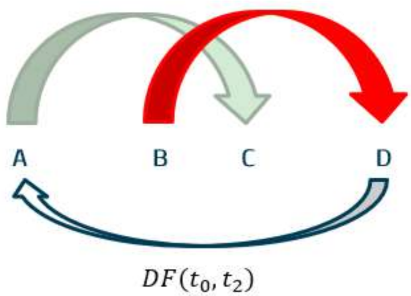
$$DF(t_0, t_{STUB}) = 1 + \frac{0.9881552 - 1}{\tau(t_0, t_1)} \tau(t_0, t_{STUB}) =$$

$$= 1 + \frac{0.9881552 - 1}{90} \cdot 53 = 0.993025$$

The third step is to estimate the discount factor between A and D, $DF(t_0, t_2)$.

$$DF(t_0, t_2) = DF(t_0, t_{STUB}) \times DF(t_{STUB}, t_2) =$$

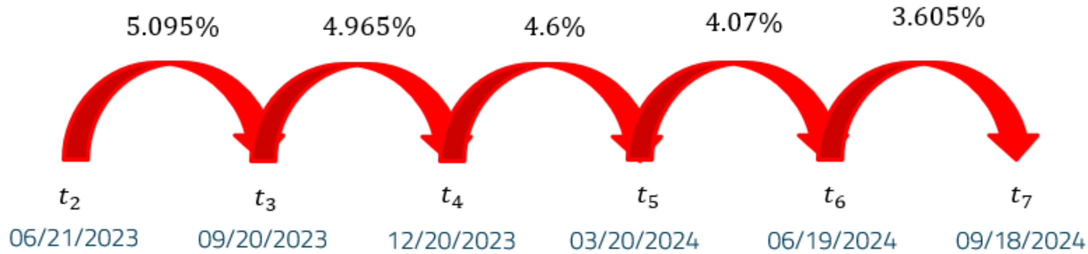
$$= 0.993025 \cdot 0.98622 = 0.97934$$



The stripped zero rate between the spot date (t_0) and t_2 is:

$$R(t_0, t_2) = \frac{365}{\tau(s, t_2)} \times \ln\left(\frac{1}{DF(s, t_i)}\right) = \frac{365}{153} \times \ln\left(\frac{1}{0.97934}\right) = 4.98\%$$

Given that the Futures are contiguous, we can work under the hypothesis that the start date of the following contracts matches the expiration date of the previous one. Thus, we are in this case:

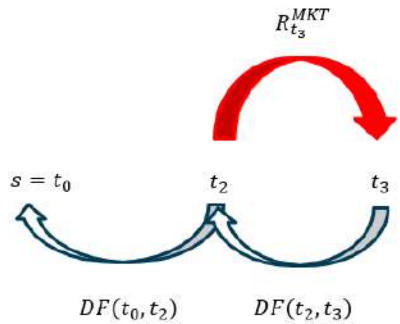


We compute the zero rate $R(s, t_i)$ with $i = 3, \dots, 7$ using the standard procedure:

1) Compute the discount factor from t_{i-1} to t_i $DF(t_{i-1}, t_i) = \left(1 + R_{t_i}^{MKT} \times \frac{\tau(t_{i-1}, t_i)}{360}\right)^{-1}$

2) Compute the discount factor from spot date to t_i $DF(s, t_i) = DF(s, t_{i-1}) \times DF(t_{i-1}, t_i)$.

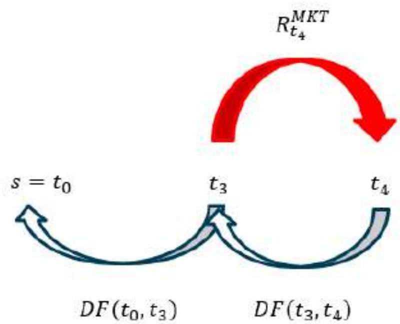
3) Compute the zero rate from discount factor $R(s, t_i) = \frac{365}{\tau(s, t_i)} \times \ln\left(\frac{1}{DF(s, t_i)}\right)$



$$DF(t_2, t_3) = \left(1 + R_{t_3}^{MKT} \times \frac{\tau(t_2, t_3)}{360}\right)^{-1} =$$
$$= \left(1 + 0.05095 \times \frac{91}{360}\right)^{-1} = 0.98728$$

$$DF(s, t_3) = DF(s, t_2) \times DF(t_2, t_3)$$
$$= 0.97934 \cdot 0.98728 = 0.96689$$

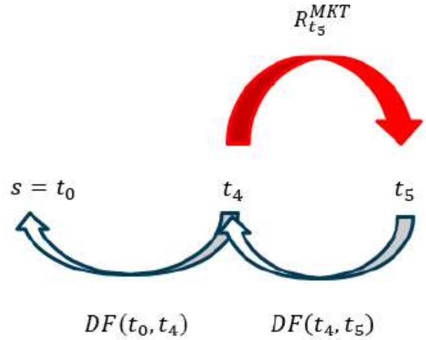
$$R(s, t_3) = \frac{365}{\tau(s, t_3)} \times \ln\left(\frac{1}{DF(s, t_3)}\right) = \frac{365}{244} \times \ln\left(\frac{1}{0.96689}\right) = 5.037\%$$



$$DF(t_3, t_4) = \left(1 + R_{t_4}^{MKT} \times \frac{\tau(t_3, t_4)}{360}\right)^{-1} =$$
$$= \left(1 + 0.04965 \times \frac{91}{360}\right)^{-1} = 0.987605$$

$$DF(s,t_4) = DF(s,t_3) \times DF(t_3,t_4)$$
$$= 0.96689 \cdot 0.987606 = 0.95490$$

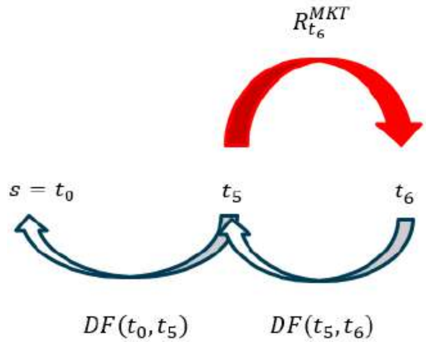
$$R(s,t_4) = \frac{365}{\tau(s,t_4)} \times \ln\left(\frac{1}{DF(s,t_4)}\right) = \frac{365}{335} \times \ln\left(\frac{1}{0.9549}\right) = 5.028\%$$



$$DF(t_4,t_5) = \left(1 + R^{MKT}_{t_5} \times \frac{\tau(t_4,t_5)}{360}\right)^{-1} =$$
$$= \left(1 + 0.046 \times \frac{91}{360}\right)^{-1} = 0.98851$$

$$DF(s,t_5) = DF(s,t_4) \times DF(t_4,t_5)$$
$$= 0.95490 \cdot 0.98851 = 0.94393$$

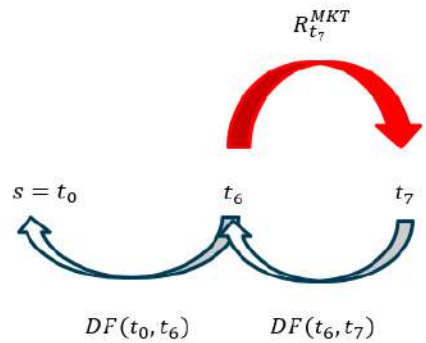
$$R(s,t_5) = \frac{365}{\tau(s,t_5)} \times \ln\left(\frac{1}{DF(s,t_5)}\right) = \frac{365}{426} \times \ln\left(\frac{1}{0.94393}\right) = 4.944\%$$



$$DF(t_5,t_6) = \left(1 + R^{MKT}_{t_6} \times \frac{\tau(t_5,t_6)}{360}\right)^{-1} =$$
$$= \left(1 + 0.0407 \times \frac{91}{360}\right)^{-1} = 0.98982$$

$$DF(s,t_6) = DF(s,t_5) \times DF(t_5,t_6)$$
$$= 0.94393 \cdot 0.98982 = 0.93432$$

$$R(s,t_6) = \frac{365}{\tau(s,t_6)} \times \ln\left(\frac{1}{DF(s,t_6)}\right) = \frac{365}{517} \times \ln\left(\frac{1}{0.93432}\right) = 4.797\%$$



$$DF(t_6,t_7) = \left(1 + R^{MKT}_{t_7} \times \frac{\tau(t_6,t_7)}{360}\right)^{-1} =$$
$$= \left(1 + 0.03605 \times \frac{91}{360}\right)^{-1} = 0.99097$$

$$DF(s,t_7) = DF(s,t_6) \times DF(t_6,t_7)$$
$$= 0.93432 \cdot 0.99097 = 0.92588$$

$$R(s,t_7) = \frac{365}{\tau(s,t_7)} \times \ln\left(\frac{1}{DF(s,t_7)}\right) = \frac{365}{608} \times \ln\left(\frac{1}{0.92588}\right) = 4.623\%$$

Table IV.11 below summarizes the results obtained from the Short-Medium Curve stripping.

Time	Maturity	Instruments	Par Rates	Zero Rates	Discount Factors
t1	04/19/2023	Depo	4.79471%	4.8324%	0.988155
t2	06/21/2023	Future	5.030%	4.9800%	0.97934
t3	09/20/2023	Future	5.095%	5.0370%	0.96689
t4	12/20/2023	Future	4.965%	5.0280%	0.9549
t5	03/20/2024	Future	4.600%	4.9440%	0.94393
t6	06/19/2024	Future	4.070%	4.7970%	0.93432
t7	09/18/2024	Future	3.605%	4.6230%	0.92588

Table IV.11 Short-Medium Term stripping

Bootstrapping – Swap (Long-term Interest rates curve)

A numerical optimization technique for finding zero rates is not needed for deposits and futures, as their zero rates are directly recovered using the previously discussed formulas.
The most difficult part of the process is to bootstrap zero rates from market swap rates for IRS. Deposits and futures have one bullet payment at maturity, but IRS has in-between cash flows as well. For this reason, we have to program a series of goal-seeking routines for stripping the zero curve from Swap Par Rates.
We illustrate this process through an example, considering the first swap that provides a par coupon of 4.4671%. This means that a plain vanilla swap "fixed vs floating rate" is worth zero if we use 4.4671% as the fixed coupon rate.
Given that only swaps with standard characteristics are used for stripping, the fixed leg has a semi-annual payment frequency with a Day Basis 30/360, while the floating leg has quarterly payments with a ACT/360 day basis. The features for a standard IRS might obviously depend on the underlying currency.
Since the spot date for the curve is 19th January 2023, all the future dates of the swap must be computed accordingly.

Bearing this in mind, we can only partially complete the NPV schedule for both legs of the swap. We assume we have a principal of USD 10 million as the basis for interest estimation.

The NPV Schedule for the Fixed Leg of the 2-years swap is shown in the table below:

Start Date	End Date	Notional	Coupon	Days	30/360	Cash Flow	Zero Rate	DF	NPV
01/19/2023	07/19/2023	10000000	4.4671%	180	0.5	223355	Z1	DF1	NPV1
07/19/2023	01/19/2024	10000000	4.4671%	180	0.5	223355	Z2	DF2	NPV2
01/19/2024	07/19/2024	10000000	4.4671%	180	0.5	223355	Z3	DF3	NPV3
07/19/2024	01/21/2025	10000000	4.4671%	182	0.505556	225836.7	Z4	DF4	NPV4

Table IV.12 NPV schedule for the fixed leg of 2-years swap. Step 1

The values of DFi can be computed starting from the continuous compounded zero rates, Zi, using exp(-Zi * Ti), where Ti is the year fraction between the End Date and the Spot Date (01/19/2023). The values of NPVi are the product between the CFi and the Dfi, while the NPV of the overall leg is the sum of each NPVi.

Since Zi, with i=1,2,3 fall inside the range of time previously stripped, i.e., from 19th January 2023 to 18th September 2024, they can be directly interpolated from the known spot rates. In this case, Z4 is our unknown. Thus, we estimate Z1, Z2, Z3 and the other dependent results (DF and NPV) through a linear interpolation.

Z1 = 4.9975% then DF1 = exp (-0.049975 * (07/19/2023-01/19/2023)/365) = 0.975522.

NPV1 = CF1*DF1 = (100,000,000 * 0.044671 * 0.5) * 0.975522 = 217,887.8.

Z2 = 5.0003% then DF2 = exp (-0.050003 * (01/19/2024-01/19/2023)/365) = 0.951226.

NPV2 = CF2*DF2 = (100,000,000 * 0.044671 * 0.5) * 0.951226 = 212,461.2.

Z3 = 4.7396% then DF3 = exp (-0.047396 * (07/19/2024-01/19/2023)/365) = 0.931434.

NPV3 = CF3*DF3 = (100,000,000 * 0.044671 * 0.5) * 0.931434 = 208,040.5.

Start Date	End Date	Notional	Coupon	30/360	Cash Flow	Zero Rate	DF	NPV
01/19/2023	07/19/2023	10000000	4.4671%	0.5	223355	0.049975	0.975522	217887.8
07/19/2023	01/19/2024	10000000	4.4671%	0.5	223355	0.050003	0.951226	212461.2
01/19/2024	07/19/2024	10000000	4.4671%	0.5	223355	0.047396	0.931434	208040.5
07/19/2024	01/21/2025	10000000	4.4671%	0.505556	225836.7	x	f(x)	f(x)

Table IV.13 NPV schedule for the fixed leg of 2-years swap. Step 2

We notice that DF4 and NPV4 depend only on the unknown variable Z4 (x), while the other terms are all known in this leg. The NPV Schedule for the Floating Leg of the 2-years swap is shown in the table below. Using the information known through the short-medium term stripping we can partially fill the cash-flow schedule of the floating leg.

Start Date	End Date	Notional	Days	Days/360	Zero Rate	DF	Forward	Cash Flow	NPV
01/19/2023	04/19/2023	10000000	90	0.25	Z1	DF1	FWD1	CF1	NPV1
04/19/2023	07/19/2023	10000000	91	0.252778	Z2	DF2	FWD2	CF2	NPV2
07/19/2023	10/19/2023	10000000	92	0.255556	Z3	DF3	FWD3	CF3	NPV3
10/19/2023	01/19/2024	10000000	92	0.255556	Z4	DF4	FWD4	CF4	NPV4
01/19/2024	04/19/2024	10000000	91	0.252778	Z5	DF5	FWD5	CF5	NPV5
04/19/2024	07/19/2024	10000000	91	0.252778	Z6	DF6	FWD6	CF6	NPV6
07/19/2024	10/21/2024	10000000	94	0.261111	Z7	DF7	FWD7	CF7	NPV7
10/21/2024	01/21/2025	10000000	92	0.255556	Z8	DF8	FWD8	CF8	NPV8

Table IV.14 NPV schedule for the floating leg of 2-years swap. Step 1

FWD1 is the fixing rate on the curve settlement date (17th January 2023) of the USD Libor, i.e., 4.79471% and we know exactly the correspondent zero rate, that is Z1=4.8324%. Discount factors (DF), Cash Flows (CF) and NPVs are estimated using the previous formulas. Regarding the Zero rates indexed with 2,3,…,6, their values can be interpolated from the known stripped zero rates as we have done in the fixed leg.

Start Date	End Date	Notional	Days/360	Zero Rate	DF	Forward	Cash Flow	NPV
01/19/2023	04/19/2023	10000000	0.25	0.048324	0.988155	0.04795	119867.8	118447.9
04/19/2023	07/19/2023	10000000	0.252778	0.049975	0.975522	FWD2	CF2	NPV2
07/19/2023	10/19/2023	10000000	0.255556	0.050341	0.963047	FWD3	CF3	NPV3
10/19/2023	01/19/2024	10000000	0.255556	0.050003	0.951226	FWD4	CF4	NPV4
01/19/2024	04/19/2024	10000000	0.252778	0.048955	0.940672	FWD5	CF5	NPV5
04/19/2024	07/19/2024	10000000	0.252778	0.047396	0.931434	FWD6	CF6	NPV6
07/19/2024	10/21/2024	10000000	0.261111	Z7	DF7	FWD7	CF7	NPV7
10/21/2024	01/21/2025	10000000	0.255556	Z8	DF8	FWD8	CF8	NPV8

Table IV.15 NPV schedule for the floating leg of 2-years swap. Step 2

The implied forward rates (FWDi, i=2,...,8) can be estimated using: $FWD_i = \frac{1}{\tau_i}\left(\frac{DF_{\tau_{i-1}}}{DF_{\tau_i}} - 1\right)$.

For instance, for i=2: $FWD_2 = \frac{1}{0.252778}\left(\frac{0.975522}{0.988155} - 1\right) = 5.123\%$.

We can consequently compute Cash flows and NPVs up to i =7.

Start Date	End Date	Notional	Days/360	Zero Rate	DF	Forward	Cash Flow	NPV
01/19/2023	04/19/2023	10000000	0.25	0.048324	0.988155	0.04795	119867.8	118447.9
04/19/2023	07/19/2023	10000000	0.252778	0.049975	0.975522	0.05123	129499.4	126329.6
07/19/2023	10/19/2023	10000000	0.255556	0.050341	0.963047	0.05069	129534.1	124747.5
10/19/2023	01/19/2024	10000000	0.255556	0.050003	0.951226	0.04863	124271.1	118209.9
01/19/2024	04/19/2024	10000000	0.252778	0.048955	0.940672	0.04439	112201	105544.4
04/19/2024	07/19/2024	10000000	0.252778	0.047396	0.931434	0.03924	99177.93	92377.7
07/19/2024	10/21/2024	10000000	0.261111	Z7	DF7	FWD7	CF7	NPV7
10/21/2024	01/21/2025	10000000	0.255556	Z8	DF8	FWD8	CF8	NPV8

Table IV.16 NPV schedule for the floating leg of 2-years swap. Step 3

The objective now is to express all the remaining variables in function of the zero rate Z8, which is the spot rate that we are seeking.

We call this spot rate "x", and we show the way in which we can express the relationships. Z7 can be estimated using a linear relationship between the previous known spot rate (4.7396%) and the unknown x:

z(x)=4.7396%+((x-4.7396%)/(01/21/2025-07/19/2024))*(10/21/2024-07/19/2024).

DF7 = exp (-z(x)*(10/21/2024-01/19/2023)/365).

FWD7 = 1/0.26111 * (0.931434/DF7-1); CF7 = 10,000,000*FWD7*0.26111; NPV7 = CF7*DF7.

The same relationships can be found for i=8.

DF8 = exp (-x*(21/01/2025-19/01/2023)/365).

FWD8 = 1/0.25556 * (DF7/DF8-1); CF8 = 10,000,000*FWD8*0.25556; NPV8 = CF8*DF8.

Both the Fixed leg and the Floating leg can be expressed in function of the only unknown, that is the zero rate with maturity 01/21/2025.

Start Date	End Date	Notional	Days/360	Zero Rate	DF	Forward	Cash Flow	NPV
01/19/2023	04/19/2023	10000000	0.25	0.048324	0.988155	0.04795	119867.8	118447.9
04/19/2023	07/19/2023	10000000	0.252778	0.049975	0.975522	0.05123	129499.4	126329.6
07/19/2023	10/19/2023	10000000	0.255556	0.050341	0.963047	0.05069	129534.1	124747.5
10/19/2023	01/19/2024	10000000	0.255556	0.050003	0.951226	0.04863	124271.1	118209.9
01/19/2024	04/19/2024	10000000	0.252778	0.048955	0.940672	0.04439	112201	105544.4
04/19/2024	07/19/2024	10000000	0.252778	0.047396	0.931434	0.03924	99177.93	92377.7
07/19/2024	10/21/2024	10000000	0.261111	f(x)	f(x)	f(x)	f(x)	f(x)
10/21/2024	01/21/2025	10000000	0.255556	x	f(x)	f(x)	f(x)	f(x)

Table IV.17 NPV schedule for the floating leg of 2-years swap. Step 4

This allows to design a goal seeking that has the objective to set the NPV of the swap (i.e. the difference of the two discounted legs) equal to zero, as the unknown x varies. Let us remember that the rate of the fixed leg is the so-called equilibrium coupon of the swap, i.e. the rate for which the swap NPV is zero.

Start Date	End Date	Coupon	Days	30/360	Cash Flow	Zero Rate	DF	NPV
01/19/2023	07/19/2023	4.4671%	180	0.5	223355	0.049975	0.975522	217887.8
07/19/2023	01/19/2024	4.4671%	180	0.5	223355	0.050003	0.951226	212461.2
01/19/2024	07/19/2024	4.4671%	180	0.5	223355	0.047396	0.931434	208040.5
07/19/2024	01/21/2025	4.4671%	182	0.505556	225836.7	**0.043969**	0.915486	206750.4

Table IV.18 NPV schedule for the fixed leg of 2-years swap. Step 5

Start Date	End Date	Days/360	Zero Rate	DF	Forward	Cash Flow	NPV
01/19/2023	04/19/2023	0.25	0.048324	0.988155	0.04795	119867.8	118447.9
04/19/2023	07/19/2023	0.252778	0.049975	0.975522	0.05123	129499.4	126329.6
07/19/2023	10/19/2023	0.255556	0.050341	0.963047	0.05069	129534.1	124747.5
10/19/2023	01/19/2024	0.255556	0.050003	0.951226	0.04863	124271.1	118209.9
01/19/2024	04/19/2024	0.252778	0.048955	0.940672	0.04439	112201	105544.4
04/19/2024	07/19/2024	0.252778	0.047396	0.931434	0.03924	99177.93	92377.7
07/19/2024	10/21/2024	0.261111	0.045664	0.922937	0.03526	92067.93	84972.9
10/21/2024	01/21/2025	0.255556	**0.043969**	0.915486	0.03185	81388.32	74509.9

Table IV.19 NPV schedule for the floating leg of 2-years swap. Step 6

Setting x = 0.043969, we reach a Swap NPV = 0. As a result, $R(s, t_8) = 4.397\%$. A similar procedure must be adopted for solving all the other zero rates that coming from swap rates. We show the strip for the spot rates up to the 4 year swap. All the stripped rates are very close to those estimated by the SWDF Bloomberg® module (the difference is less than 1 basis point).

The table below displays the NPV schedule for the Fixed Leg of the 3-years swap:

Start Date	End Date	Coupon	Days	Days/360	Cash Flow	Zero Rate	DF	NPV
01/19/2023	07/19/2023	4.0218%	180	0.5	201090	0.049975	0.975522	196167.8
07/19/2023	01/19/2024	4.0218%	180	0.5	201090	0.050003	0.951226	191282.1
01/19/2024	07/19/2024	4.0218%	180	0.5	201090	0.047396	0.931434	187302.1
07/19/2024	01/21/2025	4.0218%	182	0.505556	203324.3	0.04397	0.915485	186140.4
01/21/2025	07/21/2025	4.0218%	180	0.5	201090	0.041757	0.900717	181125.2
07/21/2025	01/21/2026	4.0218%	179	0.497222	199972.8	**0.039507**	0.88804	177853.8

Table IV.20 NPV schedule for the fixed leg of 3-years swap.

The NPV Fixed Leg is 1,119,601.41, the Equilibrium Coupon, i.e. the par rate is $R_{t_9}^{MKT} = 4.0218\%$ and the zero rate, $R(s, t_9) = 3.9507\%$. For interpolating the zero rate with maturity 21st July 2025, $R(s, t_8)$ and $R(s, t_9)$ have been used. In the same way, we can continue to solve a 1-dimensional goal seeking problem.

The table below shows the NPV Schedule for the Floating Leg of the 3-years swap. Given that we use the par rate for the fixed leg, after solving for the unknown $R(s, t_9)$, we obtain the same NPV for the floating leg, that is 1,119,601.41.
The strip of zero rates that mature on 04/22/2025, 07/21/2025 and 10/20/2025 must be linearly interpolated starting from $R(s, t_8)$ and $R(s, t_9)$.

Start Date	End Date	Forward	Days	Days/360	Cash Flow	Zero Rate	DF	NPV
01/19/2023	04/19/2023	0.047947	90	0.25	119867.8	0.048324	0.988155	118447.9407
04/19/2023	07/19/2023	0.051231	91	0.252778	129499.4	0.049975	0.975522	126329.5524
07/19/2023	10/19/2023	0.050687	92	0.255556	129534.1	0.050341	0.963047	124747.5084
10/19/2023	01/19/2024	0.048628	92	0.255556	124271.1	0.050003	0.951226	118209.9795
01/19/2024	04/19/2024	0.044387	91	0.252778	112201	0.048955	0.940672	105544.3572
04/19/2024	07/19/2024	0.039235	91	0.252778	99177.93	0.047396	0.931434	92377.72239
07/19/2024	10/21/2024	0.035049	94	0.261111	91516.7	0.045633	0.922987	84468.76604
10/21/2024	01/21/2025	0.032068	92	0.255556	81951.13	0.043970	0.915485	75025.0259
01/21/2025	04/22/2025	0.033545	91	0.252778	84795.1	0.042854	0.907787	76975.91777
04/22/2025	07/21/2025	0.031337	90	0.25	78343.4	0.041751	0.900731	70566.30477
07/21/2025	10/20/2025	0.029133	91	0.252778	73642.63	0.040635	0.894146	65847.25679
10/20/2025	01/20/2026	0.026906	92	0.255556	68759.39	**0.039507**	0.88804	61061.07604

Table IV.21 NPV schedule for the floating leg of 3-years swap.

The below table displays the NPV Schedule for the Fixed Leg of the 4-years swap:

Start Date	End Date	Coupon	Days	Days/360	Cash Flow	Zero Rate	DF	NPV
01/19/2023	07/19/2023	3.7884%	180	0.5	189420	0.049975	0.975522	184783.4239
07/19/2023	01/19/2024	3.7884%	180	0.5	189420	0.050003	0.951226	180181.3232
01/19/2024	07/19/2024	3.7884%	180	0.5	189420	0.047396	0.931434	176432.2832
07/19/2024	01/21/2025	3.7884%	182	0.505556	191524.7	0.04397	0.915485	175337.9423
01/21/2025	07/21/2025	3.7884%	180	0.5	189420	0.041751	0.90073	170616.2778
07/21/2025	01/20/2026	3.7884%	179	0.497222	188367.7	0.039508	0.888038	167277.6898
01/20/2026	07/20/2026	3.7884%	180	0.5	189420	0.038348	0.874355	165620.3234
07/20/2026	01/19/2027	3.7884%	179	0.497222	188367.7	**0.037174**	0.861743	162324.4487

Table IV.22 NPV schedule for the fixed leg of 4-years swap.

The NPV Fixed Leg is 1,382,573.71, the Equilibrium Coupon, i.e. the par rate is $R_{t_{10}}^{MKT} = 3.7884\%$ and the zero rate, is $R(s, t_{10}) = 3.717\%$.

The zero rate with maturity 20th July 2026 has been interpolated using $R(s, t_9)$ and $R(s, t_{10})$. In the same way, we can continue to have only one unknown for our goal seeking problem. The following table shows the NPV

Schedule for the 4-years Floating Swap Leg and for the 5-years Swap Fixed Leg:

Start Date	End Date	Forward	Days	Days/360	Cash Flow	Zero Rate	DF	NPV
01/19/2023	04/19/2023	0.047947	90	0.25	119867.8	0.048324	0.988155	118447.9407
04/19/2023	07/19/2023	0.051231	91	0.252778	129499.4	0.049975	0.975522	126329.5524
07/19/2023	10/19/2023	0.050687	92	0.255556	129534.1	0.050341	0963047	124747.5084
10/19/2023	01/19/2024	0.048628	92	0.255556	124271.1	0.050003	0.951226	118209.9795
01/19/2024	04/19/2024	0.044387	91	0.252778	112201	0.048955	0.940672	105544.3572
04/19/2024	07/19/2024	0.039235	91	0.252778	99177.93	0.047396	0.931434	92377.72239
07/19/2024	10/21/2024	0.035049	94	0.261111	91516.7	0.045633	0.922987	84468.76604
10/21/2024	01/21/2025	0.032068	92	0.255556	81951.13	0.043970	0.915485	75025.0259
01/21/2025	04/22/2025	0.033547	91	0.252778	84798.56	0.042855	0.907787	76979.03207
04/22/2025	07/21/2025	0.031339	90	0.25	78347.57	0.041751	0.90073	70570.008
07/21/2025	10/20/2025	0.029135	91	0.252778	73647.6	0.040636	0.894145	65851.62206
10/20/2025	01/20/2026	0.026908	92	0.255556	68765.19	0.039508	0.888038	61066.11685
01/20/2026	04/20/2026	0.031585	90	0.25	78961.46	0.038931	0.881081	69571.44619
04/20/2026	07/20/2026	0.030432	91	0.252778	76926.35	0.038348	0.874355	67260.93722
07/20/2026	10/19/2026	0.029273	91	0.252778	73995.14	0.037764	0.867933	64222.80075
10/19/2026	01/19/2027	0.028108	92	0.255556	71832.23	**0.037174**	0.861743	61900.89659

Start Date	End Date	Coupon	Days	Days/360	Cash Flow	Zero Rate	DF	NPV
01/19/2023	07/19/2023	3.65935%	180	0.5	182967.5	0.049975	0.975522	178488.8666
07/19/2023	01/19/2024	3.65935%	180	0.5	182967.5	0.050003	0.951226	174043.5342
01/19/2024	07/19/2024	3.65935%	180	0.5	182967.5	0.047396	0.931434	170422.2034
07/19/2024	01/21/2025	3.65935%	182	0.505556	185000.5	0.04397	0.915485	169365.1408
01/21/2025	07/21/2025	3.65935%	180	0.5	182967.5	0.041751	0.90073	164804.3175
07/21/2025	01/20/2026	3.65935%	179	0.497222	181951	0.039508	0.888038	161579.4568
01/20/2026	07/20/2026	3.65935%	180	0.5	182967.5	0.038347	0.874355	159978.6019
07/20/2026	01/19/2027	3.65935%	179	0.497222	181951	0.037174	0.861743	156795.0674
01/19/2027	07/19/2027	3.65935%	180	0.5	182967.5	0.036538	0.848426	155234.4358
07/19/2027	01/19/2028	3.65935%	180	0.5	182967.5	**0.035892**	0.835639	152894.8426

Table IV.23 NPV schedule for the floating leg of 4-years swap and for the fixed leg of 5-years swap

The NPV Fixed Leg is 1,643,606.47, the Equilibrium Coupon, i.e. the par rate is $R_{t_{11}}^{MKT} = 3.65935\%$ and the zero rate is $R(s, t_{11}) = 3.5892\%$. In this case, the zero rate with maturity 19th July 2027 has been interpolated using $R(s, t_{10})$ and $R(s, t_{11})$, and the goal attainment problem continues to be mono-dimensional.

Start Date	End Date	Forward	Days	Days/360	Cash Flow	Zero Rate	DF	NPV
01/19/2023	04/19/2023	0.047947	90	0.25	119867.8	0.048324	0.988155	118447.9407
04/19/2023	07/19/2023	0.051231	91	0.252778	129499.4	0.049975	0.975522	126329.5524
07/19/2023	10/10/2023	0.050687	92	0.255556	129534.1	0.050341	0.963047	124747.5084
10/19/2023	01/19/2024	0.048628	92	0.255556	124271.1	0.050003	0.951226	118209.9795
01/19/2024	04/19/2024	0.044387	91	0.252778	112201	0.048955	0.940672	105544.3572
04/19/2024	07/19/2024	0.039235	91	0.252778	99177.93	0.047396	0.931434	92377.72239
07/19/2024	10/21/2024	0.035049	94	0.261111	91516.7	0.045633	0.922987	84468.76604
10/21/2024	01/21/2025	0.032068	92	0.255556	81951.13	0.043970	0.915485	75025.0259
01/21/2025	04/22/2025	0.033547	91	0.252778	84798.56	0.042855	0.907787	76979.03207
04/22/2025	07/21/2025	0.031339	90	0.25	78347.57	0.041751	0.90073	70570.008
07/21/2025	10/20/2025	0.029135	91	0.252778	73647.6	0.040636	0.894145	65851.62206
10/20/2025	01/20/2026	0.026908	92	0.255556	68765.19	0.039508	0.888038	61066.11685
01/20/2026	04/20/2026	0.031584	90	0.25	78959.87	0.038931	0.881081	69570.06171
04/20/2026	07/20/2026	0.030432	91	0.252778	76924.5	0.038347	0.874355	67259.34132
07/20/2026	10/19/2026	0.029272	91	0.252778	73993.04	0.037764	0.867933	64221.01159
10/19/2026	01/19/2027	0.028107	92	0.255556	71829.85	0.037174	0.861743	61898.89483
01/19/2027	04/19/2027	0.031414	90	0.25	78535.85	0.036858	0.855028	67150.37024
04/19/2027	07/19/2027	0.030784	91	0.252778	77814.45	0.036538	0.848426	66019.82185
07/19/2027	10/19/2027	0.030146	92	0.255556	77040.15	0.036215	0.84194	64863.18105
10/19/2027	01/19/2028	0.029504	92	0.255556	75398.74	**0.03589**	0.835639	63006.15294

Table IV.24 NPV schedule for the floating leg of 5-years swap

A script can be quite helpful to iteratively solve the computation of the zero rates implied by a swap curve.

Time	Maturity	Market rate	Instruments	Zero rates	Discount Factors
t8	01/21/2025	4.4671%	Swap	4.3970%	0.915486
t9	01/20/2026	4.0218%	Swap	3.9508%	0.887943
t10	01/19/2027	3.7884%	Swap	3.7174%	0.861743
t11	01/19/2028	3.6594%	Swap	3.5892%	0.835639

Table IV.25 Long Term stripping

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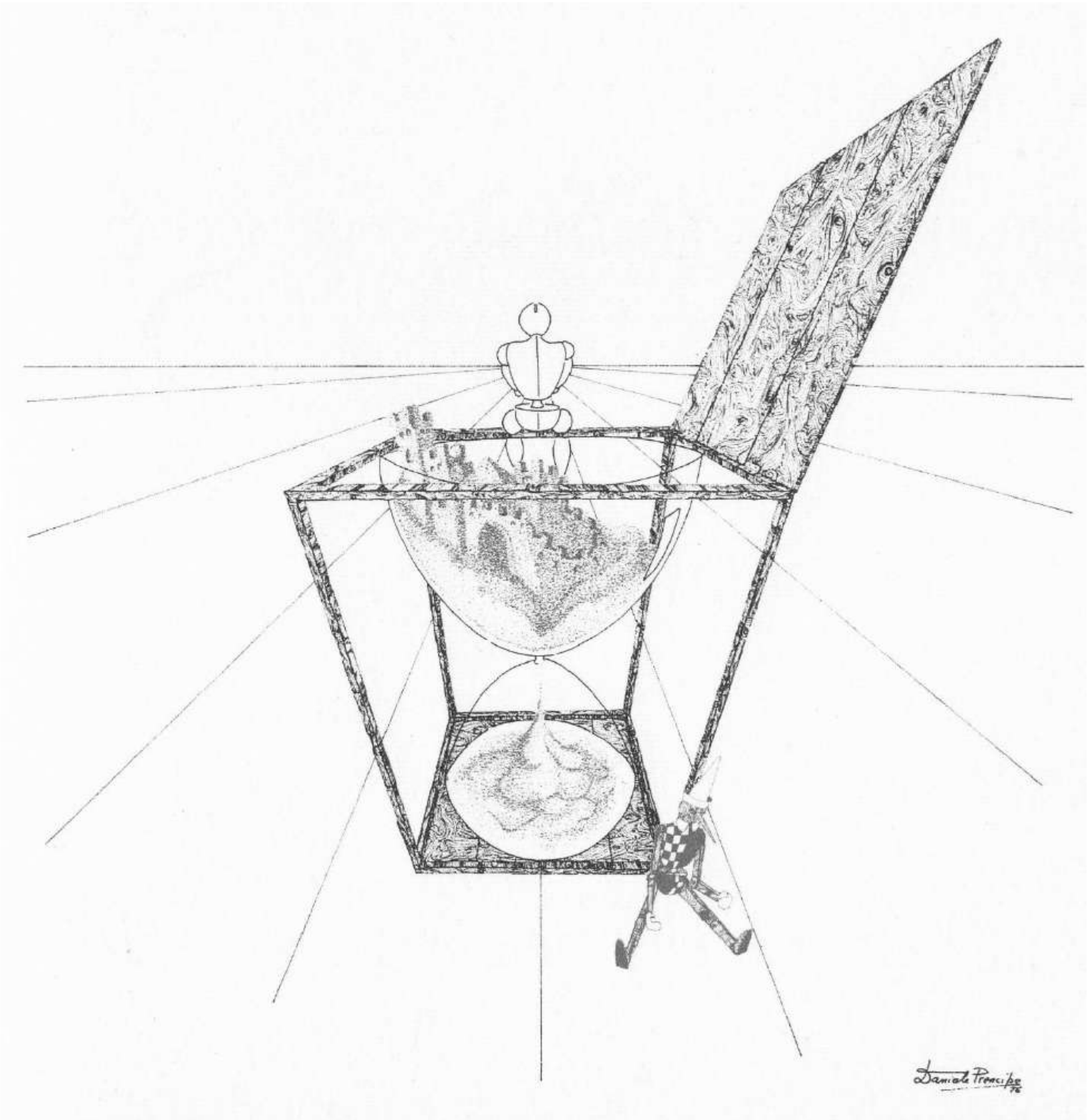

PART V: CREDIT DERIVATIVES

Chapter 1 – Credit Default Swap

Introduction and definitions
Classification of credit derivatives
Payoff and mechanics
Trigger events
Settlement
CDS market quotes
CDS and bond yields
Theoretical pricing and sensitivity
Probability term structure

V.1 CREDIT DEFAULT SWAP

A credit derivative is a financial contract whose pay-off is linked to the creditworthiness of one or more companies or states. Such instruments may be used for hedging, trading or as a means of investing in government/corporate credit. They help isolate credit risk and allow for its efficient transfer between market participants. In fact, the quick rise of credit derivatives constituted one of the most interesting market developments that marked the end of the 1990s. According to statistics from the International Swaps and Derivatives Association (ISDA), the notional of credit-related contracts in 2008 was about 54.6 trillion dollars, ten times as much as those present on the market at the beginning of 2001. Among the many types of credit derivatives, **Credit Default Swaps** (or CDS) on a single issuer are the most common.

A credit default swap is essentially a bilateral agreement in which one counterparty (**protection buyer**) pays a periodic premium to another counterparty (**protection seller**) to buy protection against losses arising from the default of the issuer of a security (**reference bond**) for a defined period of time. The Credit Default Swap acts as an **insurance** on a debt security: an investor who wants to cover the risk of insolvency generated by the issuer of the security in his portfolio (typically bankruptcy and debt restructuring) pays an insurance premium to a counterparty that intervenes only if the default event occurs. The premium paid by the protection buyer is often called **spread**, and it is quoted in basis points per annum and generally paid quarterly.

As an example, let us assume a trader holds a bond issued by company XYZ, and a 5-years CDS for company XYZ is quoted at 160 bps p.a. on January 10, 2018. This means that if he wanted to buy protection on the potential default of company XYZ for an exposure equal to USD 10 million, the trader would have to pay 40 basis points (or 0.40%) of 10,000,000 i.e., USD 40,000 each quarter as an insurance premium for the protection he receives. If the credit event does not occur for the insured period, the protection seller will not make any payment on the CDS. On the other hand, if the credit event does occur, the protection buyer stops paying the premium to the counterparty: it is sufficient that he pays the accrued premium up to the day of the credit event. This allows both counterparties to close out their respective positions immediately after the credit event has occurred, eliminating any administrative costs that might have arisen.

Although the most common and standardized CDS provide for quarterly payments, some swap contracts may provide for up-front payments. The mechanics of the derivative is outlined below.

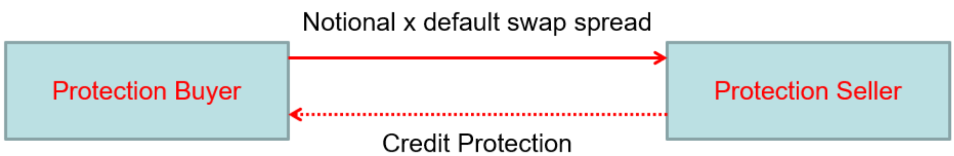

Figure V.1 CDS – Before the potential credit event

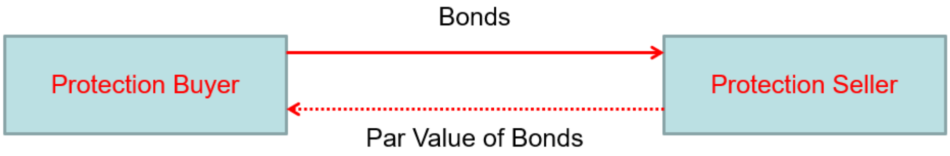

Figure V.2 CDS – After the credit event, Physical settlement

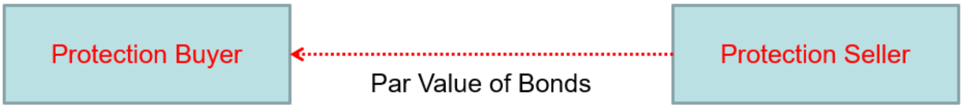

Figure V.3 CDS – After the credit event, Cash settlement

From a market-instruments perspective, an analyst can identify different types of CDS contracts:

- CDS: instrument that indicates that the protected underlying are senior unsecured bonds issued by corporates or government entities.

- LCDS: Loan-only CDS are contracts where protection is bought/sold on syndicated secured leveraged loans. They are characterized by higher recovery rates than standard CDS.

- MCDS: the protected entity is a municipality, so the reference security is a municipal bond.

- ABCDS: Asset Backed Security CDS.

From an economic perspective, all CDS are very similar to buying credit event insurance. These contracts can therefore also be considered closer to options than to swaps. The protection buyer essentially has the right to sell his bond at par if the credit event materializes, but the premium paid is not upfront. The precise definition of the credit event, the contractual duties, and the settlement mechanisms are negotiated between the counterparties and legally formalized upon inception of the "insurance" contract. To facilitate transactions, most of the CDS contracts are standardized by the ISDA, and the most common credit events, also called trigger events, have been classified. These are the most important:

Bankruptcy: it is the inability of a company to repay its debt. This serious insolvency must be formalized in a written and official form.

Failure to pay: this event occurs when the reference entity, after a certain grace period, fails to pay the principal and any interest due. Typically, a threshold amount is defined, and if the accumulation of debt exceeds this threshold, the institution enters this state.

Debt restructuring: this event occurs when the debt conditions are changed due to contingent difficulties. ISDA provides four options for dealing with the problem of correctly defining the debt restructuring from a contractual point of view:

No Restructuring: this option excludes the case of debt redefinition from the contract. In this way, the possibility for the protection seller of having to pay the buyer for a credit event less than the declared default is eliminated.

Full Restructuring: this option allows the protection buyer to deliver bonds of any maturity after the debt restructuring, whatever its form.

Modified Restructuring: once the debt restructuring has occurred, this option allows the protection buyer to deliver bonds with a maturity of less than 30 months. This form of insurance has become the most common standard in North America.

Modified Modified Restructuring: once the debt restructuring has occurred, this option allows the protection buyer to deliver bonds with a maturity of less than 60 months.

The first step undertaken when the credit event occurs is the official communication between the counterparties of the contract (**Credit Event Notice**). This formal communication details the reasons why the **trigger event** is proven to have occurred. Once this notice has been accepted by both counterparties, the contract enters its settlement phase.
The amount must be paid by the protection seller to the protection buyer by way of physical settlement or cash settlement, as specified in the contract.
The physical settlement is the market standard. Following the occurrence of the credit event, it provides that the protection buyer delivers the reference security to the protection seller in exchange for the notional amount of the bond, paid in cash. On the other hand, cash settlement only involves the exchange of cash flows. This last feature allows traders to take short credit positions.

Let us examine a practical example, considering a trader who has a USD 10 million bond which yields 5%. The issuer of this fixed income instrument is an automotive company. The creditworthiness of the issuer is deteriorating and this leads the trader to buy a 3-years CDS.
The spread at the time of stipulation is quoted on the market at 150 bps. If the firm does not go bankrupt, the protection buyer pays the protection seller USD 10,000,000 x 0.015 x 0.25 = USD 37,500 quarterly.

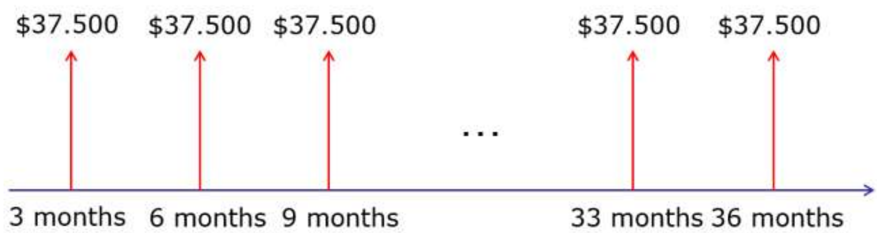

Figure V.4 No default – CDS cash flows from the protection seller perspective

In the event of default, the premium that the protection buyer must pay to the seller matures until the date on which the credit event occurs, while the protection seller must pay the notional amount of the security. Let us suppose the **recovery rate** is 40%.

In the event of default and assuming a cash settlement, the protection seller will have to pay (1-0.40) x 10,000,000 = USD 6,000,000 to the trader and will receive from him the premium accrued up to the default date. Under physical settlement, the trader would deliver USD 4,000,000 and receive USD 10,000,000.

Figure V.5 Default at t – CDS cash flow from the protection seller perspective

Figure V.6 CDS description. Source: Bloomberg®

Tenor	Name	MSG Quotes Bid	Ask	Time	CMAN Bid	Ask	Time	My PCS (Pricing Sources) Bid	Ask	Time	Source
1) Curve	ENELIM CDS EUR SR CRV										
2) 6 Mo	ENELIM CDS EUR SR 6M D14				26.55	45.64	01/27/23	31.90	44.34	14:00:55	CBIN
3) 1 Yr	ENELIM CDS EUR SR 1Y D14	36.00	56.00	09:04:40	33.70	58.15	01/27/23	36.00	56.00	09:04:40	MSG1
4) 2 Yr	ENELIM CDS EUR SR 2Y D14	40.00	58.00	09:04:40	40.58	60.89	01/27/23	40.00	58.00	09:04:40	MSG1
5) 3 Yr	ENELIM CDS EUR SR 3Y D14	57.00	74.00	09:04:40	61.11	74.25	01/27/23	57.00	74.00	09:04:40	MSG1
6) 4 Yr	ENELIM CDS EUR SR 4Y D14	80.00	92.00	09:04:40	80.30	89.92	01/27/23	80.00	92.00	09:04:40	MSG1
7) 5 Yr	ENELIM CDS EUR SR 5Y D14	102.00	110.00	14:29:42	100.36	109.28	01/27/23	102.00	110.00	14:29:42	MSG1
8) 7 Yr	ENELIM CDS EUR SR 7Y D14	128.00	144.00	09:04:40	127.75	139.75	01/27/23	128.00	144.00	09:04:40	MSG1
9) 10 Yr	ENELIM CDS EUR SR 10Y D14	154.00	174.00	09:04:40	153.57	170.45	01/27/23	154.00	174.00	09:04:40	MSG1

Figure V.7 CDS market quotes. Source: Bloomberg®

We now consider a portfolio that consists of a corporate bond maturing in five years with a yield (**par yield**) of 5%, and a long position in a 5-year CDS that costs 250 bps per year. This portfolio is approximately equivalent to a long position in a risk-free instrument paying 2.5% per annum.

The effect of the CDS is to "transform" a corporate bond into a risk-free bond. Indeed, if the issuer of the bond does not go bankrupt, the investor earns 2.5% per year (5%-2.5%=2.5%). If the debt issuer goes bankrupt, on the other hand, the investor would earn 2.5% until default and would obtain the full initial notional back thanks to the CDS. This amount can then be re-invested at the risk-free rate for the time between the credit event and the maturity.

Theoretically, the CDS spread, s, over n years should be very close to the excess yield between a corporate bond over n years (y) and a risk-free bond of the same maturity (r).

In mathematical terms: $s = y - r$.

If this were not the case, arbitrage opportunities would arise. It should be noted though that the above relationship is of an approximate nature for various reasons, among which:

- Market participants are not always allowed to go short on corporate bonds.

- CDS have a default risk of the counterparty selling protection.

- For tax or liquidity reasons, an investor could be not indifferent to buying a risk-free security rather than a corporate bond and a CDS.

- Arbitrage assumes that interest rates are constant over time.

Another interesting point is to define the risk-free interest rate to be used in the formula. In fact, bond traders usually derive the risk-free zero-rate curve from government bond yields, while derivative traders usually use the LIBOR zero curve.

The purpose of the valuation of a CDS is to obtain an equilibrium premium paid on a regular basis by the protection buyer towards the protection seller which equals the value of the two legs of the swap (premium leg and contingent leg).

To achieve this objective we need:

a) The probability of default estimation of the counterparty that issued the security.

b) An assumption on the recovery rate.

c) A financial model based on discounted cash flows.

Regarding point a), the analyst receives this data from the credit risk management office (see the part of the Notes dedicated to Credit Risk). This figure is provided via a term structure if the company is publicly traded. If such detailed market information is not available, the one-year probability of default is provided and prospective values are generated from it.

If we call the one-year default probability p_D, we obtain the term structures (unconditional default and survival probabilities) shown in the table below:

Time (year)	Default Probability	Survival Probability
1	p_D	$1 - p_D$
2	$p_D(1 - p_D)^1$	$(1 - p_D)^2$
3	$p_D(1 - p_D)^2$	$(1 - p_D)^3$
4	$p_D(1 - p_D)^3$	$(1 - p_D)^4$
5	$p_D(1 - p_D)^4$	$(1 - p_D)^5$

Table V.1 Default and Survival probability

For instance, for $p_D = 2\%$, the table becomes:

Time (year)	Default Probability	Survival Probability
1	0.02	0.98
2	0.0196	0.9604
3	0.019208	0.941192
4	0.01882384	0.92236816
5	0.018447363	0.903920797

Table V.2 Default and Survival probability: example

As regards point b), the recovery rate is established on the date of signing the CDS contract. It is usually set equal to 40%, $R = 0.4$.

Point c) concerns a model based on discounted cash flows. We assume a continuously compounded risk-free rate (LIBOR) of 1% and we calculate the discount factors based on the formula: $DF(t) = \exp(-r \cdot t)$.

Time (year)	Risk Free rate	Discount Factors
0.5	0.01	0.995012479
1	0.01	0.990049834
1.5	0.01	0.98511194
2	0.01	0.980198673
2.5	0.01	0.975309912
3	0.01	0.970445534
3.5	0.01	0.965605416
4	0.01	0.960789439
4.5	0.01	0.955997482
5	0.01	0.951229425

Table V.3 CDS Theoretical pricing: risk free rates and discount factors

We now have all the relevant information to calculate the CDS spread, s. The present value of the payments expected from the protection seller are summarized in the table:

Time	Survival Prob.	Expected Payment	Discount Factors	PV (Expected Payment)
1	0.98	0.98 x s	0.990049834	0.970248837074185 x s
2	0.9604	0.9604 x s	0.980198673	0.941382805843808 x s
3	0.941192	0.941192 x s	0.970445534	0.913375572611587 x s
4	0.92236816	0.92236816 x s	0.960789439	0.88620158713836 x s
5	0.903920797	0.9039207968 x s	0.951229425	0.859836059334291 x s
			Total	4.57104486200223 x s

Table V.4 CDS Theoretical pricing Expected Payments. Step 1

The table shows the calculation of the discounted value of payments made on the CDS at rate s on a symbolic notional of USD 1. For example, there is a 92.237% chance that the fourth payment of s will be made. The expected payment is therefore (probability x payment) = 0.92237 x s and its present value is:

0.92237 x s x DF (4) = 0.92237 x s x 0.96078 = 0.8862 x s.

We assume a recovery rate equal to 40%, $R = 0.4$ and that the issuer of the security defaults in the middle of the considered time interval (every 6 months). In this case, the expected present values of the pay-offs are summarized in the following table:

Time	Default Prob.	Rec. Rate	Expected Payoff	Discount Factors	PV (Expected Payment)
0.5	0.02	0.4	0.012	0.995012479	0.01194015
1.5	0.0196	0.4	0.01176	0.98511194	0.011584916
2.5	0.019208	0.4	0.0115248	0.975309912	0.011240252
3.5	0.01882384	0.4	0.011294304	0.965605416	0.010905841
4.5	0.018447363	0.4	0.011068418	0.955997482	0.01058138
				Total	0.056252539

Table V.5 CDS Theoretical pricing Expected Payments. Step 2

For example, there is a probability of default of 1.88% on the fourth year. Given a 40% recovery rate, the expected payoff at this time is 0.0188 x (1-0.4) x USD 1 = USD 0.0113.

The present value of the expected pay-off is equal to 0.0113 x DF (3.5) = 0.0113 x 0.9656 = USD 0.01091.

The total value of all discounted expected payoffs is USD 0.05625.

The last step is to estimate the accrued premium of payments in the event of default. As shown in the table, for example, there is a 1.882% probability that there will be a six-month accrued premium payment in the middle of the third year.

The accrued premium is equal to 0.5 x s.

The expected value of the payment at this time is 0.01882 x 0.5 x s = 0.00941 x s. Its present value is 0.00941 x s x DF (3.5) = USD 0.009088.

The present value of the sum of all cash flows is 0.046877 x s.

Time	Default Prob.	Expected Accrual Payments	Discount Factors	PV (Expected Accrual Payment)
0.5	0.02	0.01 x s	0.995012479	0.00995012479192682 x s
1.5	0.0196	0.0098 x s	0.98511194	0.00965409700811001 x s
2.5	0.019208	0.009604 x s	0.975309912	0.00936687639512011 x s
3.5	0.01882384	0.00941192 x s	0.965605416	0.00908820092938291 x s
4.5	0.018447363	0.0092236816 x s	0.955997482	0.0088178163828303 x s
			Total	0.0468771155073701 x s

Table V.6 CDS Theoretical pricing Expected Payments. Step 3

The discounted value of all expected cash flows for the CDS is therefore equal to:

4.571 x s + 0.0469 x s = 4.6179 x s

The discounted value of the expected payoff is equal to USD 0.05625, as seen above. The value of the equilibrium CDS spread is therefore:

4.6179 x s = 0.05625   s=0.05625/4.6179=0.01218 thus s=121.8 bps.

It should be noted that the proposed example is simplified: in a real market context, payments take place more frequently than annually (typically quarterly) and the default can occur at any time and not only in the middle of the period.

Figure V.8 CDSW (Credit Default Swap Valuation) module. Source: Bloomberg®

FURTHER READINGS

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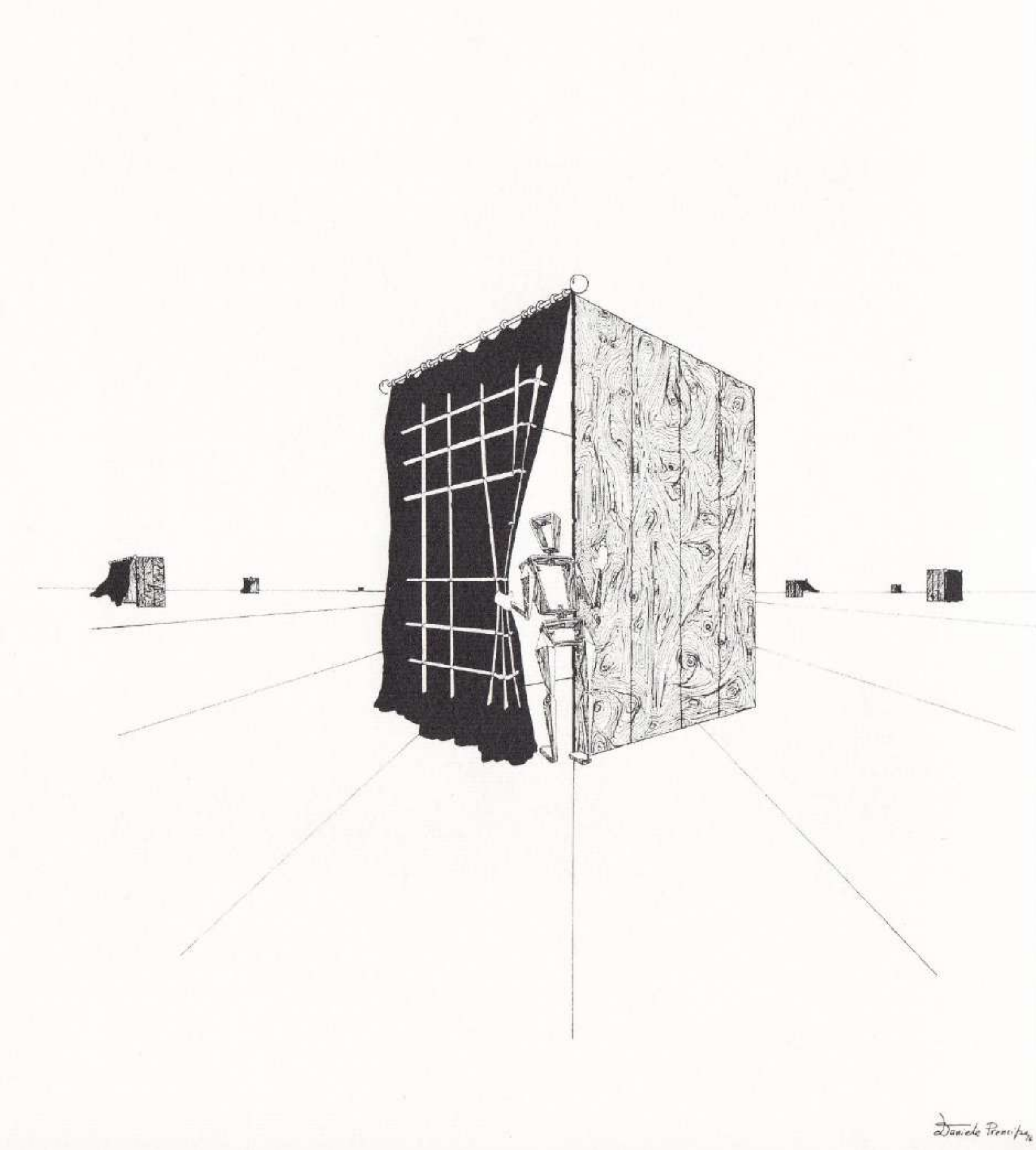

PART VI: INFLATION DERIVATIVES

Chapter 1 – Inflation indexed swaps

Zero-Coupon Inflation-Indexed Swap (ZCIIS)
Year-on-Year Inflation-Indexed Swap (YYIIS)
CPI simulation
Historical and normalized seasonality
Daily and monthly CPI interpolation
Exotic IIS: "BTP Italia" case study

VI.1 INFLATION INDEXED SWAPS

An Inflation-Indexed Swap (IIS) is a swap deal in which, for each future payment date, $T_1, \ldots, T_M$, counterparty A pays the inflation rate of the relevant period to counterparty B, while counterparty B pays a fixed rate to counterparty A (or a floating rate typically indexed to EURIBOR plus a spread). The inflation rate is calculated as the percentage return of the CPI (Consumer Performance Index) over the reference time frame. So, like any swap, this one is also composed of two legs, of which the non-inflation-indexed one is valued using the traditional formulas used for IRS (Interest Rate Swap) legs.
Therefore, in this section we will only focus on the pricing of the exotic leg indexed to the CPI. The valuation of the instrument will then be given by the algebraic sum of the prospective discounted cash flows generated by the two legs.
Generally, there are two main types of IIS: the Zero-Coupon Inflation-Indexed Swap (ZCIIS) and the Year-on-Year Inflation-Indexed Swap (YYIIS).
In a ZCIIS on the maturity date T_M, counterparty A pays a variable amount to counterparty B equal to:

$$CF_M = N\left[\frac{I(T_M)}{I_0} - 1\right] \quad \textit{(Eq. VI.1)}$$

Where I_0 is the baseline CPI and $I(T_M)$, or written in short form I_M, is the CPI value at time T_M.
Let us note that an inflation bond, which can form a leg of an IIS, can be modeled as a portfolio of inflation Zero Coupons, similarly as a bullet bond can be represented as a portfolio of Zero Coupon Bonds.
Eq. VI.1 can therefore be rewritten to determine a cash flow of an inflation coupon bearing bond characterized by a generic range T_i:

$$CF_i = N_i \cdot \frac{I(T_i)}{I_0} \cdot C_i \cdot \varphi_i \quad \textit{(Eq. VI.2)}$$

φ_i is the year fraction calculated according to the stipulated day base between the previous payment date (T_{i-1}) and the next one (T_i). If no cash flow exchanges have occurred yet, the start date is considered, or in any case, what is specified in the contractual terms of the derivative. In government bonds characterized by a semi-annual coupon payment frequency, it is often set equal to 0.5 for simplicity. In a YYIIS, counterparty A pays a periodic cash flow to counterparty B at each T_i equal to:

$$CF_i = N_i \cdot \left[\frac{I(T_i)}{I(T_{i-1})} - 1\right] \cdot \varphi_i \quad \textit{(Eq. VI.3)}$$

As clearly shown in Eq. VI.3 in the case of a YYIIS, the inflation base, $I(T_{i-1})$, is not fixed, but it varies with each payment date (roll-over of the inflation base).
It should be noted that, regardless of the financial instrument analyzed, it is necessary to have a model capable

of conducting projections of the CPI over time in accordance with market prices.
If a series of listed ZCIIS with a maturity T_M is available, the prospective value of the CPI at the same time T_M, can be derived applying the formula:

$$I_M(0) = I_{REF}(0) \cdot [1 + K(T_M)]^{T_M} \quad \textit{(Eq. VI.4)}$$

Where:
$I_M(0)$ is the prospective value of the CPI at time T_M estimated at time zero of the valuation. Unless otherwise specified, in order not to weigh down indexation, the abbreviated notation I_M will be used, assuming the observation of inflation data at the current time of valuation.
$I_{REF}(0)$ is the CPI reference value at time zero of the valuation. This corresponds to the level of inflation measured n months before the settlement date. The standard lag for inflation instruments is n = 3 months.
$K(T_M)$ is the fixed interest rate (coupon) between the settlement date and the maturity of the non-inflation-indexed leg of the ZCIIS which allows the financial balance of the financial instrument. In other words, it is the interest rate to be applied to the fixed leg that allows the total NPV of the ZCIIS to be zeroed. This makes it possible to intuitively derive Eq.VI.4 as:

$$NPV_{REC} = NPV_{PAY} \rightarrow N[[1 + K(T_M)]^{T_M} - 1] = N\left[\frac{I(T_M)}{I_0} - 1\right] \rightarrow [1 + K(T_M)]^{T_M} = \frac{I(T_M)}{I_0} \rightarrow$$

$$I(T_M) = I_0[1 + K(T_M)]^{T_M}$$

T_M is the time to maturity of the ZCIIS considered for the estimation of I_M.
A rigorous demonstration of the pricing formula for a ZCIIS and a YYIIS derivative is provided in the Brigo and Mercurio milestone.

Market contributions for $K(T_M)$ have a yearly frequency until the tenth year, then the frequency becomes every five years until the thirtieth year. For certain countries, as in Europe, Bloomberg also contributes the fiftieth year. Since the contributions do not have a sufficient granulometry to have simulated prospective values of I for each month, which is typical of the frequency with which the inflation figure is published, it is necessary to use an interpolation to derive these values. The commonly adopted approach is to estimate a constant monthly intra-period inflation increase equal to:

$$\Delta I_{T_M,T_{M+1}} = \frac{\ln\left(\frac{I_{M+1}}{I_M}\right)}{12 \cdot \varphi_{M+1}} \quad \textit{(Eq. VI.5)}$$

φ_{M+1} is the year fraction calculated according to the day basis between the previous (T_M) and the following (T_{M+1}) ZCIIS contribution date. For instance, in the case of European, Italian and French inflation, the day basis that is used is ACT/360.

Considering Eq. VI.5 in the time range between T_M and T_{M+1}, the inflation values can be simulated with the traditional monthly granulometry applying the following relation:

$$I_{i+1} = I_i \exp\left(\Delta I_{T_M,T_{M+1}}\right) \textit{ (Eq. VI.6)}$$

with $I_M < I_i < I_{M+1}$ and i ranging from 1 to the number of months between I_M and I_{M+1}.

Inflation is characterized by seasonality, so it is good market practice to consider it in order to obtain more precise forward values, in line with macroeconomic expectations.
The standard approach consists of calculating the normalized monthly residuals obtained from the historical CPI values according to the following formula:

$$\mathcal{R}_i = \frac{\sum_{j=1}^{S} \ln\left[\frac{I_i^j}{I_{i-1}^j}\right]}{S} - \bar{I} \textit{ (Eq. VI.7)}$$

Where:
$\mathcal{R}_i$ is the normalized residual for month i, therefore $i = 1, \dots, 12$. Intuitively, 1 corresponds to the month of January, 2 to the month of February, and so on.
S is the number of years considered to calculate historical seasonality. In the standard model, 5 years are considered, equal to the value suggested by the Bloomberg® SWIL inflation module.
I_i^j is the CPI recorded in year j and for month i.
I_{i-1}^j is the CPI recorded in year j an for month $i-1$. Clearly, for the calculation of $\mathcal{R}_1$, the value of I_{i-1}^j is equal to that of December of the previous year ($I_0^j = I_{12}^{j-1}$).
$\bar{I} = \frac{\sum_{k=1}^{12 \cdot S} \ln\left[\frac{I_k}{I_{k-1}}\right]}{12 \cdot S}$ is the great mean of all logarithmic returns of inflation values over the considered time period of 12 months for the total number of years considered for the seasonality of the model. Thus, in the standard market case, the contribution $\bar{I}$ corresponds to the average of the returns of the last 5 years.

The first term of equation Eq. VI.7 is the historical monthly residual, i.e. the logarithmic variation in CPI values is measured exclusively in the considered month, while the second term $\bar{I}$ represents the overall average logarithmic variation over the entire time period for the calculation of seasonality. The difference between these two contributions gives rise to the normalised residual for the considered month i.
A special case for the calculation of residuals occurs when CPI values are present between the reference date and the valuation date. In this circumstance, and only for the first time interval from the valuation date to the date of the first contribution of $K(T_M)$, an estimate of $\mathcal{R}$ is made, excluding from the calculation the contributions of inflation for the months between the reference month and the current month. For example, if we wanted to evaluate an inflation derivative at the end of December, reasonably the CPI values between the

reference month (if the lag is the market standard it would therefore be September) and the current period would already have been published. In particular, the CPI values for the months of October and November are expected to be available. Referring to Eq. VI.7, for the first contribution, i.e. the historical residuals, the $\mathcal{R}$ of the months between the reference date and the present time are set to zero (in the example $\mathcal{R}_i = 0, i = \{10,11\}$) and consequently in the great mean of all returns, $\bar{I}$, the contributions of the months discarded in the calculation of the historical residuals must be excluded. Obviously, it should be highlighted that this particular expedient, also known in jargon as "truncated seasonality", only applies for the first time interval that goes up to the first market contribution of the ZCIIS. After the first $K(T_M)$, formula Eq. VI.7 in its full version continues to be applied for all twelve months.
A linear reproportioning that considers the missing months exclusively for the first period must also be done for the interpolation of $\Delta I_{T_M,T_{M+1}}$. In order to take into account the seasonality $\mathcal{R}_i$, Eq. VI.6 for the monthly inflation estimate is generalized as follows:

$$I_{i+1} = I_i \exp\left(\Delta I_{T_M,T_{M+1}} + \mathcal{R}_{MONTH(I_{i+1})}\right) \quad \textit{(Eq. VI.8)}$$

with $I_M < I_i < I_{M+1}$ and i ranging from 1 to the number of months between I_M and I_{M+1}.

The "$MONTH$" function inserted as a subscript of $\mathcal{R}_{MONTH(I_{i+1})}$, intuitively allows to derive the month to which the CPI projection refers, I_{i+1}.
Given the way normalized residuals have been constructed, it is important to note that the forward inflation values simulated using the recursive formula Eq. VI.8 continue to generate CPI values for the various maturities of the ZCIIS that are internally consistent with Eq. VI.4 and, therefore, in line with market expectations.
It is evident that if we set all monthly normalized residuals to zero, Eq. VI.8 degenerates into Eq. VI.6. This recursive formula allows to derive all the forward levels of inflation on a monthly basis up to the last maturity date for which the ZCIIS are contributed by the market. Having these values, the inflation values that appear in formulas Eq. VI.1, Eq. VI.2 and Eq. VI.3 can be calculated for the determination of the cash flows of the inflation-indexed leg for the different types of IIS. Clearly, once the valuation for both swap legs has been calculated, the overall valuation is given by the algebraic sum of the contributions of the two legs.
Up to this point in the discussion, the main types of IIS present in the global financial markets have been discussed, i.e. the ZCIIS and YYIIS, which allow us to outline the fundamental principles for the pricing of this type of inflation-linked derivatives.

The equations presented have been implemented in a numerical environment like Matlab, which allows the valuation of a hedge of a generic inflation-linked security in accordance with the pricing framework currently considered as the market standard.
If a Bloomberg calculation module (SWPM) that replicates the calculation is present for the observed hedging, the result obtained by the program will be compared to the market benchmark. This comparison allows to validate the programmed pricing routines and to be confident about the correct implementation of the standard valuation framework. All market data have a reference date of 30 June 2023, and the source of market data is

the info-provider Bloomberg®. The validation process of the models begins with the derivation of the forward values of inflation corresponding to the quotations of the forward-looking ZCIIS.
Let us consider the market data relating to the ZCIIS of Italian inflation shown in Figure VI.1 with a reference date of 30 June 2023.

1) Curve | 2) Convexity & Seasonality | 3) YOY Volatility | 4) ZC Volatility | 5) Bonds

Mode User(Default) · Country Italy · Date 06/30/23 · Market Side Mid · CPI Index 11) ITCPIUNR
Zero Coupon Curve · Type Contributed · Contributor BGN
Lag 3 Months · Interpolation Monthly · Base Index 3/2023 118

Tenor	ZC Bid	ZC Mid	ZC Ask	CPI Bid	CPI Mid	CPI Ask
1 YR	0.34450	0.82375	1.30300	118.40651	118.97203	119.53754
2 YR	1.72646	1.88250	2.03854	122.10962	122.48452	122.85999
3 YR	1.84479	2.00500	2.16521	124.65178	125.24096	125.83199
4 YR	1.86249	1.99450	2.12651	127.03961	127.69945	128.36185
5 YR	1.91825	2.03250	2.14675	129.76028	130.48922	131.22144
6 YR	1.95480	2.04450	2.13420	132.53419	133.23539	133.93968
7 YR	1.96718	2.06750	2.16782	135.23991	136.17405	137.11371
8 YR	1.94071	2.02250	2.10429	137.61418	138.49998	139.39076
9 YR	1.97866	2.08000	2.18134	140.75560	142.01949	143.29347
10 YR	2.02345	2.11252	2.20159	144.17245	145.43601	146.70953
12 YR	2.05456	2.13564	2.21672	150.61588	152.05811	153.51298
15 YR	2.06959	2.16550	2.26142	160.44539	162.72190	165.02853
20 YR	2.10441	2.20153	2.29865	178.96647	182.40215	185.90043
25 YR	2.14594	2.23225	2.31855	200.63646	204.91779	209.28676
30 YR	2.17679	2.28243	2.38807	225.13844	232.22723	239.53155

Figure VI.1 Contribution of the ZCIIS linked to Italian inflation $K(T_M)$ and strip of the related forward values of the CPI. Reference date 30 June 2023. Source: Bloomberg® – SWIL module

The figure above shows the value of the inflation base, $I_{REF}(0)$ which is equal to the March 2023 CPI, i.e. a lag of three months is applied prior to the reference date, the market contributions for Zero Coupons, $K(T_M)$ for the different maturities of the term structure, $T_M = \{1,2,3,4,5,6,7,8,9,10,12,15,20,25,30\}$ and the related prospective values of the CPI calculated according to Eq. VI.4.

The output of the strip recursive formula is perfectly aligned with the values of the SWIL module (see Table VI.1). A similar procedure and similar results are obtained for European inflation, which data and values are shown in Figure VI.2 and Table VI.2.

Tenor	T	ZC Mid	CPI Mid	Model Gap
1Y	1	0.82375	118.972	0.00000
2Y	2	1.8825	122.4845	0.00000
3Y	3	2.005	125.241	0.00000
…	…	…	…	…
30Y	30	2.28243	232.2272	0.00000

Table VI.1 Forward CPI implied by the Italian ZCIIS: CPI Mid = CPI Ref * (1+ZC Mid)^T. CPI Ref = 118

Tenor	ZC Bid	ZC Mid	ZC Ask	YoY Bid	YoY Mid	YoY Ask	CPI Bid	CPI Mid	CPI Ask	Cvx (bp)
1 YR	2.73501	2.76000	2.78499	2.73501	2.76000	2.78499	125.33671	125.36720	125.39769	0.00
2 YR	2.51807	2.53880	2.55953	2.52108	2.54188	2.56269	128.22145	128.27331	128.32517	0.00
3 YR	2.47307	2.49630	2.51953	2.47647	2.49968	2.52288	131.27713	131.36643	131.45577	0.00
4 YR	2.46225	2.48625	2.51025	2.46528	2.48923	2.51318	134.46688	134.59293	134.71907	0.00
5 YR	2.45378	2.47500	2.49622	2.45667	2.47799	2.49931	137.72087	137.86355	138.00635	0.00
6 YR	2.44649	2.46900	2.49151	2.44941	2.47192	2.49444	141.04000	141.22605	141.41231	0.00
7 YR	2.44999	2.47100	2.49201	2.45226	2.47338	2.49451	144.52511	144.73270	144.94054	0.00
8 YR	2.45204	2.47300	2.49396	2.45388	2.47495	2.49601	148.08964	148.33220	148.57511	0.00
9 YR	2.45911	2.48125	2.50339	2.46007	2.48219	2.50431	151.81514	152.11063	152.40663	0.00
10 YR	2.46736	2.49000	2.51264	2.46730	2.48986	2.51243	155.67369	156.01802	156.36304	0.00
12 YR	2.49865	2.51880	2.53895	2.49462	2.51500	2.53538	164.05054	164.43794	164.82617	0.00
15 YR	2.54941	2.57100	2.59259	2.53828	2.55984	2.58140	177.97431	178.53720	179.10174	0.00
20 YR	2.60475	2.62750	2.65025	2.58561	2.60812	2.63063	204.03703	204.94382	205.85444	0.00
25 YR	2.64812	2.67000	2.69187	2.62222	2.64408	2.66594	234.49595	235.74847	237.00741	0.00
30 YR	2.68368	2.70380	2.72392	2.65195	2.67249	2.69303	270.02396	271.61591	273.21693	0.00
50 YR	2.90156	2.92170	2.94184	2.81840	2.83835	2.85830	509.86850	514.88161	519.94301	0.00

Figure VI.2 Contribution of the ZCIIS linked to European inflation $K(T_M)$ and strip of the related forward values of the CPI. Reference date 30 June 2023. Source: Bloomberg® – SWIL module

Tenor	T	ZC Mid	CPI Mid	CPI Mid (Bloo)	Model Gap
1Y	1	2.76	125.3672	125.3672	0.00000
2Y	2	2.5388	128.2733	128.27331	0.00000
3Y	3	2.4963	131.3664	131.36643	0.00000
4Y	4	2.48625	134.5929	134.59293	0.00000
5Y	5	2.475	137.8636	137.86355	0.00000
6Y	6	2.469	141.2261	141.22605	0.00000
7Y	7	2.471	144.7327	144.7327	0.00000
8Y	8	2.473	148.3322	148.3322	0.00000
9Y	9	2.48125	152.1106	152.11063	0.00000
10Y	10	2.49	156.018	156.01802	0.00000
12Y	12	2.5188	164.4379	164.43794	-0.00001
15Y	15	2.571	178.5372	178.5372	-0.00002
20Y	20	2.6275	204.9438	204.94382	0.00000

Table VI.2 Forward CPI implied by the European ZCIIS. CPI Ref = 122

The next step is to calculate the prospective values for the CPI with the correct monthly granulometry, which corresponds to the frequency of publication of the index. The constant monthly intra-period inflation increase, $\Delta I_{T_M,T_{M+1}}$, is calculated, according to Eq. VI.4 for both time series. The values thus calculated are shown in Table VI.3.

T	EU CPI Mid	ITA CPI Mid	EU ΔI	ITA ΔI	EU Model Gap	ITA Model Gap
0	122	118				
1	125.3672	118.97203	0.002269	0.000684	**0.000207974**	**0.000370457**
2	128.27331	122.48452	0.00191	0.002425	0.00000000	0.00000000
3	131.36643	125.24096	0.001986	0.001855	0.00000000	0.00000000
…	…	…	…	…	…	…
25	235.74847	204.91794	0.002334	0.00194	0.00000000	0.00000000
30	271.61591	232.22722	0.00236	0.002085	0.00000000	0.00000000

Table VI.3 Monthly intra-period inflation increases for the Italian and European CPI, $\Delta I_{T_M,T_{M+1}}$ with no values between the Reference CPI Date and the valuation date.

It is worth to note that the $\Delta I_{T_M,T_{M+1}}$ values, for $T_M \geq 2$, record a very low error compared to the one committed for $T_M = 1$. The reason for this discrepancy is that we have worked under the hypothesis of the absence of values between the CPI reference date (i.e. March 2023) and the valuation date (i.e. June 2023).
If we investigate further on this aspect, we notice that two further values are provided for March and April: 122.79 and 122.81 for the CPTFEMU Index and 118.4 and 118.6 for the ITCPI index.


Eurostat Eurozone HICP Ex Tobacco Unrevised Series NSA
Range 04/30/2004 - 06/30/2023 Period Monthly
Market Last Price Mid Line Currency
View Price Table

Date	Last Price	Mid Line	Date	Last Price
12/31/23			12/31/22	120.240000
11/30/23			11/30/22	120.700000
10/31/23			10/31/22	120.790000
09/30/23			09/30/22	118.990000
08/31/23			08/31/22	117.550000
07/31/23			07/31/22	116.830000
06/30/23			06/30/22	116.700000
05/31/23	122.810000	122.810000	05/31/22	115.740000
04/30/23	122.790000	122.790000	04/30/22	114.780000
03/31/23	122.000000	122.000000	03/31/22	114.120000
02/28/23	120.940000	120.940000	02/28/22	111.350000
01/31/23	119.960000	119.960000	01/31/22	110.300000

Italy CPI FOI Ex Tobacco NSA
Range 04/30/2004 - 06/30/2023 Period Monthly
Market Last Price Mid Px Currency
View Price Table

Date	Last Price	Mid Px	Date	Last Price
12/31/23			12/31/22	118.200000
11/30/23			11/30/22	117.900000
10/31/23			10/31/22	117.200000
09/30/23			09/30/22	113.500000
08/31/23			08/31/22	113.200000
07/31/23			07/31/22	112.300000
06/30/23			06/30/22	111.900000
05/31/23 H	118.600000	118.600000	05/31/22	110.600000
04/30/23	118.400000	118.400000	04/30/22	109.700000
03/31/23	118.000000	118.000000	03/31/22	109.900000
02/28/23	118.500000	118.500000	02/28/22	108.800000
01/31/23	118.300000	118.300000	01/31/22	107.700000

Figure VI.3 Contribution of Eurozone HICP Ex Tobacco and Italy CPI Ex Tobacco. Source: Bloomberg®

It is important to consider this information for the CPI projections, as a result, the formula has to be scaled only for the first period, i.e. $T_M = 1$.

T	EU CPI Mid	ITA CPI Mid	EU Delta I	ITA Delta I	EU Gap	ITA Gap
2 Months	**122.81**	**118.6**				
1	125.3672	118.97203	0.002061	0.000313	0.00000000	0.00000000
2	128.27331	122.48452	0.00191	0.002425	0.00000000	0.00000000
3	131.36643	125.24096	0.001986	0.001855	0.00000000	0.00000000
…	…	…	…	…	…	…
20	204.94382	182.40215	0.002299	0.001903	0.00000000	0.00000000
25	235.74847	204.91794	0.002334	0.00194	0.00000000	0.00000000
30	271.61591	232.22722	0.00236	0.002085	0.00000000	0.00000000

Table VI.4 Monthly intra-period inflation increases for the Italian and European CPI, $\Delta I_{T_M,T_{M+1}}$ with two values between the Reference CPI Date and the valuation date.

The following step concerns the modelling of seasonality according to the standard market methodology. Figures VI.4 and VI.5 represent the estimation provided by Bloomberg® SWIL computation module for the European and the Italian case, respectively.

Figure VI.4 Historical and normalized seasonality for European inflation, Source: Bloomberg® - SWIL


Date 06/30/23 Market Side Mid CPI Index 11) ITCPIUNR
Seasonality
Apply Seasonality
Seasonality Type: Historical
Years of Average: 5

Month	Historical	Normalized
Jan	0.004729	0.002249
Feb	0.002370	-0.000110
Mar	0.002334	-0.000147
Apr	0.001085	-0.001395
May	0.001583	-0.000897
Jun	0.002918	0.000438
Jul	0.001874	-0.000607
Aug	0.004695	0.002214
Sep	-0.003363	-0.005843
Oct	0.007562	0.005081
Nov	0.001743	-0.000737
Dec	0.002234	-0.000246

Figure VI.5 Historical and normalized seasonality for Italian inflation, Source: Bloomberg® - SWIL

The code replication is conducted implementing relation Eq. VI.7 and it is shown in Table VI.5. The model gap measured as the difference between the values is less than the sixth decimal place in all cases.

Month	ITA Hist. Model	ITA Norm. Model	EUR Hist. Model	EUR Norm. Model
1	0.004729476	0.002248967	-0.004238252	-0.007066837
2	0.002370192	-0.000110317	0.004928341	0.002099756
3	0.002333568	-0.000146941	0.011557235	0.008728651
4	0.001085469	-0.001395041	0.005579979	0.002751394
5	0.001582952	-0.000897557	0.002241828	-0.000586757
6	0.002919389	0.00043888	0.003414507	0.000585922
7	0.001873697	-0.000606812	-0.002354244	-0.005182829
8	0.0046948	0.002214291	0.001691948	-0.001136637
9	-0.003362644	-0.005843153	0.004916095	0.00208751
10	0.007561807	0.005081298	0.005826278	0.002997693
11	0.001743094	-0.000737415	-0.00104946	-0.003878045
12	**0.002234309**	-0.0002462	0.001428763	-0.001399822

Table VI.5 Calculation of Italian and European seasonality, $\mathcal{R}_i$

For instance, in order to compute $\mathcal{R}_{i=12}$ for the Italian CPI, we have to compute the log-returns for all the Decembers in the sample period (the market standard convention is to consider the last 5 years).

The average of these five returns constitute the historical seasonality.

In order to get the normalized historical seasonality we have to subtract $\bar{I}$, i.e. the average of all the log-returns on the considered period.

In this case, considering the 60 past CPI values $\bar{I} = 0.0024805$.

Consequently $\mathcal{R}_{i=12} = 0.0022343 - 0.0024805 = -0.0002462$.

It is worth to note that the sum of all the twelve normalized values return one, $\sum_{i=1}^{12} \mathcal{R}_i = 1$.

Date	ITA CPI	logreturns
30/11/2018	102.2	
31/12/2018	102.1	-0.00098
30/11/2019	102.3	
31/12/2019	102.5	0.001953
30/11/2020	102	
31/12/2020	102.3	0.002937
30/11/2021	105.7	
31/12/2021	106.2	0.004719
30/11/2022	117.9	
31/12/2022	118.2	0.002541
	Average	**0.0022343**

Table VI.6 Example for the calculation of the historical seasonality

Once the contributions $\Delta I_{T_M,T_{M+1}}$ and $\mathcal{R}_i$ have been estimated, the recursive formulas Eq. VI.6 and Eq. VI.8 can be applied in order to derive the prospective values of the CPI with monthly frequency, respectively without and with the contribution of seasonality. For illustrative purposes, Figure VI.6 highlights the inflation values directly derived from the ZCIIS on European inflation using black dots, while the green dot shows the reference inflation of March 2023. It is essential that all market-oriented simulation models provide forward-looking CPI values that necessarily pass through these points, otherwise market expectations would be disregarded. The red line represents the projection of the CPI without the seasonal contribution calculated according to Eq. VI.6, while the blue line also considers normalized residuals and it is estimated using Eq. VI.8. As we have all I_i values for a generic month i, we can valuate inflation-linked financial products and consequently design the appropriate hedging swap. Here we only analyze the hypothetical inflation leg of an IIS though.

The first valuation concerns the inflation leg consisting of an IIS that hedges a bond issued on 30 June 2023 and maturing on 30 June 2033. Its financial characteristics are summarized in Figure VI.7.
Assuming that we want to evaluate the leg of a hedging swap at the reference date of the market data (30 June 2023) with a reference notional equal to EUR 10 million, we obtain the flows shown in Table. VI.7.
The formula for calculating cash flow Payment CF_i is Eq. VI.2 which corresponds to the table value identified with [C].
The inflation base I_0, i.e. the inflation level at the date of issuance of the bond, is 92.21738.
$I(T_i)$ corresponds to the prospective value of the European CPI recorded three months before the payment date and reported in column [D].
Therefore, the ratio $\frac{I(T_i)}{I_0}$ called index ratio is shown in column [B].

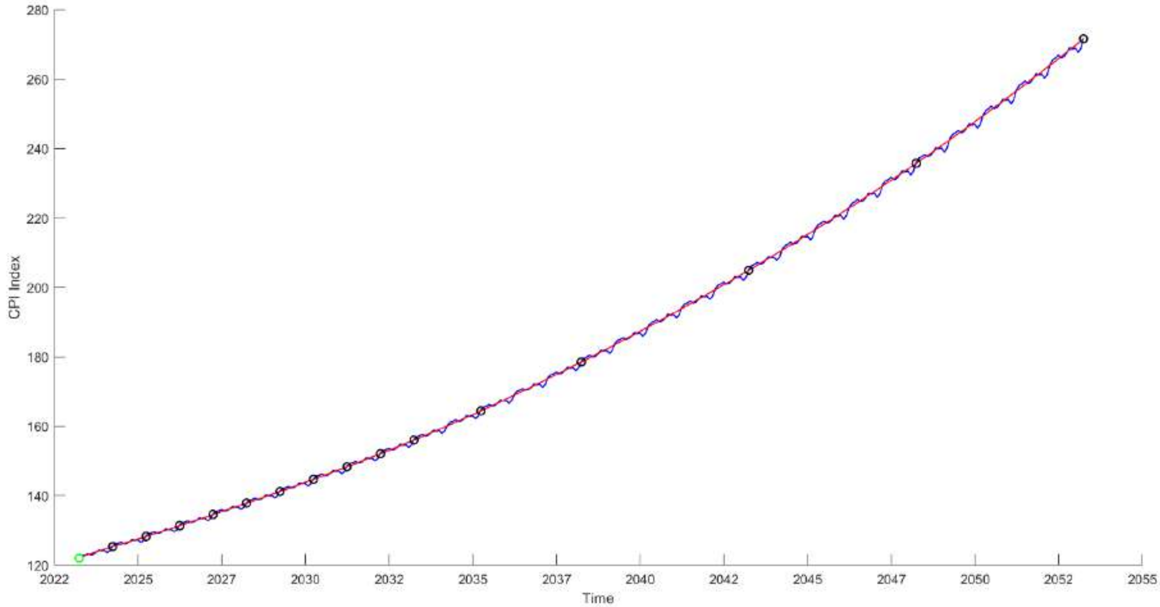

Figure VI.6 Simulation of the forward values of the European CPI using the market standard method

Figure VI.7 Characteristics of the BTPi security with ISIN code: IT0004545890. Source: Bloomberg®

Pay Date	Days	Real Notional [A]	Index Ratio [B]	Payment [C]	CPI [D]	Discount [E]	Zero Rate [F]	PV [G]
09/15/2023	180	13355631.98	1.33556	170284.31	123.1621	0.992474	3.581046	169002.74
03/15/2024	180	13451491.22	1.34515	171506.51	124.0461	0.973453	3.791694	166953.58
09/16/2024	180	13710515.73	1.37105	174809.08	126.4348	0.955236	3.764864	166983.85
03/17/2025	180	13778495.29	1.37785	175675.81	127.0617	0.939146	3.660775	164985.16
09/15/2025	180	14031532.01	1.40315	178902.03	129.3951	0.925061	3.518775	165495.38
03/16/2026	180	14107529.29	1.41075	179871	130.0959	0.912198	3.388176	164077.92
09/15/2026	180	14371451.37	1.43715	183236	132.5298	0.899987	3.278935	164910
03/15/2027	180	14452446.48	1.44524	184268.69	133.2767	0.888927	3.173951	163801.35
09/15/2027	180	14723491.83	1.47235	187724.52	135.7762	0.877929	3.089677	164808.78
03/15/2028	180	14804585.45	1.48046	188758.46	136.524	0.86735	3.020007	163719.66
09/15/2028	180	15081605.57	1.50816	192290.47	139.0786	0.856798	2.962814	164754.05
03/15/2029	180	15165337.29	1.51653	193358.05	139.8508	0.846563	2.915988	163689.75
09/17/2029	180	15451105.28	1.54511	197001.59	142.486	0.835989	2.879183	164691.07
03/15/2030	180	15540224.36	1.55402	198137.86	143.3079	0.825672	2.853817	163596.9
09/16/2030	180	15834910.79	1.58349	201895.11	146.0254	0.815075	2.832396	164559.57
03/17/2031	180	15926554.55	1.59266	203063.57	146.8705	0.804679	2.815723	163400.95
09/15/2031	180	16231110.75	1.62311	206946.66	149.6791	0.794334	2.802328	164384.72
03/15/2032	180	16329847.04	1.63298	208205.55	150.5896	0.783992	2.792357	163231.59
09/15/2032	180	16645434.25	1.66454	212229.29	153.4998	0.773657	2.783618	164192.67
03/15/2033	180	16748447.33	1.67484	213542.7	154.4498	0.763596	2.776268	163060.37
09/15/2033	180	17076935.72	1.70769	217730.93	157.479	0.753218	2.77322	163998.91
03/15/2034	180	17190502.72	1.71905	219178.91	158.5263	0.742718	2.775889	162788.2
09/15/2034	180	17531681.61	1.75317	223528.94	161.6726	0.732142	2.779	163654.9
03/15/2035	180	17648272.8	1.76483	225015.48	162.7477	0.721806	2.782736	162417.42
09/17/2035	180	18003669.09	1.80037	229546.78	166.0251	0.711465	2.784776	163314.45
03/17/2036	180	18133735.97	1.81337	231205.13	167.2246	0.701678	2.784502	162231.64
09/15/2036	180	18504183.13	1.85042	235928.33	170.6407	0.692027	2.784248	163268.67
03/16/2037	180	18637865.94	1.86379	237632.79	171.8735	0.682508	2.784012	162186.16
09/15/2037	180	19018611.78	1.90186	242487.3	175.3846	0.673068	2.783792	163210.48
03/15/2038	180	19156011.07	1.9156	244239.14	176.6517	0.66386	2.783589	162140.7
09/15/2038	180	19548155.77	1.95482	249238.99	180.268	0.655172	2.777954	163294.33
03/15/2039	180	19691020.34	1.9691	251060.51	181.5854	0.647575	2.764512	162580.51
09/15/2039	180	20094953.91	2.0095	256210.66	185.3104	0.639943	2.751689	163960.13
03/15/2040	180	20241814.67	2.02418	258083.14	186.6647	0.632482	2.739766	163232.88
09/17/2040	180	20657047.01	2.0657	263377.35	190.4939	0.624947	2.728294	164596.82
03/15/2041	180	20808015.73	2.0808	265302.2	191.8861	0.61778	2.717877	163898.42
09/16/2041	180	21234862.89	2.12349	21505607.39	195.8223	0.61046	2.7077	13128304.29

Table VI.7 Pricing of the inflation leg of a IIS hedging the BTP inflation IT0004545890

The revalued notional [A] is equal to the nominal notional of the swap (EUR 10 million) multiplied by the index ratio.
The cash flow CF_i which must therefore be paid at payment date T_i is equal to the notional revalued $N_i \cdot \frac{I(T_i)}{I_0}$ by the coupon of the bond (2.55%), reproportioned to the half-year according to Eq. VI.2.

The discount factors [E] to be applied to each payment date are calculated according to the traditional formula:

$$DF(0,T_i) = \exp\left(-r_{T_i}T_i\right) \quad \textit{(Eq. VI.9)}$$

$DF(t_0, T_i)$ is the discount factor to be applied from the valuation time t_0 to the time related to the amount to be discounted T_i. If t_0 coincides with the current time, $t_0 = 0$, the abbreviated notation DF_{T_i} is used.
r_{t_0,T_i} is the spot rate (also called zero rate) between time t_0 and the time related to the amount to be discounted T_i. If t_0 coincides with the current time, $t_0 = 0$, the abbreviated notation r_{T_i} is used. The discount rate is shown in the table in column [F].
T_i is a future time expressed in year fractions in which an amount of money has to be considered for the discount. In this case, therefore, an ACT/365 day convention applies between the reference date for pricing (30 June 2023) and the date of the interest payment.

Zero rates are linearly interpolated from the interest rate term structure that best represents risk. Therefore, in the case of collateralised derivatives, the risk-free yield curve to be considered is the 1-day yield curve (OIS-ESTR), while for non-collateralised derivatives, the reference level of the yield curve should reflect the payment frequency.
Therefore, if the flows were exchanged on a monthly basis, the curve with a tenor of 1 month would be selected; if the interest flows were exchanged on a quarterly basis, the tenor would be 3 months, and so on.
Assuming that the swap in question is collateralised, as per financial best practice to reduce counterparty risk, the most correct choice is to interpolate the discount rate using the OIS-ESTR curve.
The potential counterparty risk is therefore less than 24 hours and, consequently, it is assumed that the fair value does not require further adjustments for counterparty risk such as CVA (Credit Valuation Adjustment) or DVA (Debt Valuation Adjustment). These contributions would anyhow be negligible.
The term structure EUR OIS ESTR from which the zero rates [F] have been linearly interpolated is shown in Figure VI.8. The NPV of the inflation leg is given by the discounted sum of the single contributions and it is equal to EUR 19,033,379. The replication of the net present value with the calculation libraries is less than 50 Euro cents compared to the market benchmark (Figure VI.9).

The subsequent valuation concerns the pricing of a YYIIS (see Figure VI.9 for the financial characteristics) and in this case the valuation formula to be implemented for the determination of cash flow is Eq. VI.3. It is therefore assumed to have a leg characterized by a notional of EUR 10 million and a maturity equal to the previous bond. In this case, a monthly interpolation of the CPI is used as in the previous case, while the frequency payment for the inflation leg is set to be yearly. The Cash flows are shown in Table. VI.8.

Figure VI.8 Interest rates term structure used to calculate discount factors. Source: Bloomberg®

Pay Date	Days	Reset Date	Reset Rate [A]	Reset Price [B]	Payment [C]	Discount	Zero Rate	PV
09/15/2023	365	06/01/2023	5.53739	123.16214	561430.24	0.992474	3.581046	557204.87
09/16/2024	367	06/01/2024	2.65718	126.43478	270885.17	0.955236	3.764864	258759.16
09/15/2025	364	06/01/2025	2.34139	129.39511	236740.28	0.925061	3.518775	218999.31
09/15/2026	365	06/01/2026	2.42254	132.52976	245618.55	0.899987	3.278935	221053.48
09/15/2027	365	06/01/2027	2.44958	135.77618	248360.38	0.877929	3.089677	218042.75
09/15/2028	366	06/01/2028	2.43226	139.07862	247279.86	0.856798	2.962814	211868.83
09/17/2029	367	06/01/2029	2.45	142.48604	249764.15	0.835989	2.879183	208799.96
09/16/2030	364	06/01/2030	2.484	146.0254	251160.04	0.815075	2.832396	204714.16
09/15/2031	364	06/01/2031	2.50207	149.67905	252986.7	0.794334	2.802328	200955.88
09/15/2032	366	06/01/2032	2.55265	153.49983	259519.45	0.773657	2.783618	200779.04
09/15/2033	365	06/01/2033	2.59231	157.47903	262831.56	0.753218	2.77322	197969.53
09/15/2034	365	06/01/2034	2.66292	161.67257	269990.94	0.732142	2.779	197671.68
09/17/2035	367	06/01/2035	2.6922	166.02512	274454.57	0.711465	2.784776	195264.77
09/15/2036	364	06/01/2036	2.78007	170.64073	281095.64	0.692027	2.784248	194525.65
09/15/2037	365	06/01/2037	2.78007	175.38465	281867.89	0.673068	2.783792	189716.3
09/15/2038	365	06/01/2038	2.78435	180.26797	282301.76	0.655172	2.777954	184956.13
09/15/2039	365	06/01/2039	2.79719	185.3104	283603.51	0.639943	2.751689	181489.98
09/17/2040	368	06/01/2040	2.79719	190.49388	285934.5	0.624947	2.728294	178693.84
09/16/2041	364	06/01/2041	2.79719	195.82234	282826.51	0.61046	2.7077	172654.16

Table VI.8 Pricing of the inflation leg of a IIS to hedge a YYIIS with monthly CPI interpolation

The reset rate [A] of the table corresponds to the quantity $\frac{I(T_i)}{I(T_{i-1})} - 1$ in Eq. VI.3. $I(T_i)$ is the forward inflation recorded three months before the payment date. $I(T_{i-1})$ is the forward inflation recorded one year and three months before the previous payment date. The values of the prospective CPI, I are shown in column [B] of the table. The cash flow [C] is obtained multiplying the swap notional by the reset rate by the year fraction between the two contiguous payment dates. In this case, the day basis is assumed to be ACT/360. The part relating to the discounting of the cash flows thus determined is identical to the previous case. The Bloomberg SWPM calculation module prices the inflation leg at EUR 4,194,119 and the gap in absolute terms of the NPV compared to the programmed library remains less than 50 cents.

Pay Date	Days	Notional	Reset Date	Reset Rate [A]	Reset Price	Payment	Discount	Zero Rate	PV
09/15/2023	365	10000000	06/15/2023	5.33974	122.99539	541390.54	0.992474	3.581046	537316
09/16/2024	367	10000000	06/16/2024	2.6284	126.2282	267951.29	0.955236	3.764864	255956.61
09/15/2025	364	10000000	06/15/2025	2.35617	129.20236	238235.38	0.925061	3.518775	220382.37
09/15/2026	365	10000000	06/15/2026	2.42428	132.33458	245794.69	0.899987	3.278935	221212
09/15/2027	365	10000000	06/15/2027	2.44857	135.57488	248257.66	0.877929	3.089677	217952.57
09/15/2028	366	10000000	06/15/2028	2.43261	138.87289	247315.36	0.856798	2.962814	211899.25
09/17/2029	367	10000000	06/17/2029	2.43027	142.24788	247752.87	0.835989	2.879183	207118.55
09/16/2030	364	10000000	06/16/2030	2.49489	145.79681	252261.2	0.815075	2.832396	205611.69
09/15/2031	364	10000000	06/15/2031	2.51512	149.46378	254306.57	0.794334	2.802328	202004.3
09/15/2032	366	10000000	06/15/2032	2.55349	153.28031	259604.36	0.773657	2.783618	200844.72
09/15/2033	365	10000000	06/15/2033	2.59597	157.25942	263202.07	0.753218	2.77322	198248.6
09/15/2034	365	10000000	06/15/2034	2.66292	161.44712	269990.94	0.732142	2.779	197671.68
09/17/2035	367	10000000	06/17/2035	2.67691	165.76891	272895.84	0.711465	2.784776	194155.78
09/15/2036	364	10000000	06/15/2036	2.79992	170.41031	283103.42	0.692027	2.784248	195915.09
09/15/2037	365	10000000	06/15/2037	2.78007	175.14783	281867.89	0.673068	2.783792	189716.3
09/15/2038	365	10000000	06/15/2038	2.78501	180.02572	282369.15	0.655172	2.777954	185000.28
09/15/2039	365	10000000	06/15/2039	2.79719	185.06137	283603.51	0.639943	2.751689	181489.98
09/17/2040	368	10000000	06/17/2040	2.77742	190.20131	283914.43	0.624947	2.728294	177431.4
09/16/2041	364	10000000	06/16/2041	2.80707	195.54039	283825.76	0.61046	2.7077	173264.16

Table VI.9 Pricing of the inflation leg of a IIS to hedge a YYIIS with a daily CPI interpolation

The following pricing example has the same characteristics as the YYIIS leg discussed above, but it assumes a daily interpolation of the CPI instead of a monthly one. In this case, for determining the reset rate, it is not sufficient to take the value of inflation associated with the relevant month, but it is necessary to make an interpolation.

This means that we apply the following standard formula in order to compute the daily CPI, starting from the stripped monthly values:

$$Daily\ CPI\ on\ day\ (d) = CPI_{m-lag} + \left[\frac{(n_d-1)}{ND_m}\right] \cdot \left(CPI_{m-lag-1} - CPI_{m-lag}\right)$$ *(Eq. VI.10)*

CPI_{m-lag} is the price index of month $m - lag$. Given that the lag for the European/Italian inflation is 3, we have CPI_{m-3}.

$CPI_{m-lag-1}$ is the price index of month $m - lag - 1$. Given that the lag for the European/Italian inflation is 3, we have CPI_{m-2}.

n_d is the actual number of days since the start of the month and ND_m is the number of days in month m.

Table VI.9 shows the cash flows, while Table VI.10 shows the determinants for calculating the reset rate [A] by applying the daily interpolation of inflation.

Numerator				Denominator				
I(m-2)	I(m-3)	(n-1)/D	Reset Price	I(m-14)	I(m-15)	(n-1)/D	Reset Price	Reset Rate
122.805	123.162	0.4670	122.995	116.83	116.7	0.467	116.761	5.340%
126.022	126.435	0.5000	126.228	122.805	123.162	0.467	122.995	2.628%
128.982	129.395	0.4670	129.202	126.022	126.435	0.5	126.228	2.356%
132.112	132.53	0.4670	132.335	128.982	129.395	0.467	129.202	2.424%
135.345	135.776	0.4670	135.575	132.112	132.53	0.467	132.335	2.449%
138.638	139.079	0.4670	138.873	135.345	135.776	0.467	135.575	2.433%
142.039	142.486	0.5330	142.248	138.638	139.079	0.467	138.873	2.430%
145.568	146.025	0.5000	145.797	142.039	142.486	0.533	142.248	2.495%
149.218	149.679	0.4670	149.464	145.568	146.025	0.5	145.797	2.515%
153.029	153.5	0.4670	153.28	149.218	149.679	0.467	149.464	2.554%
157.008	157.479	0.4670	157.259	153.029	153.5	0.467	153.28	2.596%
161.189	161.673	0.4670	161.447	157.008	157.479	0.467	157.259	2.663%
165.545	166.025	0.5330	165.769	161.189	161.673	0.467	161.447	2.677%
170.147	170.641	0.4670	170.41	165.545	166.025	0.533	165.769	2.800%
174.877	175.385	0.4670	175.148	170.147	170.641	0.467	170.41	2.780%
179.749	180.268	0.4670	180.026	174.877	175.385	0.467	175.148	2.785%
184.777	185.31	0.4670	185.061	179.749	180.268	0.467	180.026	2.797%
189.945	190.494	0.5330	190.201	184.777	185.31	0.467	185.061	2.777%
195.258	195.822	0.5000	195.54	189.945	190.494	0.533	190.201	2.807%

Table VI.10 Calculation of the Reset rate in a YYIIS with daily interpolation

The NPV of the YYIIS inflation leg with daily interpolation is equal to EUR 4,173,191 and the model gap still remains below the 50 cents threshold.

The screen-shots summarizing the financial characteristics of the three IIS on European inflation as well as the pricing according to the market standard are shown in Figure VI.9.

Figure VI.9 Valuation and main characteristics for the three European IIS analyzed.
Source: Bloomberg® - SWPM module. Reference date for the valuation: 30 June 2023

We have examined both the ZCIIS and the YYIIS, now starting from this pricing framework, we mention that less popular exotic variants have been created and among them, with particular reference to our country, is the BTP Italia. If a hedge has to be designed, for which one leg of the IIS is physically represented by this government bond, it is necessary to have a valuation and quantitative analysis model of the associated risk and, since this financial instrument is not so globally widespread, the usual theoretical Bloomberg® calculation modules (SWPM) cannot be used.

The BTP Italia provides the investor with protection against the increase in the Italian price levels: both the coupons, paid semi-annually, and the principal, whose revaluation is also paid half-yearly, are revalued based on Italian inflation, measured by the national institute for Statistics, ISTAT, through the national index of consumer prices for blue-collar and white-collar households (FOI), excluding tobacco (Bloomberg code: ITCPI Index).
Thanks to the indexation mechanism used, every 6 months the holder is entitled to recover the loss of purchasing power incurred in that period, through the payment of the half-yearly revaluation of the invested capital. In addition, the coupons, which are also paid semi-annually, provide a constant minimum return in real terms. In fact, the amount of each coupon is calculated multiplying half of the fixed annual real interest rate, established at issue, by the capital, revalued based on the semi-annual inflation. At maturity, the BTP Italia guarantees the return of the nominal value.

The revaluation of the nominal capital and of the coupons is implemented through the Indexation Coefficient (CI). In particular, CI is calculated based on the inflation detected by the FOI index, excluding tobacco, processed and published monthly by ISTAT. This coefficient allows to know, at a generic date (day d of month m), the value of the nominal capital revalued based on price trends. CI is calculated using the following formula:

$$CI_{d,m} = \frac{\Im_{d,m}}{\Im_{\overline{d,m}}} \quad \textit{(Eq. VI.11)}$$

Where $\Im_{d,m}$ is the index on day d of month m of the coupon payment, while $\Im_{\overline{d,m}}$, called the basic index, means the index on the payment date of the previous coupon (six months earlier). The CI value thus obtained is truncated to the sixth decimal place and rounded to the fifth.
In the case of the first coupon payment, when the accrual date of the coupon coincides with the accrual date of the bond, then the base index of CI is taken as of the accrual date of the bond.
Since the FOI ex-tobacco price index is published by ISTAT in the second half of the month following the reference month, in order to calculate the price index at a generic date (day d of month m), a calculation method is used for the daily interpolation (see also Eq. VI.10):

$$\Im_{d,m} = I_{m-3} + \frac{d-1}{D} \cdot (I_{m-2} - I_{m-3}) \quad \textit{(Eq. VI.12)}$$

$\Im_{d,m}$ is the index of day d and month m.
I_{m-3} is the FOI index ex-tobacco that precedes the one for which the calculation is made by three months.
I_{m-2} is the FOI index for tobacco that precedes the one for which the calculation is made by two months.
d is the day of the month for which the calculation is made.
D is the number of actual days in month m.

The index at the coupon payment date is therefore calculated based on the ISTAT FOI ex-tobacco indices relating to three months and two months preceding the month for which the calculation is made. The value thus obtained is truncated to the sixth decimal place and rounded to the fifth.
As mentioned above, the determination of the monthly prospective values of I is implemented with the same procedure already described above. See in particular the recursive formula in Eq. VI.8.
The total half-yearly remuneration CF_i of the swap leg indexed to Italian inflation will therefore be given by the sum of the coupon $\bar{C}_i$ plus the revaluation of capital of the reference period $\bar{N}_i$:

$$CF_i = \bar{C}_i + \bar{N}_i \quad \textit{(Eq. VI.13)}$$

The variable amount of the semi-annual coupons is calculated multiplying the annual real interest rate, divided by two, by the nominal principal revalued on the coupon payment date (equal to the nominal capital multiplied by the Indexation Coefficient modified on the coupon payment date).

$$\bar{C}_i = \frac{C_i}{2} \cdot N_i \cdot \max(CI_i, 1) \quad \textit{(Eq. VI.14)}$$

C_i is the annual real coupon rate.
N_i is the current reference notional.
CI_i is the inflation coefficient calculated on the payment date of amount T_i.
$\max(CI_i, 1)$ represents the so-called "modified" Indexation Coefficient.

In the event that the half-yearly Indexation Coefficient is less than one, i.e. in the event that there is a reduction in prices on a semi-annual basis, which would theoretically correspond to a devaluation of the capital, it is assumed that the price index is the same as that of the previous period (so-called coupon floor mechanism).
As a result, the Indexation Coefficient becomes equal to one (modified Indexation Coefficient), and then the real coupon rate, which constitutes the guaranteed minimum return, is paid.
In the following period, if the Indexation Coefficient on a half-yearly basis returns above one, the price index of the previous half-year is taken as the basis, provided that the latter is higher than the last maximum value recorded in the previous half-years. Otherwise, the basis will continue to be the last maximum value.
In addition to the payment of the semi-annual coupon, the payment of the revaluation of the capital, accrued in the reference half-year, is also envisaged:

$$\overline{N}_i = N_i \cdot \max(CI_i - 1{,}0) \quad \textit{(Eq. VI.15)}$$

In the event that the half-yearly Indexation Coefficient is less than one, no revaluation is paid (so-called capital floor mechanism). In the following period, if the Indexation Coefficient on a half-yearly basis returns above one, the price index of the previous half-year is taken as the basis, provided that the latter is higher than the last maximum value recorded in the previous half-year. Otherwise, the basis will continue to be the last maximum value.
For payment purposes, the result obtained from formulas Eq. VI.14 and Eq. VI.15 for the coupon calculation and the capital revaluation calculation, respectively, is rounded to two decimal places.
The presence of an optionality for determining $\overline{N}_i$ and $\bar{C}_i$ makes a stochastic modelling of the forward values of inflation necessary. In particular, the characteristic of observation of the previous values of CI, in the different payment dates T_i, makes such optionality path-dependent and consequently difficult to solve in a closed formula, if not by introducing approximations.
For pricing all the floor options incorporated in the financial instrument (both on the coupon and on the notional) and due to the characteristic of the retrospective observation of the values of the Inflation Coefficient, a Monte Carlo method has therefore to be implemented.
As an example, the simulations are made implementing a very simple Brownian arithmetic motion applied to forward rates calculated in the interval T_M, T_{M+1} and derived from the ZCIIS according to the traditional financial mathematics relation:

$$[1 + K(T_M)]^{T_M}[1 + F^I(T_M, T_{M+1})]^{(T_{M+1}-T_M)} = [1 + K(T_{M+1})]^{T_{M+1}} \rightarrow$$

$$F^I(T_M, T_{M+1}) = \left[\frac{[1+K(T_{M+1})]^{T_{M+1}}}{[1+K(T_M)]^{T_M}}\right]^{\frac{1}{T_{M+1}-T_M}} - 1 \quad \textit{(Eq. VI.16)}$$

Stochastic dynamics is then applied to $F_K(T_M, T_{M+1})$ according to the following stochastic differential equation (SDE):

$$dF_t^I = \mu dt + \sigma dW_t \quad \textit{(Eq. VI.17)}$$

dF_t^I indicates the variation in the inflation forward rate at time t. The forward follows the granulometry of the ZCIIS on inflation. Since the BTP Italia has a maturity below ten years, the tenor is equal to one year. Therefore, the notation here adopted F_t^I represents the synthetic form for $F_{t,t+tenor}^I = F_{t,t+1Y}^I = F_t^I$.

μ is the drift of the stochastic process, in this case the slope of the curve calculated between two contiguous ZCIIS is used, i.e. $\frac{K(T_{M+1})-K(T_M)}{T_{M+1}-T_M}$.

dt is the time instant at which the random perturbation is applied.

σ is the volatility to be applied to the stochastic process. In this case, the implied volatility is used on the floorlets of actively traded YYIIS derivatives. Since it is a Brownian arithmetic stochastic process, which is the underlying assumption behind the normal Bachelier model, the normal implied volatility quoted by Bloomberg and expressed in basis points will be used. Since there are no liquid option contributions on Italian inflation, the implied normal volatility for the YYIIS floorlets of French inflation is used as a proxy in the calculation model, after verifying that this value does not differ much from the historical volatility recorded by the Italian CPI.

dW_t is a Wiener stochastic process.

The solution of equation Eq. VI.17 is quite straightforward since all the variables are well separated, so it can be integrated directly from time T_M to time T_{M+1}.

$$\int_{T_M}^{T_{M+1}} dF_t^I = \int_{T_M}^{T_{M+1}} \mu dt + \int_{T_M}^{T_{M+1}} \sigma dW_t = \mu \int_{T_M}^{T_{M+1}} dt + \sigma \int_{T_M}^{T_{M+1}} dW_t \rightarrow \quad \textit{(Eq. VI.18)}$$
$$F_{T_{M+1}}^I - F_{T_M}^I = \mu(T_{M+1} - T_M) + \sigma\left(W_{T_{M+1}} - W_{T_M}\right) \rightarrow F_{T_{M+1}}^I = F_{T_M}^I + \mu(T_{M+1} - T_M) + \sigma\left(\Delta W_{T_{M+1}-T_M}\right)$$

Remembering that:

$$\Delta W_{T_{M+1}-T_M} = \epsilon\sqrt{T_{M+1} - T_M} = \epsilon\sqrt{\Delta T} \quad \textit{(Eq. VI.19)}$$

ϵ represents an extraction from a standard normal distribution (of mean zero and variance one); we reach the definition of a stochastic dynamics suitable for the simulation of forward Year-on-Year inflation rates that can be implemented in a programming language:

$$F_{T_{M+1}}^I = F_{T_M}^I + \mu\Delta T + \sigma\epsilon\sqrt{\Delta T} \quad \textit{(Eq. VI.20)}$$

Once the stochastic perturbation is made to the forward, $F_{T_{M+1}}^I$, and once the previous $K(T_M)$ is known, the next simulated ZCIIS, i.e. $K(T_{M+1})$ is calculated with Eq. VI.16.

In such a way, it is possible to reach a simulation for the ZCIIS along all T_M and, consequently, an entire simulated term structure is available. It is possible to verify that if a sufficiently large number of simulations of Eq. VI.20 is implemented and Eq. VI.16 is applied to obtain $K(T_{M+1})$, the expected value coincides with the initial values contributed by the market.
Once the simulations of the ZCIIS at the different maturities are available, Eq. VI.4-8 can be applied to derive the simulated forward values of inflation.
Once the CPIs have been estimated, Eq. VI.11-15 that characterize the pay-off of the BTP Italia are applied in order to derive the cash flows (thus including the exotic optionality incorporated in the security) at the different payment dates, T_i. In accordance with the Monte Carlo methodology, the expected value will be obtained as the mean calculated on the number of simulations implemented.
The sum of such cash flow discounted using the appropriate discount factor constitutes the value of the swap leg indexed to the BTP Italia.
As an example of valuation of a structured financial product linked to inflation, a swap hedging a BTP Italia is valued. More specifically, one leg of the derivative is the IT0005410912 bond maturing on 26 May 2025. The financial characteristics of this instrument are shown in Figure VI.10.

Figure VI.10 Characteristics of the BTP ITALIA bond with ISIN code: IT0005410912. Source: Bloomberg®

The inflation leg has been valued considering the Monte Carlo model described above. The volatility proxy used in the pricing model is the normal one implied in the floorlets written on French inflation at the valuation date. The data are shown in Figure VI.11.

Market View
View Floorlet Normal Vol Output (%)
Interpolated

Tenor	-2.00%	-1.00%	-0.50%	0.00%	0.50%	1.00%	1.50%	2.00%	2.50%	3.00%	3.50%	4.00%
1 YR	2.73	2.74	2.67	2.61	2.56	2.52	2.53	2.54	2.55	2.56	2.46	2.35
2 YR	2.28	2.29	2.21	2.13	2.07	2.00	1.96	1.91	1.92	1.94	2.01	2.06
3 YR	1.92	1.93	1.85	1.77	1.70	1.63	1.57	1.50	1.50	1.50	1.57	1.64
4 YR	1.92	1.93	1.85	1.77	1.70	1.63	1.57	1.50	1.50	1.50	1.57	1.64
5 YR	1.92	1.93	1.85	1.77	1.70	1.63	1.56	1.50	1.50	1.50	1.57	1.64
7 YR	1.66	1.67	1.59	1.51	1.44	1.37	1.31	1.26	1.25	1.25	1.28	1.33
10 YR	1.45	1.46	1.38	1.31	1.24	1.16	1.12	1.07	1.05	1.04	1.08	1.11
12 YR	1.15	1.16	1.09	1.02	0.95	0.88	0.82	0.75	0.73	0.70	0.74	0.77
15 YR	1.15	1.16	1.09	1.02	0.95	0.88	0.82	0.75	0.73	0.70	0.74	0.77
20 YR	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Figure VI.11 Volatility implied in the floorlets written on the French inflation index. Source: Bloomberg®

The leg valuation is equal to EUR 1,539,587.74 with a standard deviation of 4,978 out of a number of implemented simulations equal to 5,000.

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PART VII: AGGREGATE RISK MEASURES

Chapter 1 – Risk Measures

VII.1 RISK MEASURES

The traditional approach to measuring market risk was based on the use of metrics related to the sensitivity of the position value to changes in the underlying risk factors: the so-called **sensitivities**. The sensitivity measures are specific for each type of financial instrument and the most used for balance sheet exposures are:

For Bonds: the duration, the convexity and the credit point value, i.e. the change in the value of the bond following a one basis point widening of the credit spread.

For Equities: the beta, with respect to an index, or a plurality of coefficients as expected in multifactor models that predict the relevance of several macroeconomic factors capable of influencing the variability of share prices.

For Options: the Greeks.

The main limitations of sensitivity measures are that they do not allow for comparing the risks assumed in different types of financial instruments, for measuring the riskiness of a portfolio made up of different types of instruments and for aggregating the risks even within the same category of tools. For example, the duration of a bond denominated in euros cannot be added to that of a bond denominated in dollars. Around the 1980s, the attempt to overcome the problems mentioned above led some financial institutions to introduce and develop the first models which made it possible to quantify, compare and aggregate the risk associated with different portfolios. Such models were initially introduced by major US banks and are generally referred to as **Value-at-Risk** or **VaR models**. One of the first institutions to develop a VaR model and the first to make it public was JP Morgan, author of the RiskMetrics model.

In the 1990s, the publication of the methodology underlying the VaR model and of the market data needed to feed it was one of the main factors that contributed to the birth and development of VaR methodologies for measuring risk. Subsequently, the decisive affirmation of VaR models in the operational reality of financial institutions essentially derived from the multiple uses and the relative simplicity of implementation. The first advantage consists in the possibility of homogeneously measuring the risk deriving from holding positions in different financial instruments and in allowing the various players to communicate their position in terms of a common risk unit. To clarify this concept we can give the example of two traders engaged respectively in long positions in shares and in securities. The first trader who wants to communicate the potential exposure of his portfolio, made up of BTPs, to adverse movements in market factors, can refer to the nominal value of the position and the relative duration. On the other hand, the other operator, who we assume is engaged in the trading of derivatives on shares, can report, in addition to the value of the position, also the possible volatility of the underlying or of the risk indicators, indicative of the sensitivity of the derivative itself with respect to the variations of the risk factors characterizing this particular financial instrument (for example, delta, vega, gamma, etc.). Although both provide a fairly objective measure of the riskiness of the investment, neither of the two indicators can establish the convenience/opportunity of closing the position or simply assess which of them is risking more at that moment. In these terms, the advantage of VaR is clear. The use of VaR techniques in the context of normal banking operations allows both horizontal communication, between operators of different desks, and vertical communication with senior management, who are able to evaluate the overall exposure of the bank's portfolio in a homogeneous manner and act promptly in the event that the risk profile is not tolerable according to the bank's risk policies.

Another advantage of VaR techniques is that they allow the determination of limits on risk assumption, which can, for example, be imparted by senior management to the various operating desks by communicating the maximum exposure that the individual trader can assume through a value expressed in the desired currency. Based on the assigned capital and the degree of sensitivity of the single position, the operator can autonomously evaluate the overall risk of the portfolio without being deprived of the necessary autonomy.

Let us now enter into the merits of quantifying VaR, which answers a very simple question, in its formulation, although complex in its solution: "What is the maximum loss that could be incurred over a certain time horizon, such that there is a very low probability, for example 1%, that the actual loss will exceed this amount?". This question implies some hypotheses that must be ascertained in empirical terms and continuously monitored: first of all, the concept of maximum loss, then the time horizon, and finally, also the level of confidence which can be deemed consistent with the bank's risk appetite. In general terms, the choice of the time horizon and the level of confidence is subjective; in particular, the confidence level defines the degree of protection from the risk of adverse market movements. The standard values for this variable are 99% which constitutes a very conservative estimate or 95%. Similarly, the periods normally adopted are one day or 10 days. The underlying assumption is that the composition of the portfolio remains unchanged during this time period, therefore the choice of the reference interval for calculating the VaR must depend on the period necessary for the liquidation of the portfolio itself.

The two main approaches for measuring VaR are:

- **parametric approaches**, based on the so-called risk factor volatility-correlation matrices;

- **full-evaluation approaches**, based on simulations that use historical data on the returns of risk factors (historical simulation) or on simulations of hypothetical scenarios generated with statistical methods (Monte Carlo simulation).

With the **parametric approach** based on volatility-correlation matrices, it is assumed that the probability distributions of risk factor returns are governed by the normal probability distribution function. The return on a portfolio, equal to the weighted algebraic sum of the returns of the individual assets, would therefore in turn be governed by a law of normal distribution. If the mean of the Gaussian distribution is 0 ($\mu = 0$) and the standard deviation is 1 ($\sigma = 1$), the normal distribution is standardized. All values of the standard normal distribution can be expressed as a product $k \cdot \sigma$, where k is a multiple consistent with the probability associated with the value to be extracted. Therefore, if we want to obtain the value that limits a certain confidence interval, we need to use:

$k = 1$ if we want to obtain the most extreme value in 68.27% of the cases.

$k = 1.25$ if we want to obtain the most extreme value in 80% of the cases.

$k = 1.65$ if we want to obtain the most extreme value in 90% of the cases.

$k = 2.33$ if we want to obtain the most extreme value in 98% of the cases.

$k = 2.58$ if we want to obtain the most extreme value in 99% of the cases.

Depending on the different levels of k it is possible to determine the worst case scenario by setting a certain level of probability. Thus, based on the properties of the normal distribution, the VaR of a position is estimated as the product of three elements:

- The market value of the position (VM).

- The sensitivity of the market value of the position to changes in the market factor (δ) e.g. interest rate, exchange rate, …

- The potential adverse change in the market factor, obtained as the product of the estimated volatility of that market factor (σ) and a scalar factor (α) that corresponds to the desired confidence level.

In mathematical notation, we have:

$$VAR = VM \cdot \delta \cdot (\sigma \cdot \alpha) \textit{ (Eq. VII.1)}$$

Therefore, VaR measures the loss of the security's position in the scenario of maximum negative variation of yields. The estimation of the VaR requires the determination of the market value of the position, its sensitivity to the risk factor, the volatility and, finally, the scaling factor which is strictly linked to the statistical distribution and to the probability (or confidence interval).

Let us analyze an example, supposing a bank has invested in a zero coupon with an exposure of EUR 1,000,000, a 1-year maturity and a 1-year market yield of 3% p.a.

The market value of such position is: $VM = \frac{1{,}000{,}000}{(1+0.03)^1} = \text{EUR } 970{,}873.79$.
As the yield increases, the market value of the exposure decreases, and vice versa as it decreases: in the case of an increasing variation of the interest rate of 2%, VM becomes equal to EUR $952{,}380$.

Therefore, the market return represents the risk factor and its volatility, measured by the standard deviation, determines the market value of the portfolio in correspondence with the evolution of the changing scenarios.

But how sensitive is the market value of the securities portfolio to the variability of the market return? If we compare the change in the market value of the portfolio before and after the 2% rate hike, the market value of the portfolio changed by 1.90%: the difference between the change in returns and that in the market value of the portfolio depends on the sensitivity coefficient δ with respect to the variation of the market return. In this case, the coefficient is given by the **modified duration**, in fact: [1/(1+0.03)] x 2% = 1.93% is the local approximation of the variation in market values of the zero-coupon bonds portfolio.

Since the risk manager is interested in predicting losses caused by extreme changes in the market return, i.e., the worst in 99% of cases, the parametric approach determines the extreme change in the market return through the product $\alpha \cdot \sigma$. The values of α are codified in accordance with a standard normal distribution:

- 2.05 for a 98% confidence interval, scaled with respect to the k seen previously, since the distribution is symmetric, and we want to isolate 2% of the extreme cases.

- 2.33 for a 99% confidence interval, scaled with respect to k given that we want to isolate 1% of extreme cases.

Assuming that σ is equal to 0.40%, it is possible to estimate the VaR as follows:

$$VaR = VM \cdot \delta \cdot (\sigma \cdot \alpha) = 970{,}873.79 \cdot 1.93\% \cdot 0.40\% \cdot 2.33 = € \, 174.64$$

When calculating the VaR of a portfolio, it must be remembered that the volatility of a portfolio is not the weighted average of the volatilities of the positions included in the portfolio, since, according to Markowitz's theory, it benefits from the diversification effect produced by the correlations. For example, if we have a portfolio characterized by n assets, the σ_P^2 would be given by:

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{i,j} \omega_i \omega_j \sigma_i \sigma_j \quad \textit{(Eq. VII.2)}$$

Where: ω is the weight invested in the financial instrument and ρ the correlation coefficient.

Then, there are two ways to extend the VaR estimation from a short-term time horizon (e.g. one business day) to a longer time horizon (e.g. 10 business days): one way consists in using the risk factor volatility matrices estimated directly on returns over the longer period considered. The second way consists in applying the so-called rule of the root of T, i.e. multiplying the estimate of the daily VaR by a scale factor equal to the root of the number of days contained in the longest period considered (for example, 10 working days). In mathematical terms: $VAR_T = VAR_1 \cdot \sqrt{T}$.

In order to understand the rule of the square root of time, let us suppose we have a stock in our portfolio with a market value of EUR 2,500,000. We assume that the sensitivity factor of the stock to risk factors is $\beta = 1.245$ and the volatility of the stock returns of the market index is 2.15%. We select a confidence interval of 99%, which corresponds to a scaling factor of 2.326 and a time horizon of 1 day. The VaR of this equity position in the portfolio is:

$$VAR = 2{,}500{,}000 \cdot 1.245 \cdot 2.15\% \cdot 2.326 \cdot \sqrt{1} = \text{EUR } 155{,}653$$

In other words, there is a 1% probability that the loss will be greater than EUR 155,653 in one day.

Taking the values obtained for the daily VaR of the equity investment, if we want to estimate the VaR for a time horizon of 10 days, we obtain:

$$VAR_{T=10} = VAR_1 \cdot \sqrt{T} = 155{,}653 \cdot \sqrt{10} = \text{EUR } 492{,}218$$

The square root rule allows to quickly transform daily measurements into calibrated measurements for longer time horizons. However, the validity of the measure obtained from the transformation is subject to the fact that the returns of the asset considered are independent on a daily basis within the time horizon considered, in our case 10 days.

One of the advantages of the parametric approaches for estimating VaR based on risk factor volatility-correlation matrices is the simplicity of implementation: by now there are several management software producers who offer various off-the-shelf solutions - i.e. standardized and at reduced costs - for this type of VaR estimation models. Another advantage consists in the extreme transparency and possibility of analyzing the positions from which the risk originates. In particular, the portfolio VaR estimate can be easily broken down according to various axes of analysis, like the business unit or trader, the product, the risk factor. One of the major criticisms is, however, the acceptance of the hypothesis of normality. In fact, the probability distributions of risk factors (stock prices, interest rates, exchange rates, etc.) actually observed on the markets are not "normal", they are instead characterized by the so-called "**fat tails**" phenomenon. This means that extreme events, e.g., particularly large negative returns, tend to occur with a much higher frequency than that predicted by the normal distribution. Therefore, the adoption of an assumption of normality of the returns of the risk factors can underestimate the portfolio risk measured through the VaR, especially if estimated on high confidence levels above 95%.

To solve this potential issue, an idea can be to adopt a statistical distribution different from the gaussian which has characteristics more consistent with the empirical observations of the returns on financial assets. Alternatively, we can decide to switch to a model that is not based on a predefined distribution of returns, and to simulate its dynamics. Precisely, by removing the a-priori assumption on the distribution of losses, the VaR can also be determined using **historical simulations** and **Monte Carlo**. The calculation procedure in the **historical simulation method** is very simple and disregards any hypothesis on the type of yield distribution function. We may decide to retrieve the series of risk factor returns observed over a given historical period, for example, over the last 2 or 3 years. Historical returns are applied to existing portfolio positions and the corresponding gains and losses are calculated. These results are reported in descending order starting from the highest losses and the worst-case scenario corresponding to a certain percentile (quantile) is extracted: that is, that level of loss which is exceeded only in a given percentage (for example, 5% or 1%) of the worst cases.

The application of this method naturally implies the full revaluation of the portfolio on each day of the reference time horizon and the subsequent calculation of the daily returns of the portfolio by difference with respect to the current value. The method is quite simple to communicate and explain and is currently the most popular VaR estimation method among banks worldwide. Furthermore, it does not require to estimate volatility and correlation matrices between risk factors: the dependence structure between them is in fact implicit in the historical yield series used. Lastly, the method works well in the treatment of option contracts, as it correctly captures the "non-linear" value changes of these contracts in response to changes in risk factors. However, it should be said that it is based on a strong assumption of "stability" over time of the (joint) probability distribution of the risk factors. In other words, it assumes that the returns observed in the past will tend to be reproduced in the future with unchanged characteristics. Several empirical studies show that this is not true, and VaR estimates based on historical data are unable to effectively predict the future evolution of portfolio risks, as the recent crisis has also highlighted. Another problem with the historical simulation method is that the VaR estimates it produces over a one-day interval are not easily extendable (or scalable) to longer time intervals, for example, 10 days. To overcome the limitations of historical simulation VaR estimates, in recent years VaR estimates based on Monte Carlo-type simulations have been introduced and are becoming increasingly important in the banking industry. Estimating VaR using the **Monte Carlo method** involves simulating returns or asset prices using stochastic processes. For each risk factor that influences the value of a given portfolio (share prices, interest rates, exchange rates, futures prices on commodities, etc.), a mathematical model is constructed which describes the possible future evolution, characterized, for example, for simpler models, by an expected value and a certain volatility. A specific dependency structure between the risk factors is then established by the developer of the simulation model. In other words, it is a question of mathematically describing how the various risk factors vary jointly, influencing each other. As an example, usually when stock prices fall rapidly, the volatility of returns increases, and when interest rates increase, stock prices or exchange rates are also affected etc.. With the scenarios of the simulated risk factors, the changes in the value of the portfolio are constructed and once the losses recorded by the portfolio are sorted in descending order, on the basis of the quantile corresponding to the pre-selected degree of probability, the Monte Carlo VaR is estimated.

The Monte Carlo simulation method has the great advantage of being **forward looking**, i.e. it is oriented towards the future, in the sense that a skilled risk manager who knows how to correctly calibrate the mathematical-statistical models that rule the dynamics of risk factors can generate extreme scenarios and complex interactions, highlighting portfolio risk situations that otherwise would not emerge with other methods such as parametric or historical simulation methods. The VaR estimates obtained via Monte Carlo simulation are also easily "scalable", i.e. extendable over periods of time longer than a single working day, such as for example 10 days or a month. Indeed, it is sufficient to simulate the evolution of risk factors over several consecutive periods. However, the disadvantages of the Monte Carlo method consist in a lower transparency and readability of the obtained results. Substantially, the method is like a black-box, in the sense that the resulting VaR estimate cannot easily be reconnected to the single positions or single risk factors that originated it. Furthermore, there is a disadvantage due to the lack of control over the random generation. Besides, there is a tendency to base VaR measurements on variations that rarely represent extreme values, for example in the case of 10,000 trails, the 99% VaR represents the hundredth most unfavorable variation. The method is also more complex to implement and requires the presence of a team of quantitative experts in the bank capable of correctly using the mathematical-statistical models that describe the risk factors. Lastly, although the Monte Carlo method is forward looking, it still depends on some subjective choices - in terms of model selection and parameter calibration - made by the analysts who make the VaR estimates. Therefore, even the VaR estimates produced by Monte Carlo simulation methods, even if based on sophisticated techniques, are still subject to a high human error component. For this reason, many companies, in addition to calculating the VaR, implement a few checks on the behavior of their portfolio under exceptional conditions (stress tests).

Stress tests consist of estimating portfolio performance in the presence of some of the most extreme market movements seen in the last 10 or 20 years.

For example, to see the impact of an extreme movement in US stock prices, we can observe the rates of change in market variables observed on October 19, 1987. At that point in time, the rate of change of the S&P 500 was 22.3 times its standard deviation. If this case is deemed to be too extreme, we can select January 8, 1988, then the rate of change of the S&P 500 was 7.7 times its standard deviation. Stress tests are performed to account for extreme events that occur from time to time but are virtually impossible under the assumed probability distributions for market variables. A daily rate of change of a market variable equal to 5 standard deviations is an extreme event. Under the normal assumption, it should occur every 7,000 years, but in reality, it occurs once or twice every ten years. Following the credit crisis of 2007/2008, the Supervisory Authorities imposed on credit institutions the calculation of the stressed VaR, which is a VaR based on historical simulation of how market variables moved during this particularly unfavorable time range. Regardless of the VaR estimation calculation model, an important verification of its reliability is represented by back-testing.

Back-testing consists in verifying the behavior of the VaR estimates, based on historical data. Let us assume that we are calculating one-day VaR with a 99% confidence level. Back-testing means observing how often the losses were greater than the VaR, based on historical data. If the frequency is equal to approximately 1% of the days considered, we can be reasonably satisfied with the methodology for calculating the VaR. While VaR has been the most successful tool in measuring market risk, a long-debated limitation within the financial industry is the inability to estimate the magnitude of losses in those scenarios where the VaR threshold is exceeded.

In this context, the need therefore arises for a consistent risk measure even in cases of non-normal distributions: the **Expected Shortfall** (ES) describes how large the losses are on average when they exceed the level of VaR. Thus ES is the expected loss, given a greater loss (in absolute value) than the VaR. ES is also called Conditional VaR (cVaR) or Tail Loss. While VaR questions how bad financial investments can go, ES asks: "If investments go bad, what is the expected loss?". The ES represents the expected loss over a period of n days, if the loss is greater than the X-th, for example, 100th, percentile. Like VaR, ES is also a function of two parameters: the time horizon and the confidence interval.

The below Figure shows the VaR estimation from a probability distribution of changes in portfolio value.

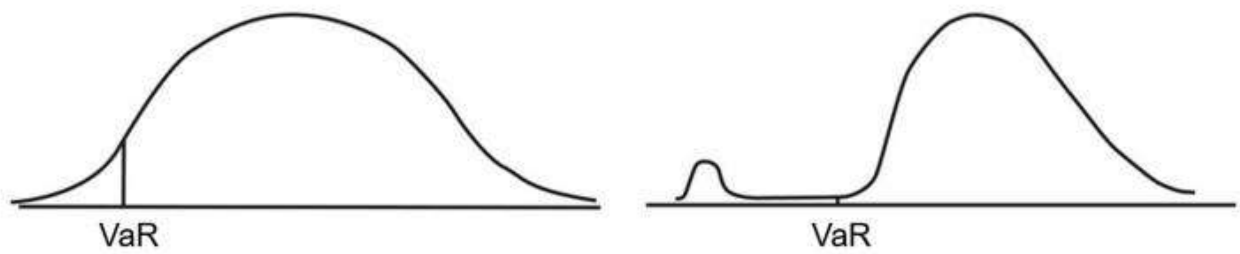

Figure VII.1 Value at Risk and Expected Shortfall

In the figure on the right, although the same VaR is highlighted, in reality the portfolio that generated this distribution is riskier than the one that generated the distribution on the left. In this case, unlike VaR, the Expected Loss is able to distinguish the two cases and therefore correctly attribute a greater risk to the portfolio that generated the right-hand probability distribution.

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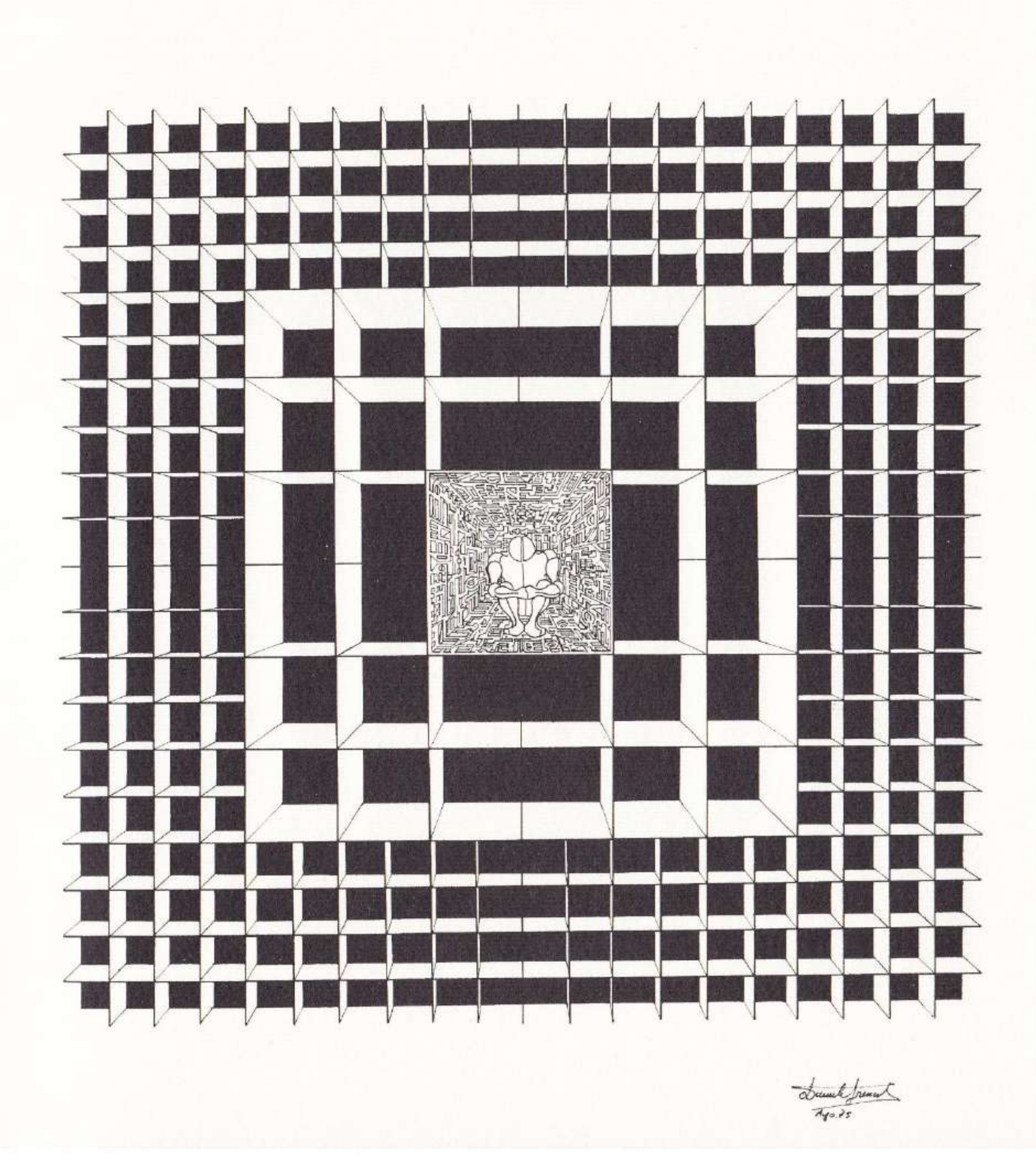

PART VIII: CREDIT RISK

Chapter 1 – Fundamentals

Demand and supply
Type of interest rates
Repayment plans
Mode of extinction
Amount and Loan-to-Value (LTV)
Guarantees
Duration
Global effective annual rate (TAEG)

Chapter 2 – Counterparty Risk

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Chapter 4 – Loss Given Default

LGD estimation
LGD prediction using an OLS cluster-robust regression
LGD prediction using a feed-forward shallow neural network

Chapter 5 – Exposure at Default

EAD estimation
EAD prediction
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Chapter 6 – Rating Systems

Rating agencies
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Chapter 7 – Credit Portfolio Risk

CreditMetrics models
Default correlations
Assets correlation using Moment matching
Assets correlation using Maximum Likelihood
Credit portfolio risk measures
Monte Carlo VaR and C-VaR

Chapter 8 – Model Validation

Cumulative Accuracy Profiles (CAP)
Receiver Operating Characteristics (ROC)
Brier Score
Binomial and other statistical tests
Tests on PD models and Credit Portfolio losses
Market versus Credit Portfolio validation methods

VIII.1 FUNDAMENTALS

In this preliminary chapter, I firstly analyze the determinants of demand and supply of credit, then I provide a summary for the core elements that constitute a mortgage or loan, i.e., interest rate, repayment plans, mode of extinction, amount and Loan-to-Value (LTV), guarantees, duration and global effective annual rate ("TAEG" - Tasso Annuo Effettivo Globale in Italian).

Demand side

Understanding the determinants of the demand for credit requires an analysis of the monetary and economic cycle of the family unit, as well as of the methods of analysis and identification of the external financial needs in the case of a corporate. A family has a characteristic cycle determined by the production of work, sale (income from work) and the acquisition of goods and services to satisfy the primary and discretionary needs of its members. The excess of current and non-current spending needs with respect to income and savings, net of a precautionary stock of money, determines the financial needs. A family with adequate financial education tends to harmonize financial flows of income and consumption throughout the entire life cycle. Therefore, the assessment of the economic-financial balance must be implemented considering a medium-long time span. Since the 1950s, the banking economy has linked the demand for consumer credit by households to three variables: the total disposable income, the propensity to consume/save and the propensity to indebtedness.
In the presence of a given disposable income, the determinants of the credit demand are the propensity to consume and the propensity to borrow. Traditional theories analyze household behavior as part of a broader project (life cycle and permanent income theory) in which the use of consumer credit is a way to anticipate consumption needs without waiting for the sum to accumulate. In relation to the fact that consumption needs (typical of the early stages of a family) and incomes (higher at the end of the professional life) do not match. In fact, there are numerous studies with different approaches that have enriched the explanation of the financial behavior of household debt, considering economic, demographic, socio-cultural, psychological and institutional variables that affect the propensity to borrow and to consume.

Consumer credit

Understanding the determinants of consumer credit behavior responds to numerous needs for risk measurement, for commercial and marketing purposes, for the protection of the financial balance of households and for the prevention of credit abuse phenomena. There are numerous factors that influence the propensity to borrow and the demand for consumer credit. A major role in the debt choices is played by the economic determinants concerning the income profile, the wealth and the level of expenses of the family. It refers to the current and expected disposable income, the nature of the income received (from employee work, self-employed or atypical jobs…), the number of family members who affect the level of actual income, the financial wealth, the existence of property real estate, including the house of residence and the existence of other debts, in particular long-term real estate loans. The institutional determinants consider the welfare and security systems enjoyed by the family and the efficiency of the judicial systems. Under the first profile, the level and methods of intervention of the health system, the insurance and social protections, the safety and breadth of the social security and assistance systems are relevant. Under the second profile, on the other hand, the ability and timing of justice in sanctioning opportunistic behavior are important. The demographic determinants concern the age of the head of the household, the phase of the family's life cycle, the degree of education, the profession, the area of residence, the type and size of the municipality of residence. Those factors are important because they influence the needs and the cultural and consumption reference patterns of the family. The psychological and behavioral determinants take into account the individual's perception of credit, to consider how personal and psychological factors influence the debt decision-making processes. Another area considers the individual factors that affect the personality of individuals, motivations, purposes, skills, preferences and perceptions, specifically those relating to the general economic or environmental situation and to the individual. The structural determinants mainly concern the characteristics of the financial system, the financial intermediaries, the financing products, and the credit distribution channels. These aspects affect the decisions to use consumer credit by changing the availability of credit, the simplicity of access to credit and the convenience of debt. A fundamental role is played by commercial distribution, in particular by large-scale distribution, which has increasingly used consumer credit as a marketing variable and as a promotional lever for its products.

Mortgages and retail estate loans

The demand for real estate mortgages can be divided into two categories: the application for financing necessary residential purposes and the one for non-necessary residential purposes. However, it is necessary to consider that the mortgage product has other areas of use, different from real estate investments and attributable to professional needs, and, last but not least, the debt consolidation in situations of family financial difficulty. The request for financing necessary residential purposes concerns the purchase, construction or renovation of a house that constitutes the habitual residence. This category constitutes the most relevant part of the mortgage applications and comes both from families who do not own a house, and from families who want to change and improve their housing situation. The application for financing for non-necessary residential purposes concerns the purchase, construction or renovation of a home intended for holidays or the future residence of the children. In this second case, temporary investment objectives are also often identified in relation to the age of the children or the time span existing before their marriage. This segment of demand is more heterogeneous than the first one. The application for financing for non-residential purposes concerns the financing of one's profession, the initiation of children into the profession, expenses of an exceptional nature that cannot be deferred. In many cases, especially in recent years, families in financial difficulty who needed to make their short-term debts sustainable by consolidating and increasing the time horizon of the amortization plan have resorted to mortgages. The determinants of the demand for real estate mortgages assume a more distinctly economic essence compared to consumer credit, since the importance of the amount, the duration and the impact on the household budget overshadow factors that may be more linked to individual or behavioral aspects. However, the high psychological value of the ownership of the residence and the high propensity towards ownership remain central, even in contexts and conditions that are scarcely rational from a financial point of view. The two fundamental factors that affect the demand for real estate mortgages are the trend of the real estate market and the level of interest rates, variables for which the cause-effect relationship is often not identifiable with certainty. The trend and situation of the real estate market relate to the level of house prices and rents and the depth of the market. The possibility of buying a home can be linked to the possibility of selling another real estate property. In certain negative periods or in specific areas, the real estate market is unable to feed the housing replacement process or this occurs over a very long period of time and to the detriment of the real estate value. The second relevant factor in the demand for mortgages is the level of interest rates, although it would be more appropriate to refer to the overall cost of the operations and the sustainability of the installments, linked not only to the level of market rates but also to other characteristics of the operation such as the expiry and the Loan to Value, which is the ratio between the amount of a loan and the value of the assets pledged as collateral. In short, the main purposes of the mortgage application are the following:

- purchase or construction of the first house and/or its renovation.
- construction of the second house.
- purchase or construction of the children's home.
- purchase or construction of housing for investment purposes.
- financing of one's profession or the profession of children.
- expenses of an exceptional nature and not deferrable.
- debt consolidation.
- improvement of the economic conditions of existing mortgages/loans.

Agents classification and clustering

Banks have developed a strong and ingrained market orientation. At the basis of this approach lies the attention paid to the customer's needs i.e. to the demand side. The latter, being expressed in various ways, determine different customer segments, that is, relatively homogeneous groups of customers (consumers or corporates). Banks aim to define their offer according to these segments in order to meet their needs more effectively or more efficiently than their competitors. The clustering of these individuals into segments is one way in which banks tend to reduce the complexity of the market with the final aim of organizing their business and achieving their goals. A customer cluster, if correctly identified and defined, must be:

- internally homogeneous, in the sense that customers belonging to the segment prefer the same characteristics of the product and/or service.
- externally differentiated, in the sense that customers in different segments have different preferences.
- characterized by a high reactivity of behaviors to the same marketing impulses.

From a strategic point of view, it is necessary not only to identify the segmentation parameters and customer segments but, above all, to choose the segments with which to operate. Segmentation can be done based on numerous parameters and criteria. The most common are:

- demographic criteria, generally corresponding to characteristics relating to age, family unit, family life cycle.
- geographical criteria, relating to the place of residence (area, region, small, medium, large municipality, …).
- socio-economic criteria, relating to income, type of disposable income, wealth, profession, level of education, riskiness, …
- psychographic criteria, relating to social class, lifestyle, reference group, psychological factors, propensity or aversion to risk.

- behavioral criteria, relating to the methods of use of the product or service, e.g. frequency of use, intensity of use, brand loyalty, sensitivity to different marketing tools, …

From an organizational point of view, suitable segmentation is the prerequisite for greater process efficiency through a high standardization of products/services and industrialization of the production phase. This is fundamental for a bank, especially for retail customer segments, characterized by small size and poor individual ability to generate value for banks, constituting the so-called mass market. In fact, retail customers are not all the same. They can be divided into two macro categories: mass retail and relational retail. Mass retail is the most standardized group since it is composed of customers (families and small economic operators), with simple needs and a high sensitivity to pricing policies. The relational retail sub-segment, on the other hand, includes affluents and small businesses, that is, customers who require more articulated assistance and a greater degree of customization, made economically sustainable by their ability to develop a greater volume of revenues. Chronologically, we have moved from segmentation criteria mainly based on geo-demographic variables towards behavioral criteria and, subsequently, towards criteria capable of identifying large groups of customers based on the distinction between individuals and companies and, within these two groupings, based on assets and turnover. However, there is no lack of more sophisticated segmentation processes that support marketing and commercial activities and product differentiation. A further approach to the segmentation of demand was that aimed at clustering the customers already acquired due to behavior in the use of banking services. This behavioral segmentation has allowed banks to get to know their customers better, both from a static and from a dynamic point of view and to identify new development opportunities in terms of cross selling or to recognize behavioral anomalies worthy of targeted interventions. Another very important segmentation criterion, which helped to redefine the business areas of the banking market, was the one known with the term "organizational segmentation". The private market has been divided into mass customers, affluent customers and private customers based on the financial assets owned. Nonfinancial Industries were divided, mainly on the basis of the size of their revenues, into small business (generally characterized by annual revenues below 2.5 million Euros), corporate (generally characterized by annual revenues above 2.5 million Euros) and large corporate (generally characterized by annual revenues greater than 50 million Euros). Like retail customers, corporate customers are also subject to segmentation. The market segmentation criteria can be used individually or through combinations. In the first case, companies are classified based on a single variable, for example, the size class, the financial complexity, the sector, the organizational characteristics, the degree of openness to the outside; in the second case they are grouped by referring to more variables at the same time: usually by crossing variables with each other (for example, size with complexity) and possibly adding further behavioral descriptors (for example, the degree of acceptance of innovative distribution channels). The most common segments are shown below:

Dimensional segmentation: the criterion traditionally used to segment the corporate market is based on size. Size can be variably measured, for example based on revenues, on the number of employees, etc..., and it allows to identify different segments of companies, typically labelled as large, medium and small. The different size of the companies corresponds to qualitative differences in the nature of the needs expressed and in the policies pursued by the bank to satisfy them. In particular, there are at least four profiles under which the companies belonging to the identified segments differ. I) The first profile is linked to the nature of the needs to be satisfied.

Traditionally, large firms tend to express complex financial needs that go beyond the mere need for financing. Conversely, small and medium-sized enterprises tend more frequently to manifest financial needs in the strict sense that translate into financing needs met through cash loans or endorsement loans. Complex financial needs, on the other hand, imply needs that also extend, for example, to the risk management determining a demand for banking services that is not exclusively oriented towards loans (credit derivatives, outright, flexible forward, swap, …). II) The second profile is related to the company's ability to access alternative sources of financing. Normally, larger companies have the possibility to draw on a greater plurality of sources of financing to meet their financial needs, such as, for example, the use of the securities market to raise risk capital or debt capital. This choice gives them greater bargaining power and consequently the opportunity to obtain more favorable price conditions. III) The third profile concerns the sensitivity to the interest rate and to other price conditions. The greater contractual strength allows larger companies to focus on the "economic dimension" of the relationship with the bank. The smaller the differentiation and complexity of the services that the bank offers to large companies, the more the company will choose its financial partner based on the most favorable price conditions it can obtain. The small business, on the other hand, is in a position of price taker in the sense that it has a limited possibility of influencing the definition of the price conditions of the services that the intermediary provides. IV) The fourth aspect refers to the complexity and articulation of the mix of financial services used by the company during its relationship with the bank. The size, the territorial distribution, the production and distribution complexity are elements that have a strong impact on the type of financial services that a company needs.

Segmentation according to the life cycle of the sector/company: the segmentation criterion based on the analysis of the life cycle of the sector and/or the company identifies the specific financial needs in relation to the phase of the life cycle in which the company, or sector, is located. In particular, this model identifies four phases, the birth, development, maturity and decline, which correspond to financial needs that differ from a qualitative-quantitative point of view. For example, in the development phase the company needs strong financial support, by way of loan capital; in the maturity phase, on the other hand, its primary financial needs are linked to the management of financial cash-flows and strategic advice. The basic hypothesis behind this criterion is that the company's growth process is accompanied by a change, an expansion and a greater complexity of the financial services it may need.

Segmentation based on organizational characteristics: this criterion is based on the organizational and managerial features that characterize the management of a company (for example "family-based management" vs. "managerial handling", private company vs. public company, etc.), recognizing the different organizational structures and different ownership structures purchasing behavior and their different needs. For example, a family-type business probably has a centralized decision-making process with mainly basic financial and credit needs. On the contrary, a managerial company is characterized by articulated, formalized and decentralized decision-making processes, in which the rational component is predominant; financial needs are more complex and lead to the need for advice in restructuring operations, support in capital market listing operations, the provision of guarantees for the issue of bonds, etc…

Segmentation based on the legal form: customer segmentation, specifically for lending purposes, can also refer to the different legal forms of the company, distinguishing between: I) the entrepreneur who, individually and professionally carries out an organized economic activity, not in a corporate form; II) the partnership, characterized by imperfect financial autonomy in which the partners are jointly liable, on a subsidiary and unlimited basis, for the obligations of the company, with certain exceptions established by law. The legal system recognizes a legal subjectivity to a partnership: it constitutes a separate subject from the partners, owner of its own legal relationships and its own assets; III) the stock company, characterized by perfect financial autonomy: the shareholders are liable for the corporate obligations within the limits of the share conferred, with some exceptions established by law. Legal personality is recognized by the legal system. The segmentation based on legal form has numerous implications from a credit profile since for each segment the information available, the administrative-accounting obligations, the separation between company and personnel assets, etc. are substantially different, and these elements affect the assessment of creditworthiness.

Segmentation based on the level of international opening: this segmentation criterion is used to consider the different management needs of companies that have a high degree of operations on foreign procurement. For this customer segment, risk management needs are particularly important, mainly linked to hedge their open positions in exchange rates. A further segmentation can be determined based on the areas of interest for the internationalization: operating in advanced foreign countries close to the country of origin is very different from operating in developing or emerging countries that can be geographically and culturally very distant.

Multi-criteria segmentation: the application of multiple criteria for market segmentation and, subsequently, to optimize the business portfolio activates a process that increases the degree of detail of the analysis. The first phase of the segmentation process consists of an economic analysis of the environment in which the company operates and the general data relating to the company. With regard to environmental data, the information commonly collected relates to the sector to which it belongs, while as regards the company, the information is limited to its size class and its geographical location. The second phase of the process involves an analysis of customers to identify elements of homogeneity in relation to economic-financial and behavioral aspects, such as: I) the preference for certain services provided by the bank and for particular distribution channels, like the preference for using the branch or on-line distribution methods; II) the possibility of establishing long-term relationships with customers, identifying "relationship-oriented" customer segments or, vice versa, the possibility of identifying companies with which to adopt risk-limiting behavior ("transaction-oriented" customers); III) greater or lesser attention to elements of customization/standardization of the service; IV) the logic that characterizes the process and the purchasing behavior of the company (short-term and price orientation, long-term orientation, relationship and quality of service).

The list of possible criteria could continue, but the identification and preference accorded to a segmentation policy must conform to the needs of the bank in relation to its functionality in managing the relationship with its customers.

Supply side

The purpose of this section is to analyze the main economic agents able to satisfy the demand for money. In this context it is essential to distinguish if the required amount of money comes from consumer lending or a real estate loan.

Consumer credit

The analysis of the structure of the supply side of the consumer credit market consists in examining the operators who offer products designed to meet the financial needs of households for consumption. From a regulatory point of view, agents who can offer loans to households and, therefore, potentially be present in the consumer credit market are:

- Banks.

- Suppliers of goods and service providers in accordance with TUB - art.122. They may conclude credit agreements in the form of a deferral of the payment of price with the exclusion of the payment of interest and other charges. For example, an appliance retailer could sell to its customers by splitting the price of the asset into multiple payments over time i.e., as if it were a loan but without adding any cost to the selling price.

- Foreign banks. The legislation allows banks and financial companies controlled by banks authorized in a Member State of the European Union (EU) to carry out banking activities admitted to mutual recognition in Italy, through a branch or under the provision of services without establishment, on the basis of the authorization issued by the authority of the country of origin and under the control of the authority itself, which remains responsible for their financial soundness. In many cases, this is the way in which financial and banking intermediaries linked to large industrial groups, especially in the automotive sector, are present in Italy, operating through the offices of dealerships and sales outlets located in Italy.

- Financial intermediaries in accordance with TUB - art. 106. The activity of granting loans to the public in any form is reserved for authorized financial intermediaries, registered in a specific list. Currently, financial intermediaries that carry out financing activities must possess specific legal and equity requirements, while their directors and shareholders must possess the requisites of professionalism, integrity and independence. Financial intermediaries are supervised entities, they must comply with the minimum capital requirements imposed by the regulations, and they must be able to carry out all risk containment activities in its various configurations. Finally, they must have an administrative and accounting organization and an internal control system capable of assisting and optimizing their risk management skills.

- Payment institutions in accordance with TUB - art. 114-sexies. Payment institutions may grant credit in close relation to payment services provided as an ancillary activity to the provision of payment services. For example, by placing a credit line at the disposal of its customers to avoid lack of funds to meet payment orders received over time. Payment institutions can grant loans as part of credit lines granted to payment service users or as part of the issue of payment instruments provided that: I) the loan is accessory and granted exclusively in relation to the execution of a payment transaction, II) the loan is short-term, not exceeding twelve months; only loans granted in relation to payments made by credit card can have a duration beyond 12 months, III) the loan is not granted using funds received or held for the purpose of executing a payment transaction.

- Microcredit companies in accordance with TUB - art. 111. Microcredit companies are a specific category of operators that can provide loans under specific conditions. They can grant loans, with specific characteristics, for starting or carrying out self-employment or micro-enterprise activities. Strictly speaking, this is not a consumer credit activity as it is a loan for professional purposes. However, they can operate in consumer credit pursuant to the third paragraph of art. 111 of TUB, which establishes that microcredit companies can also provide, on a non-prevalent basis, loans in favor of individuals in conditions of particular economic or social vulnerability, provided that the loans granted fulfill the following requirements: I) they are granted for a maximum amount of 10,000 Euro, they are not backed by collateral and have a maximum duration of 5 years, II) they are accompanied by the provision of auxiliary family budget services that provide debtors with useful information to improve the management of income and expenditure flows and are carried out during the entire duration of the loan repayment plan, III) they have the purpose of allowing the social and financial inclusion of the beneficiary, IV) their terms are more favorable than those prevailing on the market.

Please note that this classification only refers to operators and financial intermediaries. A category of activities directly implemented through the web can be added:

- Crowdfunding: a process through which a group of people financially supports an initiative (economic, social or other) or the satisfaction of a need (of a family, business or non-profit organization) through the web.

- Peer to peer lending: financing activities by numerous subjects who do not know each other, carried out without the help of financial intermediaries directly on the Internet through specialized sites that deal with putting the parties in contact and carrying out a credit check.

- Social lending: loan activity carried out by individuals to other individuals or non-profit companies on the Internet through specific platforms managed by specialized operators. Generally, the loan is split between numerous lenders.

The structure of the offer is also influenced by the type of distribution channels used by intermediaries. The legislation on consumer credit (Legislative Decree n. 141 of 2010) has significantly intervened on the organization of distribution channels, in order to improve the supervision of the customer relationship and fairness in the context of household financing. In the literature, distribution channels are defined as the set of subjects, activities, processes and technologies that allow the provision of a service to the customer. The focus is therefore on all types of channels that banks use in the context of their business. A functional classification for the analysis of the consumer credit market is that which distinguishes between:

- Direct distribution channels: branches of the company and the network bank.
- Intermediated distribution channels: commercial dealers, credit brokers, financial agents and branches of commercial partner banks.
- Virtual channels: Internet, ATM counters, credit cards, call center and smart phones.

Direct channels do not provide for the intervention of any person between intermediary and consumer. They can be defined as "short" and all those who act within these channels have a hierarchical relationship of dependence on the lender. In the case that the interaction with the consumer is mediated by technology, we are in the context of virtual channels that organize the provision of the service differently, with a greater participation of the consumer in the improvement of the service but always without any third-party intervention apart from the debtor and the lender. The intermediated distribution channels, on the other hand, provide for the intervention of one or more subjects other than the lender in the provision of the financing service. These subjects can carry out different types of activities, from the simple reporting of the customer, to the promotion of the loan and, in some cases, to the conclusion of the same. In many cases, they manage the interaction with the customer and constitute the only interface with which the consumer relates in the assignment process. The management of these negotiation phases, the most delicate for a correct development of the relationship and for the protection of the consumer, has pushed the legislative authorities to regulate the "credit intermediaries". The legislation (Legislative Decree n. 141 of 2010) regulates financial intermediaries and the intermediated distribution channels of consumer credit and the entire consumer financing activity. With reference to the distribution channels, it specifies the role of the activity and the extent of intervention of agents, credit brokers and commercial establishments both in the context of the credit process and in the individual phases and also in the financial relationship with the consumer. Particularly:

- Commercial dealers can participate in the promotion and conclusion of loan agreements solely for the purchase of their own goods and services based on specific agreements entered into with banks and financial intermediaries. However, these activities cannot be implemented if they concern contracts relating to the issuance of credit cards.

- Brokers can only promote bank loans but cannot under any circumstances conclude contracts, as well as carry out the provision of loans and any form of payment or collection of money on behalf of banks or financial intermediaries.

- Financial agents can act under a single mandate, that is, they cannot be agents of several financial intermediaries and offer similar products from different banks or financial companies. For reasons of completing the range of products, they may act under a maximum of three mandates but only if they concern different products.

From an operational point of view the classification of the products intended for the financing of households, which represents the demand side, can be carried out with reference to the purposes (finalized and non-finalized credit) and to the methods of use and duration (revolving credit or fixed-term credit). A possible classification of consumer credit products can be found below:

- Finalized credit: it includes all loans whose purpose is determined at the time of the request and, in general, concerns the purchase of a durable good, a consumer good or a service. The customer can access the financing directly from the point of sale of the supplier of goods or services.

- Non finalized credit: it includes all loans disbursed to households, intended to cover a generic financial need, the cause of which is not directly linked to the purchase of a specific good or service.

- Fixed-term loan: a loan with a specific duration and a debt amortization plan predefined in the installments and payment deadlines.

- Loan with an unspecified maturity: it includes all loans without any defined deadline and with high flexibility of use in which the debtor is granted the right to decide the amounts and the period of use in relation to his own financial needs. Generally, they are revolving credit lines where the return of the sums used restores the amount to be used.

The most important financing product used to be the classic installment loan, used mainly to finance the purchase of durable goods and, in particular, the purchase of new and used cars. Over time, thanks to a progressive change in consumer behavior and dominant orientations on the supply side, a significant disintermediation of consumer credit has been generated. The role of commercial dealers decreased significantly, and the weight of non-finalized loans recorded a significant increase. It should be specified that this is not due to a contraction in the volumes of finalized credit but to a higher growth of non-finalized forms of financing, like personal loans, salary-backed loans and revolving cards. The increase in the importance of non-finalized credit is a trend shared both by retail banks and specialized intermediaries. The main household financing products are:

- Personal loan: the personal loan is a non-finalized loan agreement that provides for the disbursement of a sum to a customer, who undertakes at the time of signing the contract to repay it according to a predefined amortization plan. The contractual scheme of the personal loan consists of the loan agreement, therefore the personal loan is perfected with the delivery of the money by the bank. The financed party assumes the obligation to repay the loan plus interest within a period normally not exceeding 10 years. Generally, in consumer credit, personal loans are mostly fixed-rate with French repayment but, with the lengthening of the duration of personal loans, floating-rate loans are also spreading. The personal loan allows to enhance the customer relationship since the applicant can contact the bank with which he has a fiduciary relationship and obtain conditions that are expressive of the actual economic and financial situation. This type of financing proves to be quite useful to satisfy financial needs of greater amounts and maturities or customer segments with greater risk, which therefore require a more in-depth assessment capable of enhancing even the so-called qualitative information. Both consumers and lenders have shown a growing preference for this type of loan. Currently, around 40% of consumer credit is made through this technical form. The personal loan is the preferential product of the banks' offer.

- Finalized credit: the targeted loans constitute a form of consumer credit whose purpose is determined at the time of the request and consists in the financing of the purchase of a durable good, a consumer good or a service. The customer normally accesses this type of loan directly from the sales branch of the commercial distributor through different organizational methods that allow quick management of the credit investigation activated by the commercial dealer. Although the operation involves only and directly the lender and the consumer, the sum is directly credited in favor of the commercial distributor who made the sale of the asset. This represents the real difference between the finalized loan and the personal loan. The other characteristics of the operation are substantially similar to the personal loan. Finalized credit is one of the most traditional forms of consumer credit, and indeed we can say that it has been the driver of the development of the Italian market. Although reduced in importance, it currently constitutes about 31% of the total credit.

- Loan against assignment of one fifth of the salary: the salary-backed loan is a personal loan intended for those who receive an income from public or private employment, open-ended and fixed-term, from retirement and from atypical work. The disbursement of the salary-backed loan does not require any minimum working seniority and its duration must not exceed ten years. The renewal of the loan is allowed only after at least two fifths of the original duration have elapsed. The salary-backed loan must be assisted by a double insurance policy that protects the creditor in the event of death and loss of employment. In fact, the insurance policy is also a way of protecting the heirs or family members in order to limit the effects of debt in the case of one of

the two events. There are no limits on the amount, although considering the maximum maturity and the constraint of the installment, the minimum pension treatment net of the installment must be protected. Loans to retirees only provide for an insurance policy which ensures the recovery of the residual credit in the event of the death of the retiree.
The salary-backed loan is a form of financing that involves several subjects, even though the credit relationship is always between the bank and the customer. In addition to the lender and the consumer, the loan against assignment of the fifth obligatorily involves an insurance company and the assigned administration (social security institution or employer). The third-party administration is the government, state or public administration, or the private company where the transferor works, which will have to pay, after notification of the contract, the monthly payment of the portion relating to the repayment of the loan, withheld from the salary. The administration, despite having an active part in the transaction, is a figure who does not intervene for the purpose of signing the contract, which remains exclusively bilateral between the bank and the customer. The insurance company guarantees the operation through two types of policy. The legislation provides that the loan is always guaranteed by an insurance policy that covers the risk of the transferor's death or loss of employment, whether due to resignation or dismissal. With reference to pensioners, however, only the life risk policy is envisaged. In the event of a claim (i.e. the negative event for the employee), the beneficiary of the policy is the lender who will collect the residual amount of the credit. The insurance retains a right of recourse against the transferor with reference to the employment risk. The assignment of the fifth, therefore, plays a positive role in all transactions with a higher risk profile, being a product naturally guaranteed by its operating structure and by the existence of insurance policies. Salary-backed loans, although having always existed on the market, have undergone an important revitalization with the regulatory measures that have affected them since 2005. Currently they represent about 10% of the overall consumer credit disbursed in Italy.

- Revolving credit cards: credit cards are electronic documents in the form of plastic cards that allow the holder, within the agreed monthly limit, both to make payments through the POS terminals installed in the affiliated shops, and to obtain cash advances at ATMs. The settlement of the balance takes place in installments through a specific credit line. Reimbursements periodically made aim to replenish the credit line for the same amount for a further use of the card. In most cases, these cards are issued with a dual functionality, by installments and by full settlement, in order to be used by the customer according to his preferences and needs. Such cards are called "option cards". The installment credit card is a form of financing with reduced monitoring possibilities and should therefore be granted to customers who have an adequate degree of reliability. The credit repayment plan used through a credit card is normally based on an installment proportional to the amount used or a fixed installment, defined in relation to the amount of the credit granted or on the basis of a predefined repayment period that determines the amount of the installment. Interest is calculated with the nominal annual rate and relates only to the amounts actually used by the credit card holder. In the case of an unused or inactivated installment card, the bank and/or the financial company do not receive any payment as interest. Installment credit cards are the most advanced financing instrument on the market as they allow access to credit in any area of consumption and leave maximum flexibility of use but at the same time they require a high degree of financial education to be able to correctly manage credit use and the effects on the income of the family. They represent approximately 10% of the total consumer credit disbursed.

Mortgages and Retail estate loans

The structure of the supply side of the household mortgage market is mainly composed of banks and specialized operators. Banks are the main operators, with a wide range of products for families. Over time, they have differentiated their business models by operating above all through the reorganization of both the production and the distribution. Some banking groups have opted for business models that provide for a unit specialized in the production of mortgages that carries out a captive activity towards the group's customers and operates through intermediated distribution channels. Several other groups and smaller banks have adopted integrated organizational solutions, internalizing production and distribution. Some banks make use of third-party networks but the main reference is constituted by the branches. Specialized operators operating in the Italian market are diversified in nature: in some cases, they are financial intermediaries belonging to banking groups, in others, they are the emanation of foreign banking groups operating in Italy through intermediated or virtual channels, in further cases they are financial intermediaries or specialized banks which operate solely in the household financing sector. In relation to the operational characteristics, their distribution organization is equally varied, which is strongly marked by multi-channeling (third-party networks, credit brokers, agents, financial advisors, points and financial shops as well as branches). With reference to the market as a whole, however, real estate mortgages are disbursed to a greater extent (and increasing) through traditional direct channels, branches, which carry about 79% of mortgages. This reflects not only the structure of the market but also the credit policies of the intermediaries: the increasing use of branches allows for greater control of risks and greater severity of customer acquisition criteria, a consequence of the greater economic difficulties and the increased riskiness that have characterized families since 2009. Indirect channels, consisting of third-party networks and credit intermediaries, affect approximately 17% of the volumes of credit disbursed, with a decreasing significance compared to previous years. Virtual channels (direct, i.e. managed by the lender, and indirect, i.e. managed by third parties) still have a marginal role on the market. Specialized financial intermediaries use external networks to a greater extent as a distribution channel, while the situation is reversed in banks, where the branch is the main distribution channel. For operational purposes we can distinguish four types of mortgages with reference to the different types of guarantees that assist it and to the different type of financial needs of the family:

- the mortgage loan ("mutuo fondiario" in Italian): the mortgage loan is a medium and long-term loan secured by a first degree collateral on a property. It is specifically governed by the TUB which defines its technical characteristics and operating methods. This kind of mortgage loan is normally aimed at the purchase or construction of a property. The maximum amount is determined in proportion to the value of the property placed under guarantee or the cost of the works to be carried out on it. The amount of the mortgage loan relating to the purchase of residential homes must never exceed 80% of the value of the property. This percentage can be raised up to 100% if additional guarantees are given. In any case, the mortgage must respect the following relationship:

$$\frac{mortgage\ loan\ amount}{real\ estate\ value\ under\ guarantee + additional\ guarantees} \leq 80\% \quad \textit{(Eq. VIII.1)}$$

In short, the concept behind mortgage loans is based on three substantial aspects: I) the amount of the loan must never exceed 80% of the real value of the property purchased, except in cases where there are additional guarantees; II) the duration of the loan must be medium and long term, thus over 18 months; III) the guarantee must be of a mortgage nature, which means strictly first degree and relating to a property.

- unsecured mortgages ("mutui chirografari" in Italian): unsecured mortgages (or "chirographary" from the Greek "cheiros", hand) are unsecured mortgages deriving from a simple private agreement signed by the debtor. In this sense they are practically identical to a personal loan. In operational practice, however, the unsecured loan indicates a loan without mortgage guarantees but, often, backed by personal guarantees, drafts or other guarantees. The absence of collateral affects both the duration of the loan, which normally does not exceed 10 years, and the investigation times, which are longer compared to consumer credit.

- "Liquidity" Mortgage Loans: "Liquidity" mortgage loans are defined as all mortgages that aim to finance the financial needs of families other than the purchase or construction of homes. Liquidity mortgages are by definition mortgages and have the function of transforming real estate into money necessary to meet the needs of the family related to the financing of their profession, specialized studies of children, expenses related to particular events such as marriage or to cope with situations of particular gravity related to health. Real estate property constitutes the guarantee of the loan but, in any case, the amount is proportionate to the current income of the family and its repayment capacity, in order to make the loan repayment plan sustainable. Liquidity mortgages constitute the competing product to the assignment of the fifth if reference is made to employees. However, it is necessary to verify the amount of the financial requirement and the deadlines necessary to make the installments sustainable, which, in some cases, exceed the constraints of the assignment of the fifth. One of the more specific purposes of the liquidity loan is debt consolidation. It happens that some families put in place an incorrect debt process, sometimes too short-term oriented or that originates a stratification of several unsustainable loans. In other cases, the same effect is determined by unforeseen events that affect income or current expenses or the weight of financial charges (trend in interest rates). All this can result in the financial inability to repay the installments and the consequent need to dilute them over a longer period of time. In these cases, liquidity mortgages combine to consolidate all existing debts into a single debt with a duration and a repayment profile more suited to the repayment capacity of the family.

- lifetime mortgage loans (PIV) or "reverse mortgage" - Law n. 44 of 2015 defines the lifetime mortgage loan as a medium and long-term loan, granted by banks as well as by financial intermediaries (pursuant to art. 106 TUB), with annual capitalization of interest and expenses and full repayment in a single installment. The loan can only be disbursed to individuals over the age of 60. Lifetime mortgage loans are secured by a first degree mortgage on residential properties. The first degree mortgage guaranteeing the PIV cannot be registered at the same time on several properties owned by the borrower. The reimbursement can be requested in a single solution: I) at the time of the death of the financed party; II) if the ownership or other real or entitlement rights on the property given as guarantee are transferred, in whole or in part; III) if acts are carried out that significantly reduce the value of the property given as a guarantee, including the establishment of collateral real rights in favor of third parties who burden said property. When signing the contract, the financed party is allowed to agree on any procedure for the gradual reimbursement of the portion of interest and expenses, before the occurrence of the aforementioned events.

Obviously, this is allowed on the repaid sums on which the annual capitalization of interest is not applied. In short, this is a loan that advances an amount proportional to the value of the property on which the mortgage guarantee is registered, normally never exceeding 50% of the evaluation value and, in any case, linked to the characteristics of the property, of the market in which it is located and its possibility to be resold. Mostly all types of properties are allowed, but the loan is normally refused in the presence of properties with poor possibility to be resold or in the presence of risks that characterize the property or the area in which it is located. The payment of interest and the return of capital are not envisaged until the death of the owner of the property or, if jointly held, of the longer-lived of the two owners. In any case, the debtors do not lose their right to live in the home. The lifetime loan (i.e. the amount disbursed plus accrued interest) can be repaid in a single solution by the heirs and/or assignees, normally within 12 months following the death of the longest-lived of the contractors.
When this event occurs, the heirs may decide to return the loan and interest without proceeding with the sale of the property, using other financial resources. If there are no heirs, or if they do not proceed with the repayment, the loan is extinguished with the sale of the property. In this case, the lender sells the property at a value equal to the market value, determined by an independent expert appointed by the lender, using the sums obtained from the sale to extinguish the credit due to the loan itself.
Once a further twelve months have elapsed without the sale having been completed, this value is reduced by 15 percent for each subsequent twelve months until the sale of the property is finalized. The function of these types of loans is to give an elderly person the opportunity to cope with exceptional health events or to guarantee the care needs of those who find themselves in difficult situations but are owners of the house in which they reside.

Interest rate

One of the main variables that characterize a mortgage is the method of calculating the interest, since it represents the component of the biggest cost. The main options are: fixed rate, floating rate, mixed rate, capped rate.

Fixed rate

The fixed rate provides that the interest rate is defined at the signing of the contract, and it remains constant for the entire duration of the loan. The fixed rate therefore allows to know both the amounts of the single installments and the total amount of debt (principal and interest) to be repaid at the time of subscription. With the fixed rate, the interest rate risk associated with any market fluctuations remains in the bank's income statement. Banks usually set an interest rate linked to the Interest Rate Swap (IRS) rate, that is the rate offered in swap contracts relating to interest rates that discounts the rational expectations on the market. From the customer's point of view, the choice of the fixed rate substantially responds to prudential and insurance needs deriving from the need to align the installments to the level of their income, normally not of a high amount but with characteristics of stability and certainty. This type of rate is advisable for those who want to be sure of the amounts of the single installments and the total amount of the debt to be repaid from the moment the contract is signed, in the face of more onerous conditions compared to floating rate mortgages. The disadvantage is not being able to take advantage of any reductions in market rates.

Floating rate

The floating rate provides that the interest rate is indexed to a market parameter which is defined at the time of the contract to which an annual nominal increase (spread) is normally added to take into account the costs and risk of the loan. Currently, banks mainly refer to the Euribor (Euro Interbank Offered Rate, i.e. the rate at which banks with primary credit standing declare themselves willing to exchange money with each other), or to the marginal refinancing rate of the ECB (i.e. the rate applied by the ECB on loans granted to banks for liquidity reasons). The money market and interbank rates reflect the trend in the cost of money on the market. However, banks mainly use other methods to raise money and, therefore, the average cost of funding does not coincide with the market cost, and it is usually lower. The variable rate follows the trend of the benchmark parameter and therefore transfers the entire risk rate to the borrower. Banks promote fixed or floating rate mortgages not only in relation to market trends but also to their exposure to interest rate risk, which also considers the characteristics of the interest rates adopted with depositors and bondholders. Given that they are facing the interest rate risk, families generally enjoy lower rates since they do not have to discount the prudential component that banks provide for the fixed rate in the face of uncertainty. In any case, the floating rate is only recommended for families who want to take advantage of the market trend but can support any increase in the amount of the installments. Traditional mathematical finance formulas can be applied to estimate the forward rate between two payment dates starting from the benchmark rate swap curve.

Mixed rate

The mixed interest rate implies the right of the borrower to periodically change their choice regarding the methods of calculating interest, transforming the mortgage from a fixed rate to a floating rate and vice versa. This option is specifically defined by the contract, together with the dates in which it can be exercised and the methods of determining the variable rate and the fixed rate (which is recalculated at each expiry of the exercise of the option). The mixed rate mortgage has a cost and is advisable when the borrower actually can follow and predict the trend of interest rates, otherwise it becomes a counterproductive and more expensive option due to the delay with which the choices are made. The balanced interest rate is different because it provides for the existence of a fixed rate and a variable rate used "pro quota" for the calculation of interest. Basically, the mortgage is partly at a fixed rate and partly at a variable rate, in order to balance the disadvantages of both and offer an intermediate solution in terms of risk and cost. From a quantitative point of view, it is not a big problem to calculate the interests on a mixed-rate loan if there is no optionality, that is the customer (or the bank) cannot dynamically choose if there is the convenience to switch from a floating to a fixed rate, or viceversa. The problem arises if there is no pre-fixed plan of the installments, that is a counterparty has the right to choose the type of rate to be applied at certain dates. In this second case, to evaluate the right to change the nature of the interests, it is necessary to perform an algorithm able to simulate the floating rates in the dates in which the option can be exercised. These projections are then compared with the fixed rate. If the customer originally has a fixed rate to pay to the bank and the simulated forward rate is lower, he has the convenience to switch, otherwise not. This reasoning can also be applied if the bank has a long position on this option: in this case the financial institution will exercise its right if the projection of the simulated forward rates is higher than the fixed rate. In quantitative finance, a numerical (or, when it is possible, analytical) integration of a Stochastic Differential Equation (SDE) has been typically adopted to design a Monte Carlo engine. Among the numerous

short-rate dynamics that can be implemented for simulating the floating rates, we choose the Vasicek (also known as Ornstein-Uhlenbeck) model.
The stochastic motion can be represented by the following SDE:

$$dr_t = k[\Theta - r_t]dt + \sigma dW_t \text{ (Eq. VIII.2)}$$

with: r_t the short-rate observed in the market at time t; W_t the standard Wiener process; k the mean reversion speed rate; Θ the long-term mean and σ the volatility.

This model is popular among professionals for different reasons:

- it considers the mean-reversion of the underlying, which is an important feature for this kind of variable.

- it only has three constant parameters to be calibrated, i.e. the long term mean level, Θ, the speed of the mean reversion, k and the volatility σ. The financial instruments which can be used for this task are very common on the markets (zero coupon bonds). The historical time series of the benchmark short-rates can also be used.

- there is an analytical solution for the SDE. This means that the projections of the rate can be performed directly on the relevant date without the need to code a numerical integration scheme, like the Maruyama-Euler scheme. This considerably reduces the computational time.

- the core dynamics also works in a low-rate environment. For the sake of precision, the tuning of the SDE parameters may have some problems with deep negative levels as shown in (Giribone, 2020). In this case, it is reasonable to adopt a one-factor Hull-White model, which is a generalization of the Vasicek model in which the mean reversion is time-variant and linked to the interest rate term structure.

There are two techniques for calibrating the model parameters of the SDE (k, Θ and σ): the least squares regression method and the maximum likelihood method.

Most software tools have built-in functionality for the least square regression. The formulas needed to perform it without using the standard software routine are reported below. We denote with the variable S_i the *i*-th sample in the dataset of rates used for the model calibration and with n the cardinality of this set. A least square regression can be done by calculating the following quantities:

$$S_x = \sum_{i=1}^{n} S_{i-1},\ S_y = \sum_{i=1}^{n} S_i,\ S_{xx} = \sum_{i=1}^{n} S_{i-1}^2,\ S_{yy} = \sum_{i=1}^{n} S_i^2,\ S_{xy} = \sum_{i=1}^{n} S_{i-1} S_i \text{ (Eq. VIII.3)}$$

From which we obtain the following parameters of the least square fit:

$$a = \frac{n\, S_{xy} - S_x S_y}{n S_{xx} - S_x^2},\ b = \frac{S_y - a S_x}{n} \text{ and } sd(\epsilon) = \sqrt{\frac{n S_{yy} - S_y^2 - a(n S_{xy} - S_x S_y)}{n(n-2)}} \text{ (Eq. VIII.4)}$$

Where ϵ is the independent and identically distributed (i.i.d.) normal random term. The relationship between the linear fit $S_{i+1} = aS_i + b + \epsilon$ and the model parameters is given by:

$$a = \exp(-\kappa\delta), b = \theta(1 - \exp(-\kappa\delta)) \text{ and } sd(\epsilon) = \sigma\sqrt{\frac{1-\exp(-2\kappa\delta)}{2\kappa}}$$ *(Eq. VIII.5)*

where δ is the process time step. Rewriting these equations in terms of κ, θ and σ, we obtain the tuned parameters for the Vasicek dynamics in accordance with the least square regression technique.

$$\kappa = -\frac{\ln(a)}{\delta}, \theta = \frac{b}{1-a} \text{ and } \sigma = sd(\epsilon)\sqrt{\frac{-2\ln(a)}{\delta(1-a^2)}}$$ *(Eq. VIII.6)*

Now let us deal with the calibration of the model parameters using the Maximum Likelihood. The conditional probability density function is derived by combining the simulation equation *Eq. VIII.2* with the probability density function of the normal distribution: $P\left(N_{0,1} = x\right) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$. The equation of the conditional probability density of an observation S_{i+1} given a previous observation S_i (with a δ time step between them) is given by:

$$f(S_{i+1} | S_i: \theta, \kappa, \hat{\sigma}) = \frac{\exp\left[-\frac{\left(S_i - S_{i-1}\exp(-\kappa\delta) - \theta(1-\exp(-\kappa\delta))\right)^2}{2\hat{\sigma}^2}\right]}{\sqrt{2\pi\hat{\sigma}^2}}$$ *(Eq. VIII.7)*

With $\hat{\sigma}^2 = \sigma^2\frac{1-\exp(-2\kappa\delta)}{2\kappa}$. The log-likelihood function of a set of observations $\{S_0, S_1, \dots, S_N\}$ can be derived from the conditional density function:

$$\ell(\theta, \kappa, \hat{\sigma}) = \sum_{i=1}^{N}\ln f(S_i S_{i-1}: \theta, \kappa, \sigma) =$$ *(Eq. VIII.8)*

$$= -\frac{N}{2}\ln(2\pi) - N\ln(\hat{\sigma}) - \frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{N}[S_i - S_{i-1}\exp(-\kappa\delta) - \theta(1 - \exp(-\kappa\delta))]^2$$

The maximum of this log-likelihood surface can be found at the location where all the partial derivatives are zero. This leads to the following set of constraints:

$$\frac{\partial\ell(\theta,\kappa,\hat{\sigma})}{\partial\theta} = 0$$ *(Eq. VIII.9)*

$$\frac{1}{\hat{\sigma}^2}\sum_{i=1}^{N}[S_i - S_{i-1}\exp(-\kappa\delta) - \theta(1 - \exp(-\kappa\delta))] = 0 \rightarrow \theta = \frac{\sum_{i=1}^{N}[S_i - S_{i-1}\exp(-\kappa\delta)]}{N(1-exp(-\kappa\delta))}$$

$$\frac{\partial\ell(\theta,\kappa,\hat{\sigma})}{\partial\kappa} = 0$$ *(Eq. VIII.10)*

$$-\frac{\delta\exp(-\kappa\delta)}{\hat{\sigma}^2}\sum_{i=1}^{N}[(S_i - \theta)(S_{i-1} - \theta) - \exp(-\kappa\delta)(S_{i-1} - \theta)^2] = 0 \rightarrow \kappa = -\frac{1}{\delta}\ln\frac{\sum_{i=1}^{N}(S_i - \theta)(S_{i-1} - \theta)}{\sum_{i=1}^{N}(S_{i-1} - \theta)^2}$$

$$\frac{\partial\ell(\theta,\kappa,\hat{\sigma})}{\partial\hat{\sigma}} = 0$$ *(Eq. VIII.11)*

$$\frac{N}{\hat{\sigma}} - \frac{1}{\hat{\sigma}^3}\sum_{i=1}^{N}[S_i - \theta - \exp(-\kappa\delta)(S_{i-1} - \theta)]^2 = 0 \rightarrow \hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}[S_i - \theta - \exp(-\kappa\delta)(S_{i-1} - \theta)]^2$$

The problem with these conditions is that the solutions depend on each other. However, both κ and θ are independent of σ, and knowing either κ or θ will directly give the value of the other. The solution of σ can be found once both κ and θ are determined. To solve these equations, it is thus sufficient to find either κ or θ. Finding θ can be done by substituting the κ condition into the θ.

Firstly, we change the notation of the θ and κ condition using the same notation as *Eq. VIII.3* which gives us:

$$\theta = \frac{S_y - exp(-\kappa\delta)S_x}{N(1-exp(-\kappa\delta))}, \kappa = -\frac{1}{\delta}\ln\frac{S_{xy} - \theta S_x - \theta S_y + N\theta^2}{S_{xx} - 2\theta S_x + N\theta^2} \text{ (Eq. VIII.12)}$$

Substituting κ into θ gives:

$$N\theta = \frac{S_y - \left(\frac{S_{xy} - \theta S_x - \theta S_y + N\theta^2}{S_{xx} - 2\theta S_x + N\theta^2} S_x\right)}{1 - \left(\frac{S_{xy} - \theta S_x - \theta S_y + N\theta^2}{S_{xx} - 2\theta S_x + N\theta^2} S_x\right)} \text{ (Eq. VIII.13)}$$

Removing denominators:

$$N\theta = \frac{S_y(S_{xx} - 2\theta S_x + N\theta^2) - (S_{xy} - \theta S_x - \theta S_y + N\theta^2)S_x}{(S_{xx} - 2\theta S_x + N\theta^2) - (S_{xy} - \theta S_x - \theta S_y + N\theta^2)} \text{ (Eq. VIII.14)}$$

Collecting terms:

$$N\theta = \frac{(S_y S_{xx} - S_x S_{xy}) + \theta(S_x^2 - S_x S_y) + \theta^2 N(S_y - S_x)}{(S_{xx} - S_{xy}) + \theta(S_y - S_x)} \text{ (Eq. VIII.15)}$$

Moving all θ to the left:

$$N\theta(S_{xx} - S_{xy}) - \theta(S_x^2 - S_x S_y) = S_y S_{xx} - S_x S_{xy} \text{ (Eq. VIII.16)}$$

Solving for θ:

$$\theta = \frac{S_y S_{xx} - S_x S_{xy}}{N(S_{xx} - S_{xy}) - (S_x^2 - S_x S_y)} \text{ (Eq. VIII.17)}$$

κ remains the same as in *Eq. VIII.10*. Having found an expression for θ and κ and using the notation introduced in *Eq. VIII.3*, we can also derive a formula for $\hat{\sigma}$ and σ.

$$\hat{\sigma}^2 = \frac{1}{N}\left[S_{yy} - 2\alpha S_{xy} + \alpha^2 S_{xx} - 2\theta(1-\alpha)(S_y - \alpha S_x) + N\theta^2(1-\alpha)^2\right] \text{ (Eq. VIII.18)}$$

$$\sigma^2 = \hat{\sigma}^2 \frac{2\kappa}{1-\alpha^2} \text{ (Eq. VIII.19)}$$

With $\alpha = \exp(-\kappa\delta)$.

Now that we have two tools to calibrate the Vasicek dynamics, let us proceed to calibrate the model on the time series of the ICE IBOR GBP 3 month (BP0003M Index). We take into consideration the fixing of the British pound short rates from 31st December 2010 to 30th September 2021 with a quarterly sample. The red line in Figure VIII.1 shows this time series, the x-axis label are the quarterly periods (i.e. 0 is the latest value, negative integers are the number of the previous quarters counting from 9/30/2021, positive numbers are the upcoming quarters). After implementing the two techniques for the Vasicek parameters in Python, we obtain: $\kappa = 0.258, \theta = 0.323$ and $\sigma = 0.235$. By the application of Itô's lemma, *Eq. VIII.2* can be rewritten into an analytical solution:

$$r_{t+1} = r_t \exp(-\kappa\delta) + \theta(1 - \exp(-\kappa\delta)) + \sigma\sqrt{\frac{1-\exp(-2\kappa\delta)}{2\kappa}}\epsilon_{0,1} \quad \textit{(Eq. VIII.20)}$$

With $\epsilon_{0,1}$ a random draw from a standardized normal distribution: $\text{NID}(0,1)$.
25 paths are simulated in accordance with *Eq. VIII.20* and they are represented in blue in Figure VIII.1.

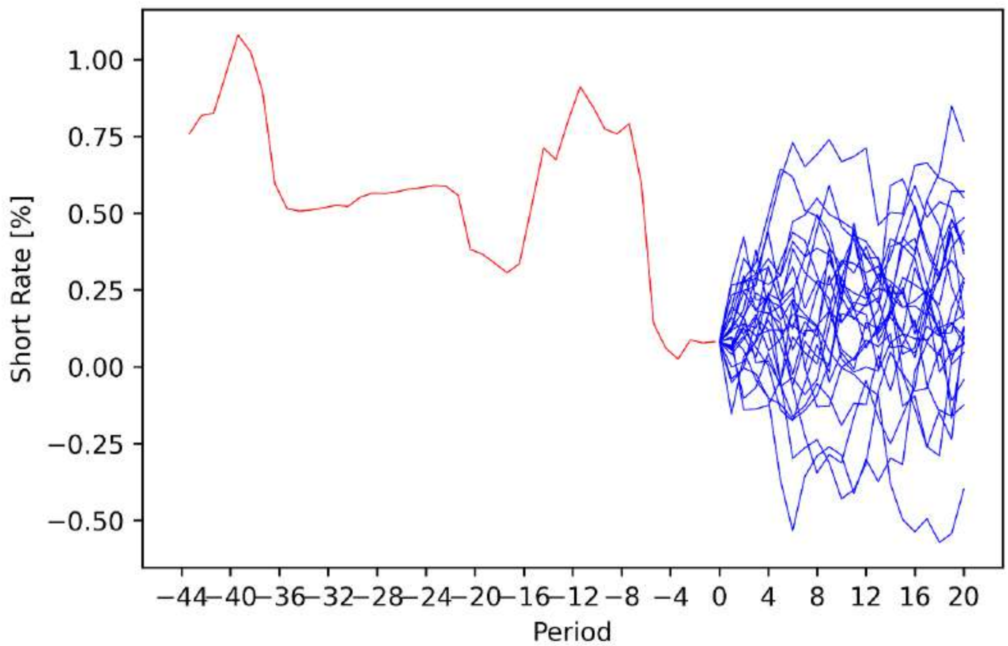

Figure VIII.1 Short rates simulations in accordance with the Vasicek model

This tool is useful to estimate the convenience to exercise the right for the holder to switch from paying an interest linked to a floating rate to a fixed rate. The main idea is to implement many replications (for instance 10,000) of the short rate for all the dates in which the analyst is interested in measuring his convenience to exercise the option, i.e., he has to compare the level of the simulated rates with the fixed level of the constant rate. The Python code is able to store the simulated values of r for each replication and for every future quarters.

Starting from this data, quants can calculate the in-the-moneyness probability of the option for each future time-step. In particular, the proposed code estimates the probability of having a short rate (assuming that BP0003M Index is equal to the value recorded on the market on 30th September 2021: 0.08188%) greater than the fixed rate equal to 0.8% within 21 quarters. This probability of having the option in-the-money at least once has been estimated in accordance with the implemented Vasicek model to be about 11.30%.

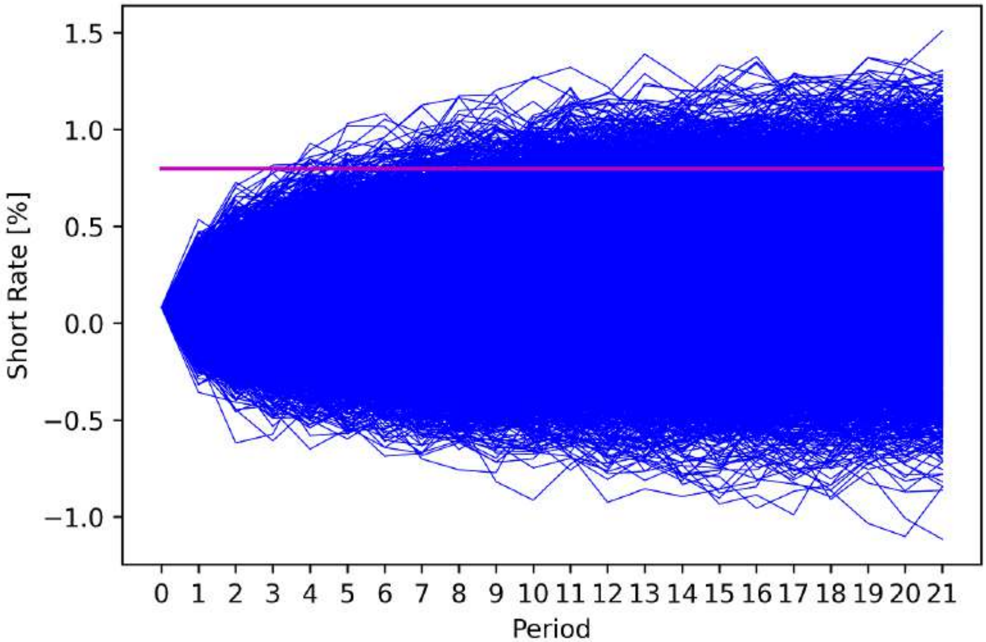

Figure VIII.2 Graphical representation of the short rates paths used for estimating the convenience to exercise the right of switching the mode for paying the interest on a mortgage

Capped rate

Banks have begun to offer households structured mortgages capable of associating the protections offered by derivative products to the loan agreement. The capped rate is a variable rate method which offers the customer the possibility of defining the maximum level of the interest rate: in the event that the level of market rates exceeds the maximum level (cap), the customer pays the target rate until interest rates return to the predefined level. In general, these mortgages have a higher spread than variable rate mortgages, but they have aroused growing interest from customers especially as a result of the instability and uncertainty that has characterized the market these past few years. The pricing formulas in accordance with the log-normal and normal pricing framework has already been discussed in Part III.

Repayment plans

The loan repayment plan (or loan amortization) summarizes the methods for repaying the loan by the debtor and specifies the amounts to be paid, the payment deadlines and the composition of the single installments in terms of interest and capital share. This feature of the loan is the area that allows the greatest degree of customization in terms of return flexibility and consistency with the family's income flows. The amortization of an undivided loan (that is a loan which cannot be divided into debt securities, such as bonds) is a form of loan in which the repayment of the capital and the payment of interest take place according to particular methods, among which the most widespread is the one that provides for the disbursement, by the debtor, of a certain number of periodic installments including capital and interests. The list of installments to be repaid, each with its own due date and in which the capital share and the interest share are usually paid, is called the "repayment plan" of the loan or the amortization plan of the loan. The essential elements of the loan with periodic repayment are:

- the initial capital (C) loaned to the borrower by the lender, which is typically a bank or a finance company.

- the amount of periodic installments (R), which can be the same for all installments or vary from one to the other.
- the frequency of the installments, indicated with (m), for instance if $m = 12$ the frequency is monthly, if $m = 2$, it is semiannual, etc….

- the interest rate (i) at which the loan is settled in a compounded interest regime.

- the total duration of the loan (n), i.e. the period from the moment of disbursement (when the sum is materially given to the debtor) to the payment of the last installment provided for in the amortization plan. It is expressed in years.

In the next subsections, we will examine the main typologies of amortization linked to loans and mortgages.

Amortisation with single reimbursement of capital

The simplest form of amortisation is that which provides for a reimbursement on a single maturity, after n years, of the full loan amount (C) and the payment of the interest i at the end of each period (Table VIII.1).

Period	Capital Share	Interest	Total Installment	Residual Debt
0	–	–	–	C
1	–	Ci	Ci	C
2	–	Ci	Ci	C
…	…	…	…	…
S	–	Ci	Ci	C
…	…	…	…	…
$n-1$	–	Ci	Ci	C
n	C	Ci	$C+Ci=C(1+i)$	–

Table VIII.1 Depreciation with single reimbursement of capital

The debtor only pays the interest Ci for $n-1$ years calculated on the residual debt (which for the entire duration of the loan is always C), while at the end of the last year (the n-th) the debtor pays both the interest and the full capital C, which is then repaid in a lump sum at the maturity of the loan, i.e. after n years. The interest payable at the end of each period is therefore the simple result of the product $C \cdot i$. Generally, loans are repaid through a depreciation plan characterized by the progression of the capital share, i.e. by the inclusion in the single payment of a share-part of the capital lent, which is added to the interest in the payment on the residual debt at the end of each period (Table VIII.2).

Period	Capital Share	Interest	Installment	Residual Debt
0	–	–	–	C
1	C_1	$(C_1+\ldots+C_n)i=I_1$	C_1+I_1	$C_2+\ldots+C_n$
2	C_2	$(C_2+\ldots+C_n)i=I_2$	C_2+I_2	$C_3+\ldots+C_n$
...	...	...	...	...
S	C_S	$(C_S+\ldots+C_n)i=I_s$	C_S+I_S	$C_{S+1}+\ldots+C_n$
...	...	...	...	...
$n-1$	C_{n-1}	$(C_{n-1}+C_n)i=I_{n-1}$	$C_{n-1}+I_{n-1}$	C_n
n	C_n	$C_n i=I_n$	C_n+I_n	–

Table VIII.2 Depreciation with periodic reimbursement of capital

Progressive French amortisation with constant installments

This is a gradual amortization in which the installments to be paid at the end of each year are calculated so that they remain constant over time for the entire duration of the loan. The installments therefore include a share of capital and a share of interest which, combining harmoniously together, maintain a constant periodic amount for all years. This is possible since the amount of capital repaid is low at the start of the amortization and then gradually increases while the loan is repaid. Vice versa the interest amount starts from a very high level and then gradually decreases during the amortization plan. It should be noted that this trend in the interest part is explained by the fact that the interest is calculated on an initially high residual debt and then increasingly lower by virtue of the progressive repayment of the capital that occurs at each installment date. Most of the loans of Italian banks are repaid using the French amortization method with constant installments which is the most widespread in Italy. Table VIII.3 briefly shows the mechanism of a standard progressive French amortisation with constant yearly installments. In accordance with (a quite old-fashioned) mathematical finance: $v=\frac{1}{1+i}$ and $a_{n\neg i}=\frac{1-(1+i)^{-n}}{i}$ (Italian actuaries read this notation as a “figurato” n at rate i). Generally, the installments of a loan have infra-yearly payments. One of the simplest methods to define the French amortisation values with infra-yearly installments is to calculate the monthly interest rate starting from the annual one, using the equivalent rate formulas valid in the compound interest regime $i_m=(1+i)^{1/m}-1$. Having no longer (i) but (i_m), the amortization is identical to that already considered, except for the need to extend the total duration of the loan to $n \cdot m$. Table VIII.4 shows this generalization.

Period	Capital Share	Interest	Total Installment	Residual Debt
0	–	0	0	C
1	$R \cdot v^{n}$	$R(1 - v^{n})$	$R = C / a_{\overline{n}\rceil i}$	$R \cdot a_{\overline{n-1}\rceil i}$
2	$R \cdot v^{(n-1)}$	$R(1 - v^{n-1})$	$R = C / a_{\overline{n}\rceil i}$	$R \cdot a_{\overline{n-2}\rceil i}$
...	...	...	...	...
S	$R \cdot v^{n-S+1}$	$R(1 - v^{n-S+1})$	$R = C / a_{\overline{n}\rceil i}$	$R \cdot a_{\overline{n-S}\rceil i}$
...	...	...	...	...
$n-1$	$R \cdot v^{2}$	$R(1 - v^{2})$	$R = C / a_{\overline{n}\rceil i}$	$R \cdot a_{\overline{1}\rceil i}$
n	$R \cdot v$	$R(1 - v)$	$R = C / a_{\overline{n}\rceil i}$	–

Table VIII.3 Progressive French amortization with constant yearly installments

Period	Capital Share	Interest	Total Installment	Residual Debt
0	–	0	0	C
1	$R \cdot v^{\frac{n \cdot m}{m}}$	$R(1 - v^{\frac{n \cdot m}{m}})$	$R = \frac{C}{\left(m \cdot a^{(m)}_{\overline{n}\rceil i}\right)}$	$R \cdot \left(m \cdot a^{(m)}_{\overline{n-1/2}\rceil i}\right)$
2	$R \cdot v^{\frac{n \cdot m - 1}{m}}$	$R(1 - v^{\frac{n \cdot m - 1}{m}})$	$R = \frac{C}{\left(m \cdot a^{(m)}_{\overline{n}\rceil i}\right)}$	$R \cdot \left(m \cdot a^{(m)}_{\overline{n-1}\rceil i}\right)$
...	...	...	...	...
$n-1$	$R \cdot v^{2/2}$	$R(1 - v^{2/2})$	$R = \frac{C}{\left(m \cdot a^{(m)}_{\overline{n}\rceil i}\right)}$	$R \cdot \left(m \cdot a^{(m)}_{\overline{1/2}\rceil i}\right)$
n	$R \cdot v^{1/2}$	$R(1 - v^{1/2})$	$R = \frac{C}{\left(m \cdot a^{(m)}_{\overline{n}\rceil i}\right)}$	–

Table VIII.4 Progressive French amortization with constant infra-yearly installments

Progressive Italian amortisation with constant capital shares

This is another particular case of progressive or gradual amortisation. It is characterized by the fact that in this repayment plan the capital shares are constant, but not the installments as in the French one. The capital shares are therefore all equal to: C/n. Therefore, the amortization plan of the loan assumes the configuration indicated in Table VIII.5. In the case the installment had an infra-yearly periodicity, we can use the equivalent rates formula as in the previous amortisation plans.

Period	Capital Share	Interest	Total Installment	Residual Debt
0	–	0	0	C
1	C/n	Ci	$(C/n) \cdot (1 + ni)$	$(C/n) \cdot (n-1)$
2	C/n	$Ci\,[(n-1)/n]$	$(C/n)\,[1 + (n-1)\,i]$	$(C/n) \cdot (n-2)$
...	...	...	...	...
$n-1$	C/n	$Ci\,(2/n)$	$(C/n)\,(1 + 2i)$	C/n
n	C/n	Ci/n	$(C/n)\,(1 + i)$	–

Table VIII.5 Italian amortization with a constant capital share

German amortisation

In this amortisation, interest is paid at the beginning of each period and not at the end; for this reason we have to use an anticipated interest rate for the calculation. Hence two cash flows are paid every year: the interest at the beginning and the capital share at the end of each period. The German amortisation works as follows:

- Upon stipulation of the loan (year 0) the debtor only pays the portion of interest of the first year.
- After one year he pays the first principal amount and the interest portion of the second year.
- After n-1 years he pays the capital share of the penultimate year and the interest share of the last year.
- After n years, the debtor only pays the last share of the capital (that of the last year).

French amortization can also be adapted to become "German", i.e. to provide for the payment of interest at the beginning of the period, rather than at the end. It is sufficient to leave the formulas for calculating the principal amount unchanged (which continues to be paid at the end of each period) and modify the formulas of the interest portion to consider that at the beginning of each year the debtor pays for the amount deriving from the application of the proper discount rate to the amount of the residual debt resulting at the end of the previous year. The latter value is therefore the result of the residual debt relating to the previous year multiplied by the discount factor, or, of the share of deferred interest (contained in the French amortization table) divided by $(1 + i)$ to anticipate it by one year. As said before, we can adopt the equivalent rates formula in the case of infra-annual cash flows.

American amortisation

The latter case is also called two-rate amortization, and is essentially built on two hypotheses:

- from the first to the $n - 1$-th maturity, the debtor reimburses only the interest portion Ci according to a rate of remuneration i, while at maturity of the loan he repays the entire sum lent in addition to interest: $C(i + 1)$;
- at the same time, the debtor also pays an installment to another subject at each period, accumulating money which, capitalized at an accumulation rate j, generally different from i, generates the sum C to be repaid on maturity of the loan.

Therefore, each total installment paid by the debtor amounts to:

$$R = \left(i + \frac{j}{(1+j)^n - 1}\right) C \quad \textit{(Eq. VIII.21)}$$

The debtor should use this method if the accumulation rate j is higher than the rate of return i. We have coded a Python function able to calculate repayment plans in accordance with the previous rules.
The coded function getAmortisationPlan(C,i,N,m,Type,j=0) takes as inputs: the capital C, the interest i of the installment, the duration of the loans in years N, and the infra-yearly frequency m. As a result, the number of installments to be paid is m * N. The function has a further optional argument: the accumulation rate j which is only taken into consideration in the American amortisation. In the following tables, the script automatically generates the plans for a loan with C = 10, 000, i = 0.05, N = 3, m = 2 varying the type of amortisation. In particular:

Table VIII.6 implements the Amortisation with single reimbursement of capital rules (Table VIII.1). The function must be called with Type='Single'.

Period	Capital Share	Interest	Total Installment	Residual Debt
0	0	0	0	10000
1	0	246.951	246.951	10000
2	0	246.951	246.951	10000
3	0	246.951	246.951	10000
4	0	246.951	246.951	10000
5	0	246.951	246.951	10000
6	10000	246.951	10247	0

Table VIII.6 Amortisation with single reimbursement of capital: C = 10.000, i = 5%, N = 3 and m = 2

Table VIII.7 implements the Progressive French amortisation with constant installments (Table VIII.4). The function must be called with Type='French'. Note that setting m=1, the code generates Table VIII.3.

Period	Capital Share	Interest	Total Installment	Residual Debt
0	0	0	0	10000
1	1566.7	246.951	1813.65	8433.3
2	1605.39	208.261	1813.65	6827.91
3	1645.03	168.616	1813.65	5182.88
4	1685.66	127.992	1813.65	3497.22
5	1727.28	86.3642	1813.65	1769.94
6	1769.94	43.7088	1813.65	0

Table VIII.7 Progressive French amortisation with constant installments: C = 10.000, i = 5%, N = 3, m = 2

Table VIII.8 implements the Progressive Italian amortisation with constant capital shares (Table VIII.5). The function must be called with Type='Italian'.

Period	Capital Share	Interest	Total Installment	Residual Debt
0	0	0	0	10000
1	1666.67	246.951	1913.62	8333.33
2	1666.67	205.792	1872.46	6666.67
3	1666.67	164.634	1831.30	5000
4	1666.67	123.475	1790.14	3333.33
5	1666.67	82.3169	1748.98	1666.67
6	1666.67	41.1585	1707.83	0

Table VIII.8 Progressive Italian amortisation with constant capital shares: C = 10.000, i = 5%, N = 3, m = 2

Table VIII.9 implements the German amortisation. The function must be called with Type='German'.

Period	Capital Share	Interest	Total Installment	Residual Debt
0	0	240.999	240.999	10000
1	1566.70	203.242	1769.94	8433.30
2	1605.39	164.552	1769.94	6827.91
3	1645.03	124.907	1769.94	5182.88
4	1685.66	84.2829	1769.94	3497.22
5	1727.28	42.6554	1769.94	1769.94
6	1769.94	0	1769.94	0

Table VIII.9 German amortisation: C = 10.000, i = 5%, N = 3 and m = 2

Table VIII.10 implements the American amortisation. The function must be called with Type='American' and specifying the accumulation rate j = 0.04.

Period	Capital Share	Interest	Total Installment	Res. Debt	Acc. Share
0	0	0	0	10000	0
1	0	246.951	246.951	10000	1557.15
2	0	246.951	246.951	10000	1557.15
3	0	246.951	246.951	10000	1557.15
4	0	246.951	246.951	10000	1557.15
5	0	246.951	246.951	10000	1557.15
6	10000	246.951	10247	0	1557.15

Table VIII.10 American amortisation: C = 10.000, i = 5%, N = 3, m = 2 and j = 5.5%

The global effective annual rate – "TAEG"

The "TAEG" is the main information parameter through which the actual cost of a loan can be verified. The TAEG identifies how much the charges relating to the loan affect, on an annual basis, the total sum of the amounts made available to the consumer. It therefore provides an indication of the actual cost of a loan since, on the one hand, it considers all the costs that the consumer has to bear, regardless of whether they are components of the lender's revenue (taxes, for example) and, on the other hand, it considers the sums actually transferred to the consumer (disbursed capital). The use of the financial discounting formula (which indicates the present value of a future cash flow) also allows to take into account the temporal distribution of costs, withdrawals and reimbursements and, therefore, to consider the actual period of use of the money (and the financial value of time). The formula is the same used in financial mathematics to calculate the internal rate of return (IRR) obtained by equating the net present value (NPV) to zero. Finally, it should be remembered that the calculation of TAEG must be done with a prudential logic that is, where there are different options - for example cost - the worst hypothesis for the customer should be used, in order to represent the most expensive possible situation of the customer's financing. The TAEG introduced by the legislation includes interest and all costs, commissions, taxes and all other expenses that the consumer must pay in order to obtain and use the loan and of which the lender is aware. Only notary fees are excluded, if they exist. The TAEG includes any fees from credit intermediaries, costs relating to ancillary services connected with the credit agreement and mandatory to obtain the credit or to obtain it at the conditions offered which the lender is aware of. It also includes - if subject to an agreement between the lender and the consumer – the costs of managing the account on which the payment transactions and withdrawals are recorded, the costs related to the use of means of payment that allow payments and withdrawals to be made and all other costs relating to payment transactions. In the event that the account can also be used for different transactions, the TAEG includes all fixed costs (even if aimed at remunerating services unrelated to the loan) and only variable costs depending on the use of the loan alone. An example of a TAEG calculation is now presented. It has been said that the TAEG is calculated as the discount rate that results from the solution of the net present value equation, which considers the (algebraic) sum of all repayments, withdrawals and costs that occur during the life of a loan. On March 28, 2013, the change in the hypothesis for calculating the TAEG (EU Directive 90/2011) was implemented, which provides for the existence of a single calculation method for all consumer loans, including current account overdrafts.

$$\sum_{k=1}^{n} \frac{F_k}{(1+TAEG)^{t_k}} = 0 \quad \textit{(Eq. VIII.22)}$$

Let us suppose we have a one-year loan of EUR 5,000:
- to be repaid in 12 constant installments, at a rate of 10%.
- with fixed commissions of various kinds in the amount of EUR 200 with advance payment.
- and monthly expenses of EUR 5.

The monthly payment, given by principal and interest, is equal to EUR 439.58. The TAEG is 22.25%, that is a value which makes the sum of F_k equal to zero (Table VIII.11).

k	0	1	2	3	4	5	6
t_k	0	0.082	0.164	0.247	0.329	0.411	0.493
F_k	4800	-444.6	-444.6	-444.6	-444.6	-444.6	-444.6
$(1+\text{TAEG})^{t_k}$	1.000	1.0167	1.0336	1.0508	1.0683	1.0861	1.1042
NPV	4800	-437.32	-430.16	-423.11	-416.18	-409.37	-402.66

k	7	8	9	10	11	12
t_k	0.575	0.658	0.740	0.822	0.904	0.986
F_k	-444.6	-444.6	-444.6	-444.6	-444.6	-444.6
$(1+\text{TAEG})^{t_k}$	1.1225	1.1412	1.1602	1.1795	1.1992	1.2192
NPV	-396.07	-389.58	-383.2	-376.92	-370.75	-364.68

Table VIII.11 TAEG computation

The correct calculation of TAEG can be checked summing the NPVs contribution in Table VIII.11. This sum should be zero. The importance of the real estate debt decision is confirmed by the regulatory attention placed on the degree of information transparency that must characterize this moment. Financial intermediaries must provide a complete information framework both in the pre-contractual phase and in the phase of signing the contract. They follow the principle of proportionality by providing a protection modulated according to the subjective characteristics of the customer, through the preparation of provisions applicable exclusively in relation to retail customers. With reference to the loan, as part of these obligations, intermediaries are required to provide:

- the TAEG;
- information on economic conditions in such a way that the overall cost can be easily understood.
- the indication that the customer will be able to consult the global average effective rate (TEGM - Tasso Effettivo Globale Medio), i.e. the reference rate for identifying the threshold relating to the hypothesis of usury.
- the comparison sheet of mortgages: it is added to the information sheet in the case of mortgage loans for the purchase of a house and it has the function of facilitating the understanding of the products allowing an optimal choice by the customer.

The TEGM indicates the average value of the rate effectively applied by the banking and financial system to homogeneous categories of credit transactions (for example: current account credit, personal loans, leasing, factoring, mortgages, etc.) in the second previous quarter. The calculation of the rate must consider the commissions, remuneration and the costs associated with the provision of credit and incurred by the customer, of which the lender is aware, also taking into account the legislation on transparency. Taxes, fees and notary fees are excluded. The recognition takes place in a differentiated manner both for the technical forms and for the amount classes, to take into account the greater incidence of fixed costs on low-value loans. The average overall effective rate is used to calculate the threshold rate, beyond which transactions are considered usurious. The TEGM results from the survey carried out every three months by the Bank of Italy on behalf of the Minister of Economy and Finance. From 14th May 2011, the limit beyond which interest is deemed usurious is calculated by increasing the average overall effective rate by a quarter, plus a margin of a further four percentage points. In any case, the difference between the threshold rate and the TEGM cannot exceed eight percentage points.

Mode of Settlement plans

The repayment of the loan can take place in different ways in relation to the evolution of the relationship between the customer and the bank and the evolution of the debtor's earning and savings capacity which could give rise to the need for a modification of the original loan or allow to repay in advance. The ordinary hypothesis is that the obligation is closed with the payment of all installments and, therefore, upon the natural expiry of the loan. The evolution of the loan can also follow other directions:

- early repayment of the loan. The customer can pay off all or part of the loan before the maturity without having to pay any penalty, fee or additional charge. Total repayment involves the termination of the contractual relationship with the repayment of the capital still due before the loan expires. The legislation has broadened the choice of the debtor, reducing the contractual rigidity of the mortgages and the cost of the changes, assuming that the cost of repayment represents a serious obstacle to competition in the credit market. In the case of mortgages other than those granted for the purchase of the house, this involves the right of the debtors to pay off early and the right of the bank to demand, as consideration for the early withdrawal, exclusively an all-inclusive fee for the repayment, which must be defined in the contract. In the case of loans granted for the purchase of the house, it involves the nullity of any agreement or clause, even after the conclusion of the contract, by which it is agreed that the borrower is required to pay a fee or penalty for early repayment (total or partial) of the loans taken for the purchase or renovation of real estate units used as homes or for the performance of their economic or professional activity by individuals.

- replacement of the creditor (subrogation). The legislation provides for the possibility of replacing the bank that granted the loan through the subrogation of the original creditor, thus creating the so-called "portability" of the loan. This allows the borrower to replace the loan with a new loan without the need for the lender's consent. The reasons are substantially linked to the possibility of obtaining more favorable economic conditions compared to the pre-existing contract which, therefore, increases competition and promotes fairness in the bank-customer relationship. In particular, thanks to the subrogation discipline (Article 120-quater TUB), the subrogate lender (i.e. the new bank) takes over the personal and real guarantees that were lent by the borrower to guarantee for the original credit (i.e. the old mortgage) without the need to cancel the old mortgage and register the new one, while the debtor does not have to incur charges, expenses or commissions for the granting of the new loan. Therefore, expenses or commissions cannot be imposed on the customer (not even indirectly) for the granting of the new loan, for the preliminary investigation and for the cadastral assessments, which must be implemented according to collaboration procedures between intermediaries, based on general criteria.

- replacement of the debtor (assumption of debt). The replacement of the debtor through the assumption of the debt determines the replacement of the debtor by a third party. The assumption occurs in the purchase and sale transactions in which the buyer decides to take over the seller's obligations against a sale price of the property which considers the debt incurred in terms of residual capital and interest. Another frequent case of replacement of the debtor may also occur in the case of separation of the spouses, when the loan agreement is jointly held by the spouses or in the case of the debtor's disappearance, if the heirs take over the obligation. The replacement is assessed by the bank which verifies the reliability and sustainability of the new debtor and may request new additional guarantees, if necessary.

- termination due to default. The bank has the right to terminate the contract in the event that the debtor is late in fulfilling to pay the installments at least seven times, even if not consecutive. For this purpose, the law considers the payment made between the thirtieth and the 180th day from the expiry of the installment to be a delayed payment.

- the renegotiation of the loan. The renegotiation of the loan is not a real method of repayment, but it gives flexibility to the bank-customer relationship, reducing the competitive demands coming from the market. The liberalization that has taken place in the mortgage market and the reduction in exit or change costs incentivize banks to favor the renegotiation of loan conditions in order not to lose the customer relationship or, in some cases, to rebalance the economic and financial situation of the debtor. In the event that the renegotiation concerns the reduction of the duration, the reduction of the spread, the variation of the indexing parameter, the reduction of the fixed interest rate or the variation of the frequency of the installments, only the consent of the parties and an exchange of correspondence are required. In the case of a mortgage loan, the extension of the duration of the loan requires, however, the use of a public deed.

The loan amount and the Loan to Value ratio

The legislation does not identify a maximum amount of the loan. The only requirement concerns the mortgage where, as seen, there is a maximum amount equal to 80% of the value of the guarantee, which can be extended to 100% in the presence of additional guarantees. Consequently, the Loan to Value (LTV), or the relationship between the amount of the loan and the value of the property placed as collateral (which often coincides with the property purchased) can assume values up to 100%. The trend of the real estate market and the dynamics of house prices have produced a generalized increase in the amounts requested in the last 15 years. Over two thirds of the required mortgages exceed EUR 100,000. The amount is a direct function of the value of the property and, above all, of the capacity and income stability of the family that must ensure the repayment of the loan. The LTV of land loans also reaches 100% of the property value. In these cases, banks may request additional guarantees from third parties, or they may require a third party to commit with their income to repay the loan in the event of non-payment by the borrower. In Italy, the average LTV is approximately 59%.

Duration

The price of houses has produced a generalized increase in financial needs and, consequently, in the amount of mortgage applications. This contributed to extending the duration of the mortgages in order to be able to distribute the loan repayment plan over a broader time span and thus make the installments sustainable. The duration of the mortgages normally varies from 5 to 30 years, and a few financial intermediaries offer mortgages with even longer durations. The duration of the loan is influenced by the socio-demographic and economic characteristics of the borrower. Age, level and characteristics of income are the main factors that determine the choice of duration. Young people with a low income level but with a "stable" growth prospect can access longer-lasting mortgages that increase the sustainability of the installment. People of advanced age encounter obvious limits in accessing very long durations. The flexibility of the duration of the loan and the possibility of suspending the payment of a specified number of installments to cope with the temporary financial difficulties of families have been highly sought after and decisive characteristics in the choice of mortgages in the past few years. Recently, with the worsening of the financial situation of many families following the negative phase of the economy, the duration has been the subject of government interventions and agreements between banks and consumer associations to improve the sustainability of the loan in times of difficulty for the households.

Guarantees

The traditional concept of guarantee in the banking sector has become rather reductive since it is necessary to consider all forms of credit risk mitigation, i.e. the set of techniques, instruments, operations, contracts and their combinations which allow to reduce the credit risk and the negative effects of lending. However, in this paragraph the focus is on the traditional forms of guarantees that assist numerous loans, divided into real guarantees, having as their object a real thing, such as a mortgage and a pledge, and personal guarantees, in which there is the possibility for the creditor to contact a person other than the original debtor to demand the fulfillment of the obligation, such as the surety and the endorsement. With reference to mortgages, the lien is the main form of guarantee to which the greatest space will be dedicated. The other forms of guarantee will also be briefly discussed. The creditor guaranteed through the lien (so-called lien creditor) acquires two related rights:

- the right to expropriate the asset pledged as security (so-called mortgaged asset), i.e. the right to ask the judge to sell the asset in order to satisfy their credit with the proceeds (or the assignment of ownership of the asset itself).
- the right to be satisfied with preference over other creditors on the proceeds of the sale.

The rights that the mortgage assigns to the creditor follow the asset; therefore, following the establishment of the mortgage, the creditor may also request expropriation against the third party who, after the establishment of the mortgage, purchased the mortgaged property from the debtor. For this reason, the mortgage is defined as a real right of guarantee, and together with the pledge it constitutes a real guarantee, to emphasize the fact that, like the property right and other real rights, it can be enforced against anyone. In any case, the mortgage is enforceable against anyone who claims a real right acquired after the constitution on the mortgaged property. Furthermore, in all cases in which several creditors claim to assert their reasons on the mortgaged asset, the mortgage creditor will have the right to be fully satisfied on the proceeds from the sale of the asset, while the other creditors (so-called "chirographary" credits) may divide the residual in proportion to the their credit.

The assets on which it is possible to establish a lien, in the most common cases, are:

- real estate and its appurtenances.
- the usufruct of real estate.
- the right of surface, that is the right that allows its owner to build and maintain a building on someone else's land, while retaining ownership.
- the State revenues.
- ships, planes and cars.

The establishment of the lien takes place through the registration in the public real estate register of the place where the property is located. This is a necessary condition for the existence of the lien. The creditor can proceed with the registration only if he has a title, that is, if there is a legal act or fact that recognizes the right to lien the property. The title can be a rule of law (legal lien), an agreement between the parties (voluntary lien) or a provision of the judicial authority (judicial lien). It should be noted that the existence of the title to proceed with the registration does not mean that the lien exists: the lien is validly constituted only once it has been registered in the public real estate register. The registration retains its effects for twenty years from the date on which it was made. These effects cease if the registration is not renewed within this period. To obtain the renewal, a new registration note must be submitted, conforming to the original one, in which the intention to renew the original registration is declared. Even if the renewal is not carried out in time, the creditor retains the right to request a new registration, which however is effective from the date in which it is implemented, with the consequence that the lien cannot be opposed to the third party purchasers of the lien property. It is possible to establish several liens on the same property. The first lien to be registered is called a first degree lien and, in the same way, the subsequent liens are named after the order in which the registration took place (for example: second degree lien, third degree lien, ...). It is possible that liens have the same degree, in the case in which several people present the registration request at the same time, against the same person or on the same properties. The degree of the lien is very important in the exercise of the banking business, because the TUB provides that the legislation on land credit applies when the lien granted to the bank is a first degree lien on real estate. Even if the lien is indivisible, weighing on all the assets on which it is registered and on each part of them, the law provides for the possibility of reducing the lien, when the guarantee is excessive compared to the credit to be guaranteed. The reduction can take place in two ways. First, it is possible to reduce the amount of credit for which the registration was made. Alternatively, it is possible to restrict the registration to only a part of the assets on which the lien was established. In land credit, each time the debtors have repaid the fifth part of the original debt, they can request a reduction of the amount registered in the lien. Furthermore, they have the right to obtain the partial release of one or more properties when it appears, from the documents produced or from appraisals, that for the sums still due the remaining tied assets constitute a sufficient guarantee to the 80% level. The lien can be extinguished for several reasons:

- for cancellation.
- for failure to renew the registration within the term of the lien.
- for the extinction of the guaranteed obligation.
- for the creditor's waiver.
- for the expiry of the term to which the lien was limited or with the occurrence of the termination condition.
- for the provision that transfers the expropriated right to the buyer and orders the cancellation of the mortgages.

The cancellation is implemented by the registrar of the real estate register in which the lien is registered, upon presentation of the deed containing the consent of the creditor, as well as the request for cancellation. The pledge gives the guaranteed creditor a series of faculties relating to an asset or a right. The so-called pledge (i.e. secured) creditor has the right to have the asset received as collateral sold and the right to satisfy its claims on the asset received as collateral with pre-emption over other creditors. The pledge may relate to assets, rights relating to assets and credits. The pledge of property takes place through the delivery of the asset, or through the delivery of the document that ensures the creditor the exclusive availability of the asset. In the first case, for example, the debtor delivers a jewel to guarantee the payment of a sum of money; in the second case, since the asset cannot be physically delivered to the creditor, a document is delivered which exclusively attributes the availability of the asset itself (for example the so-called pledge note). The creditor must keep the asset and cannot use it without the consent of the person who gave the guarantee. In the event of non-payment, he can proceed with the sale of the asset (at auction or through an authorized intermediary) or by asking the judge for the assignment, i.e., the ownership of the asset. In the case of pledge of a credit, the agreement establishing the pledge must be in writing and be communicated to the debtor of the pledged credit. In the event of non-payment, the creditor can proceed with the collection of the credit received as a pledge. The surety is a personal guarantee. It does not refer to an object but to a person other than the creditor who ensures the fulfillment of the obligation with his assets and his income capacity. If the debtor fails to pay the debt, the creditor can contact the guarantor and demand the fulfillment of the guaranteed obligation, without even bothering to act against the original debtor. However, the guarantor who fulfills the guaranteed obligation has the right to retaliate against the principal debtor and any other guarantors. The surety must be given expressly, that is, through an explicit declaration of the guarantor and indicate the maximum amount of the guarantee. Furthermore, it follows the main obligation (so-called accessory nature of the surety) for which the surety is valid if and to the extent that the guaranteed obligation (i.e. the credit) exists. The endorsement is a declaration affixed to a bill of exchange or (more rarely) to a bank check through which a person guarantees the payment of the sum, in whole or in part. The declaration consists in the signing of the security by the guarantor with an indication as an endorsement or guarantee. Unlike the surety, the endorsement is an autonomous guarantee and therefore the guarantor can only oppose the creditor with formal exceptions.

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VIII.2 COUNTERPARTY RISK

From a financial point of view, clearly a loan can be managed like a fixed income instrument and its theoretical fair value can be estimated using the discounted cash flows (DCF) methodology. As a result, the most challenging aspect is to consider the risk premium to be added to the risk-free discount rates. The first part of the chapter shows how to extend the DCF methodology with the aim of implementing a more realistic pricing model for a loan or a mortgage. In the second part, we introduce how the Probability of Default can be estimated using financial market data.
The most preferred methodology is to use Credit Default Swap (CDS) Premiums. When they are not listed, the second financial approach is to use the quotes for actively traded bonds, and this approach will be explained. As a last resort, the KMV (Kealhofer, Merton and Vasicek) model can be employed, which is heavily based on statement data.

Broadly speaking, counterparty risk can be considered as a sort of hybrid between the traditional financial risk and credit risk. One of the main risks when a bank lends money is clearly the possibility that the borrower is unable to fulfill his obligations. A traditional way to consider the risk premium is to extend the traditional method used for pricing financial instruments like bonds. If we consider the value of the loan as the sum of the present value (i.e. the current value) of the expected cash flows (principal plus interests), discounted at a rate including the bank's risk premium, we can capture the effects of changes in the debtor's risk, similarly to what happens for the value of a bond.

The interests received by the creditor are normally fixed or indexed to a market parameter to cover the variations in the cost of funding, like the coupons of a security: they are not able to remunerate the bank for the greater risk borne following a possible deterioration in the quality of the debtor.

Let us analyze a practical example and consider a five-year loan of EUR 500,000 at a fixed rate of 3% with a monthly payment of EUR 8,975.36. If there is no change (for example a deterioration in the external rating of the borrower), the theoretical value of the loan is equal to:

$$\sum_{t=1}^{60} \frac{8{,}975.36}{(1+3\%)^{\frac{t}{12}}} = 500{,}000$$

If, on the other hand, there were a worsening that required an increase in the discount rate - which would cover a greater risk - of 0.5% - the theoretical value of the loan would be equal to:

$$\sum_{t=1}^{60} \frac{8{,}975.36}{(1+3.5\%)^{\frac{t}{12}}} = 494{,}042.8$$

If the bank did not incorporate such "risk increase" in the rate, the worsening of the individual debtors would

result in an increase in the positions in default, at the loan portfolio level, therefore, in the losses borne by the bank, equal to the value of the new expected losses not transferred into the price.
A code can be easily written allowing to compute a risk-adjusted NPV generalizing the previous example: the code is written in order to take into account different discount rates to be applied at the different future payment dates, as well as variable installment amounts.

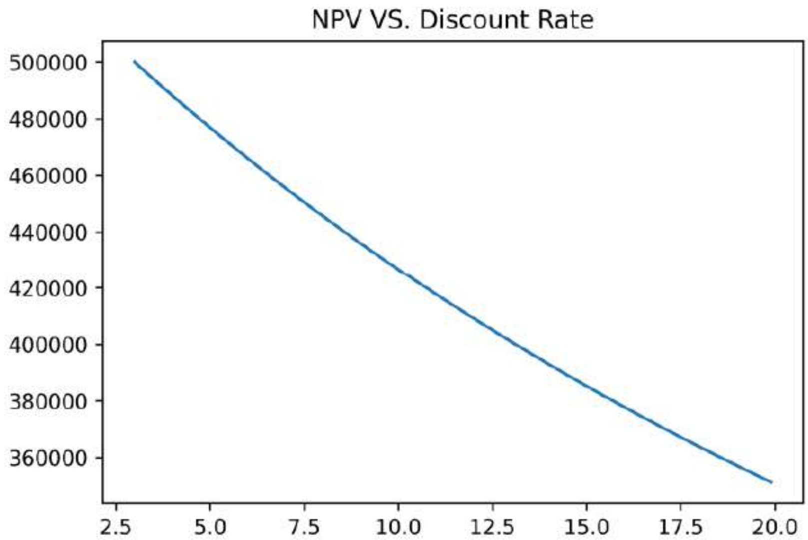


Figure VIII.3 NPV versus Discount Rate

If the counterparty has listed Credit Default Swaps (CDS) on active markets, the probability of default can be calculated using their premiums (s), in accordance with the following formula:

$$PD(T) = 1 - \exp\left(-\bar{\lambda}(T)T\right) \text{ (Eq. VIII.23)}$$

where:
$\bar{\lambda}(T) = \frac{s(T)}{1-RR}$ is the hazard rate.

T is the time to maturity, expressed in years.

RR is the recovery rate.

s is the spread, which is typically expressed in basis points.

The equation can be derived in accordance with probabilistic theory in the following way. The hazard rate $\lambda(t)$ at time t is defined so that $\lambda(t)\Delta t$ is the probability of default between time t and $t + \Delta t$ conditional on no earlier default.

If $V(t)$ is the cumulative probability of the company surviving to time t, the conditional probability of default between time t and $t + \Delta t$ is:

$$\frac{V(t)-V(t+\Delta t)}{V(t)} \quad \textit{(Eq. VIII.24)}$$

Since this equals $\lambda(t)\Delta t$, it follows that:

$$\frac{V(t)-V(t+\Delta t)}{V(t)} = \lambda(t)\Delta t \rightarrow \frac{V(t+\Delta t)-V(t)}{V(t)} = -\lambda(t)\Delta t \rightarrow \frac{V(t+\Delta t)-V(t)}{\Delta t} = -\lambda(t)V(t) \quad \textit{(Eq. VIII.25)}$$

For $\Delta t \rightarrow 0$

$$\frac{dV(t)}{dt} = -\lambda(t)V(t) \quad \textit{(Eq. VIII.26)}$$

Solving the Ordinary Differential Equation (ODE) for $V(t)$, we obtain the general solution:

$$V(t) = \exp\left[-\int_0^t \lambda(\tau)d\tau\right] \quad \textit{(Eq. VIII.27)}$$

Defining $PD(t)$ as the probability of default by time t, so that $PD(t) = 1 - V(t)$, it follows that:

$$PD(t) = 1 - \exp\left[-\int_0^t \lambda(\tau)d\tau\right] = 1 - \exp\left[-\bar{\lambda}(t)t\right] \quad \textit{(Eq. VIII.28)}$$

Where $\bar{\lambda}(t)$ is the average hazard rate (or, equivalently, the default intensity) between 0 and time t.

In the case of CDS markets, the premium can be seen as a direct compensation received by the insurer for the possibility of default up to the maturity, $t = T$. This means that the average loss rate between time 0 and T should be approximately $s(T)$ per annum.

Let us suppose that the average hazard rate during this time is $\bar{\lambda}(T)$. Considering that the Recovery Rate, RR, in a standard CDS is 40%, we can express the average loss rate with the quantity: $\bar{\lambda}(T)(1 - RR)$. This means that it is approximately true that:

$$\bar{\lambda}(T)(1 - RR) = s(T) \rightarrow \bar{\lambda}(T) = \frac{s(T)}{(1-RR)} \quad \textit{(Eq. VIII.29)}$$

As an example, let us consider Table VIII.12 which shows the EUR CDS Senior Curve for one of the most important Italian power-energy companies.

Applying the previous, we can obtain the hazard rates, $\lambda(T)$, and consequently the estimation of the probabilities of default for every tenor, $PD(T)$.

T	0.5	1	2	3	4	5	7	10
$s(T)$	11	12.8	18.5	29.8	41	55.5	77.5	100.5
$\lambda(T)$	0.0018	0.0021	0.0031	0.005	0.0068	0.0092	0.0129	0.0167
$PD(T)$	0.09	0.21	0.61	1.48	2.7	4.52	8.64	15.42

Table VIII.12 CDS and probability of default

Figure VIII.4 Probability of Default using CDS premium. Source: Bloomberg®

Let us present another example. Based on Bloomberg® CDSW module, the probability of default implied by the premium of CDS for the Novartis AG Group in 5 years is about 1.97%. The Evaluation Date is 20th January 2023, the Spread is 24 bps (0.0024), the Recovery Rate (RR) is typically equal to 40% as said above, the Hazard Rate is $\lambda = \frac{s}{1-RR} = \frac{0.0024}{1-0.4} = 0.004$. The Maturity Date is 20th December 2027, and theTime to Maturity (Day Basis: ACT/360) is $T = 4.9861111$. We can thus calculate:

Implied Probability of Default: $PD(T) = 1 - \exp(-\lambda T) = 1 - \exp(-0.004 \cdot 4.986111) = 0.0197469$

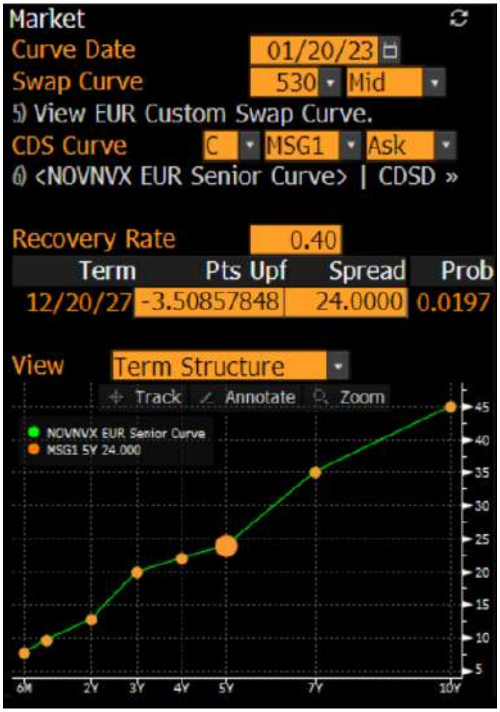


Figure VIII.5 CDSW (Credit Default Swap) module. Source: Bloomberg®

Another approach for the Probability of Default estimation is to consider the bond yield spreads. A bond yield spread is the excess of the promised yield on the bond over the risk-free rate. The usual assumption is that the excess yield is the compensation for the possibility of default.
Let us now consider a portfolio composed of a corporate bond that matures in 5 years having a par yield of 5%, and of a long position in a 5-years Credit Default Swap (CDS) which costs 250 basis points per annum. Such portfolio is approximately equivalent to a long position in a risk-free instrument that pays 2.5% per year.

The effect of the CDS is to "transform" a corporate bond into a risk-free bond. In fact, if the bond issuer does not fail, the investor yields 2.5% p.a. (that is 5% - 2.5% = 2.5%). On the other hand, if the debt issuer fails, the investor earns 2.5% up to default and the entire initial notional is then returned thanks to the Credit Derivative. This amount can then be re-invested at the risk-free rate for the time elapsing from the credit event to maturity. Theoretically, the T-year CDS spread, s, should be close to the excess yield between a T-year corporate bond and a risk-free bond of equal maturity. In mathematical terms: $s = y - r$, where s is the excess spread, y the yield of the corporate bond and r the risk-free rate. If this did not happen, arbitrage opportunities would arise.

It is worth noting that the relationship outlined here is to be considered as an approximation for several reasons, among which we mention the following:

- Market participants are not always allowed to take a short position on bonds issued by companies.

- a CDS has itself a default risk coming from the protection seller.

- For tax or liquidity reasons an investor may not be indifferent to buying a risk-free security rather than a corporate bond and a CDS.

- Arbitrage assumes that interest rates are constant over time.

Consequently, if the market provides a CDS curve, it is preferable to use this approach for the PD estimation rather than a synthetic spread computed from fixed income markets.

Once the implied spread s has been estimated, the usual Equation based on the hazard rate holds.

$$PD(T) = 1 - \exp\left(-\frac{s(T)}{1-RR}T\right) \quad \textit{(Eq. VIII.30)}$$

Another interesting point of discussion regards the definition of the risk-free rate (r) to be used in the formula. In fact, bond traders usually derive the risk-free zero-rate curve from government bond yields, while derivative traders generally use the LIBOR-swap zero curve. Given that treasury bonds yields are riskier than LIBOR-swap market yields, it is a good practice to adopt the latter, more prudential, zero-curve.

In order to calculate the $PD(T)$ for different tenors, let us consider a few bullet senior unsecured bonds issued by the same important Italian power company already used previously. They are fixed-coupon bonds contributed by the markets, as a result, the spread over the risk-free swap curve can be computed starting from the quoted Clean Prices. Using these elements for performing the queries on the Bloomberg® Database, four fixed-income instruments can be found, useful to this aim. The financial characteristics for the Zero-spread calculation over the 6-months tenor swap curve are reported in Table VIII.13.

Tenor	Bloomberg Ticker	Financial Instrument	Yield [%]
6M	EUR006M Index	Deposit	-0.515
7M	EUFR0AG BGN Curncy	Forward Contract	-0.511
8M	EUFR0BH BGN Curncy	Forward Contract	-0.506
9M	EUFR0CI BGN Curncy	Forward Contract	-0.504
10M	EUFR0DJ BGN Curncy	Forward Contract	-0.494
11M	EUFR0EK BGN Curncy	Forward Contract	-0.489
12M	EUFR0F1 BGN Curncy	Forward Contract	-0.482
13M	EUFR0G1A BGN Curncy	Forward Contract	-0.475
14M	EUFR0H1B BGN Curncy	Forward Contract	-0.467
15M	EUFR0I1C BGN Curncy	Forward Contract	-0.461
16M	EUFR0J1D BGN Curncy	Forward Contract	-0.454
17M	EUFR0K1E BGN Curncy	Forward Contract	-0.447
18M	EUFR011F BGN Curncy	Forward Contract	-0.44

2Y	EUSA2 BGN Curncy	Swap	-0.4583
3Y	EUSA3 BGN Curncy	Swap	-0.3965
4Y	EUSA4 BGN Curncy	Swap	-0.3277
5Y	EUSA5 BGN Curncy	Swap	-0.2566
6Y	EUSA6 BGN Curncy	Swap	-0.1826
7Y	EUSA7 BGN Curncy	Swap	-0.1097
8Y	EUSA8 BGN Curncy	Swap	-0.0358
9Y	EUSA9 BGN Curncy	Swap	-0.0345
10Y	EUSA10 BGN Curncy	Swap	0.1002
11Y	EUSA11 BGN Curncy	Swap	0.1619
12Y	EUSA12 BGN Curncy	Swap	0.2206
15Y	EUSA15 BGN Curncy	Swap	0.3591

Table VIII.13 IR Term Structure. Source: Bloomberg®

ID	Coupon	Cpn. Frequency	Day Basis	Maturity date	Mid MKT Price
A	5.25	Annual	ACT/ACT	20th May 2024	115.85
B	5.625	Annual	ACT/ACT	21st June 2027	132.55
C	5.675	Annual	ACT/ACT	1st October 2032	145.95
D	5.5	Annual	ACT/ACT	7th October 2033	147.56

Table VIII.14 Bond Mid MKT Price. Source: Bloomberg®

Having the financial characteristics of the fixed incomes, the market prices and the risk-free term structure, the excess yields over the risk-free can be calculated for the four bonds. The idea is to find the spread to be added to the discount rate that can match the Market price. This task can be solved handling the minimization of the following objective function:

$$\min_{s\geq 0}\left|P^{MKT}-P^{TH}\right| = \min_{s\geq 0}\left|P^{MKT}-\left[\sum_{i=1}^{N}\frac{CF_i}{\left(1+r_{\tau_i}+s\right)^{\tau_i}}-Accrued\ Int.\right]\right| \quad \textit{(Eq. VIII.31)}$$

Or a quadratic form as well, like the SSE – sum of squared error.

In the previous formula, s is the spread and the solver has the aim to find it to minimize the gap between the market price, P^{MKT}, and the theoretical price, P^{TH}, estimated using the traditional Discounted Cash-Flows (DCF) model.

In accordance with the DCF technique, the Dirty Price of a bond is equal to the discounted sum of all the N future payments (coupons and the Face Amount at maturity). Thus, the i-th CF has to be multiplied by the proper discount factors, that is $\frac{1}{\left(1+r_{\tau_i}+s\right)^{\tau_i}}$ where τ_i is the vector of all the year fractions related to the i-th future payment dates, and r_{τ_i} is the risk-free rate interpolated from the zero-rates swap curve for the considered year fraction. Given that the market provides a Clean Price, in accordance with the most popular quote convention, the Accrued Interest should be subtracted from the theoretical Dirty Price. All the four straight bullet bonds have an annual coupon payment frequency, the day basis is ACT/ACT, as a result the Accrued Interest can easily be calculated. The estimated spreads can be considered as synthetic CDS premiums to be paid for a time equal to the maturity of the analyzed bond. Using a recovery rate equal to 40% and writing the code able to solve the minimization problem, we can calculate the bond issuer's probabilities of default for the four maturities.

Bond ID	T[years]	Implied Spread	Convergence	PD(T) [%]
A	2.89	19.066	2.17E-10	0.914
B	5.98	35.248	5.09E-10	3.451
C	11.26	73.124	6.44E-10	12.826
D	12.28	73.91	3.17E-10	14.038

Table VIII.15 Goal Implied spread and PD

The coherence of the Z-spreads, i.e. the spreads applied over the deterministic risk free zero-rates of the swap curve, has been tested using the Bloomberg® Yield and spread (YAS) module. The accuracy of this second methodology fails when the market for the security is not liquid, in fact in this case s cannot be only attributed to the creditworthiness of the issuer. Unluckily the overlapping of these two effects cannot be easily distinguished. Another point of interest is the choice of the Recovery Rate to be applied for the PD estimation using bonds. In this example, the standard value of 40% is used, but it can also be calibrated from market data. In fact, the recovery rate for a bond is normally defined as the bond's market value a few days after a default, as a percentage of its face value. Typically Rating Agencies, such as Moody's, Fitch or S&P, provide historical data on average recovery rates for all categories of fixed incomes. Table VIII.16 below shows the Recovery rates on corporate bonds as a percentage of face value (1982-2012), from Moody's. (Source: Hull – Options, Futures and other derivatives).

Class	Average RR [%]
Senior secured bond	51.6
Senior unsecured bond	37
Senior subordinated bond	30.9
Subordinated bond	31.5
Junior subordinated bond	24.7

Table VIII.16 Average Recovery Rates. Source: Hull – Options, Futures and other Derivatives

Figure VIII.6 YAS (Yield and Spreads Analysis) module. Source: Bloomberg®

If CDS or listed bond quotes are not available on the financial markets, a quantitative analyst can use the KMV (Kealhofer, Merton and Vasicek) model. It is a method based on the Equity market price and on the availability of the firm statements. The assumptions under the Merton methodology can be divided into four sections:

- **Debt**: it is homogeneous with a time to maturity equal to T.

- **Capital Structure**: it is assumed that the public firm is funded using debt and equity. Consequently, it holds that $V_A(t) = D(t) + V_E(t)$, where $V_A(t)$ is the value of the company's assets at time t, $D(t)$ is the debt repayment, $V_E(t)$ is the value of the company's equity at time t.

- **Dynamics of the assets**: the model assumes that the firm's assets are tradable and they follow a geometric Brownian motion: $dV_A = \mu_A V_A dt + \sigma_A V_A dW_t$, where μ_A is the instantaneous expected rate of return, σ_A the volatility and dW_t is a standard Wiener process.

- **Market perfection**: it assumes that taxes are ignored. There is no penalty to short sales. Market is fully liquid; investors can purchase or sell any assets at the desirable market price. Borrowing and lending are at the same risk free interest rate, and this interest rate is constant through the horizon.

Based on these assumptions, in 1974, Merton proposed a model where a company's equity is an option on the assets of the company.

If $V_A(T) < D$, it is (at least in theory) rational for the company to default on the debt at time T. The value of the equity, $V_E(T)$, is then zero.

If $V_A(T) > D$, the company should make the debt repayment at time T and the value of the equity at this time is $V_E(T) = V_A(T) - D$.

In mathematical terms, the firm's equity value at time T is given by the pay-off:

$$V_E(T) = \max(V_A(T) - D, 0) \text{ (Eq. VIII.32)}$$

The analogy with the pay-off of a European call option is clear: the value of the equity is a call option on the value of the assets with a strike price equal to the repayment required on the debt.

Under the assumptions of the model, the traditional Black-Scholes-Merton (BSM) option pricing framework can be applied and consequently the following equation is valid:

$$V_E(t) = V_A(t)\phi(d_1) - \exp[-r(T-t)]D\phi(d_2) \text{ (Eq. VIII.33)}$$

$$d_1 = \frac{\ln\left(\frac{V_A(t)}{D}\right) + \left(r + \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}}, d_2 = d_1 - \sigma_A\sqrt{T-t} \text{ (Eq. VIII.34)}$$

t is the time of the valuation, σ_E is the volatility of the equity, r is the risk-free rate prevailing at time T. The value of the debt today is $V_A(0) - V_E(0)$ and, in accordance with the BSM (Black-Scholes-Merton) theory, the risk-neutral probability that the company will default on the debt is $\phi(-d_2)$.
Unfortunately, in order to estimate the probability of default from $\phi(-d_2)$, a further relation is needed, in fact there are two unknowns: the value and the volatility of the assets, V_A and σ_A.

The other variables can be directly observed or computed: the risk free rate, r, from the Libor swap curve or from the treasury bond yields; the debt, D, from the last available firm statement or from the financial bond markets; V_E can be computed multiplying the current outstanding shares by the spot value of the traded stock; σ_E can be estimated using the implied volatilities from actively traded options (if any) or, more likely, using a traditional econometric backward looking approach such as close-to-close volatility or a GARCH.

The second equation can be found applying Itô's lemma. Thanks to it, we can rewrite the stochastic differential equation followed by V_E : $dV_E = \mu_E V_E dt + \sigma_E V_E dW_t$ in this other useful term:

$$dV_E = \left(\frac{1}{2}\sigma_A^2 V_A^2 \frac{\partial^2 V_E}{\partial V_A^2} + \mu_A V_A \frac{\partial V_E}{\partial V_A} + \frac{\partial V_E}{\partial t}\right) dt + \sigma_A V_A \frac{\partial V_E}{\partial V_A} dW_t \text{ (Eq. VIII.35)}$$

Comparing the diffusion term of the two latest stochastic differential equations, we can derive that:

$$\sigma_E V_E = \sigma_A V_A \frac{\partial V_E}{\partial V_A} \text{ (Eq. VIII.36)}$$

In accordance with the BSM option theory, the term $\frac{\partial V_E}{\partial V_A}$ is the Delta Greek of a European Call Option, Δ^E, and it is equal to $\phi(d_1)$.

Thus, to find the unobservable value and volatility of the asset, the following nonlinear system of equations should be solved:

$$\begin{cases} f_1(V_E, \sigma_E) = V_A \phi(d_1) - \exp[-r(T-t)]D\phi(d_2) - V_E = 0 \\ f_2(V_E, \sigma_E) = \frac{V_A}{V_B}\phi(d_1)\sigma_A - \sigma_E = 0 \end{cases} \quad \text{(Eq. VIII.37)}$$

$\frac{\partial f_1}{\partial V_A} = \phi(d_1) > 0$, like the Delta Greek in the BSM framework, f_1 is an increasing function of V_A that implies that $f_1(V_A)$ has a unique solution. For the same reason, f_2 has a unique solution as well. This consideration about the solution of the system allows to code a routine based on a traditional steepest-descent algorithm. Once the system has been solved, we have all the data for finding the probability of default through the $\phi(-d_2)$.

It is worth noting that d_2 is also called Distance-to-Default in the KMV (Kealhofer-Merton-Vasicek) model:

$$DD = d_2 = d_1 - \sigma_A\sqrt{T} = \frac{\ln\left(\frac{V_A}{D}\right) + \left(r - \frac{1}{2}\sigma_A^2\right)T}{\sigma_A\sqrt{T}} \quad \text{(Eq. VIII.38)}$$

Considering that the assets follow a Geometric Brownian Motion and, consequently, $V_A(t)$ is log-normal distributed with expected value at time t, such that:

$$V_A(T) = V_A(t)\exp\left\{\left(r - \frac{1}{2}\sigma_A^2\right)(T-t) + \sigma_A W_{T-t}\right\} \quad \text{(Eq. VIII.39)}$$

The probability of default, $PD(T-t)$, for $t = 0$ can now be computed as follows:

$$PD(T) = Pr[V_A(T) < D] = Pr\left[V_A \exp\left\{\left(r - \frac{1}{2}\sigma_A^2\right)(T-t) + \sigma_A W_T\right\} < D\right] =$$

$$= Pr\left[W_T < \frac{\ln\left(\frac{D}{V_A}\right) - \left(r - \frac{\sigma_A^2}{2}T\right)}{\sigma_A}\right] = Pr\left[Z < \frac{\ln\left(\frac{D}{V_A}\right) - \left(r - \frac{\sigma_A^2}{2}T\right)}{\sigma_A\sqrt{T}}\right] = Pr\left[Z < -\frac{\ln\left(\frac{V_A}{\mathrm{D}}\right) + \left(r - \frac{\sigma_A^2}{2}T\right)}{\sigma_A\sqrt{T}}\right] =$$

$$= Pr[Z < -DD] = \phi(-DD) \quad \text{(Eq. VIII.40)}$$

Where $Z \sim NID(0,1)$.

A real-market application of the methodology is now presented. The example consists in estimating the probability of default of one of the most important Italian telecommunication companies using the KMV model starting from its latest statements and the price of its traded stock. For the equity value, the last price of the stock quoted on 30th June 2020 is considered, i.e., 0.0172, which is to be multiplied by the current number of shares, 5380.61 million. As a result, V_E = EUR 92.55 million. The value of debt, V_D, can be taken from the latest end-of-year (EOY) balance sheet and is equal to EUR 78.83 million. The historical volatility for the equity on 30th June 2020 is estimated as σ_E = 90%.

Considering a forecasting time window for the probability of default equal to one year, $T = 1$, and a risk-free rate equal to $r = 0.482\%$ (see the IR term structure in the previous example), we can use a script for computing the Distance to Default (1.2437) and, consequently, the Probability of Default (10.68%).

The 1-year probability of Default estimated for the company on 30th June 2020 is rather critical. Considering that this firm has been suspended by the Exchange on June 2022, we can deduce that the goodness of this estimation made with the EOY-2020 statement together with the market data two years before the suspension can be considered a good proxy of the creditworthiness of the firm considered for the analysis.

Let us now consider another case, and estimate the Probability of Default of a wealthy firm. Following the KMV model, the parameters necessary for the calculation of the default probability using this method are: the value of the equity (V_E), the value of the debt (D), the risk-free rate (r), the volatility of the equity (σ_E), the value of the assets (V_A) and the volatility of the assets (σ_A).

The value of the equity was calculated multiplying the stock price of "Eni Gas e Luce" as of 31st December 2020 by the number of shares of the company. As shown in the following figures, the value of the shares as of 31st December 2020 was 8.548, the number of shares of the company was 3,572,550,000 and, consequently, the value of the equity was equal to: V_E=8.548× 3,572,550,000= EUR 30,538,157,400. As regards the amount of the debt (D), this was selected from the balance sheet of the company itself and is equal to EUR 31,704,000,000. As for the value for the risk-free rate, the term structure of the risk-free rate was used with a tenor equal to 6 months (i.e. EURIBOR6M) and the value of the one-year zero rate is equal to $r =$ -0.533%. The last parameter needed to set up the system of equations for V_A and σ_A is the volatility of the equity. It was estimated using the Bloomberg® HVT (Historical Volatility) module and it is equal to $\sigma_E = 53.58\%$.

The most popular methodology with which Bloomberg estimates historical volatility is the close-to-close technique, i.e. the standard deviation of the daily returns of the stock, which are annualized with the factor 260.

Figure VIII.7 Euro Swap Curve. Source: Bloomberg®

ENI IM Equity | Actions | Templates | Chart | Export | Historical Volatility Table

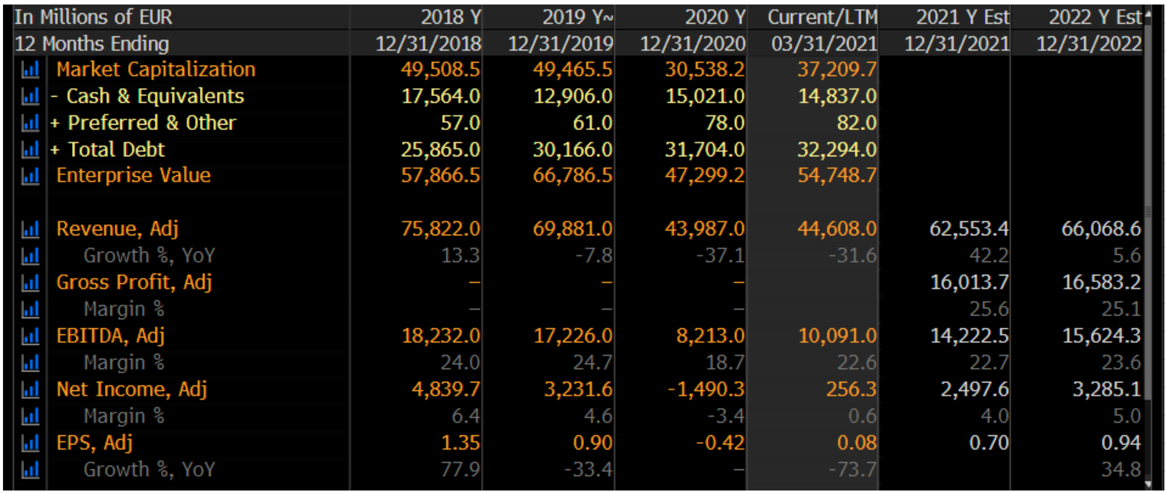
Period Daily: 1Y | Range 05/09/20 - 05/09/21 | Ann. Factor 260 | Currency LCL

Hist Vol 260 | Trade | Model CLV | Normal | Y/P Price | IVOL

	Date	Hist Vol (260)	Price (L)	Implied Vol
Thu	01/07/21	53.699	9.068	29.204
Wed	01/06/21	53.705	9.048	28.440
Tue	01/05/21	53.700	8.761	32.590
Mon	01/04/21	53.578	8.448	31.884
Fri	01/01/21			
Thu	12/31/20	53.577		
Wed	12/30/20	53.577	8.548	29.691
Tue	12/29/20	53.577	8.562	29.852
Mon	12/28/20	53.579	8.596	31.190
Fri	12/25/20			
Thu	12/24/20	53.590		
Wed	12/23/20	53.486	8.584	31.201
Tue	12/22/20	53.420	8.360	34.397
Mon	12/21/20	53.402	8.244	35.397
Fri	12/18/20	53.228	8.621	30.175
Thu	12/17/20	53.221	8.711	28.911
Wed	12/16/20	53.223	8.756	29.133
Tue	12/15/20	53.224	8.752	30.472
Mon	12/14/20	53.216	8.678	34.458
Fri	12/11/20	53.207	8.808	28.144
Thu	12/10/20	53.203	8.936	30.399

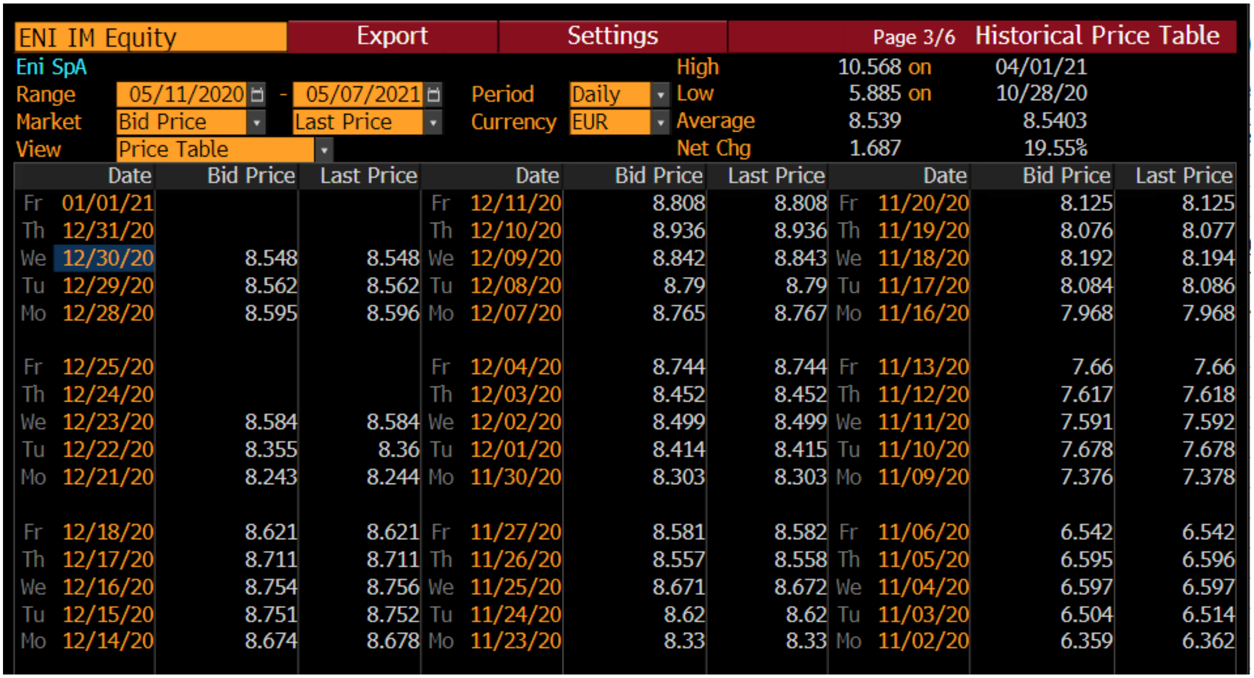
Suggested Functions | **OVME** Price equity & equity index options | **GIP** Chart intraday price movements

Figure VIII.8 Historical Volatility Table (HVT). Source: Bloomberg®



In Millions of EUR	2018 Y	2019 Y	2020 Y	Current/LTM	2021 Y Est	2022 Y Est
12 Months Ending	12/31/2018	12/31/2019	12/31/2020	03/31/2021	12/31/2021	12/31/2022
Market Capitalization	49,508.5	49,465.5	30,538.2	37,209.7		
- Cash & Equivalents	17,564.0	12,906.0	15,021.0	14,837.0		
+ Preferred & Other	57.0	61.0	78.0	82.0		
+ Total Debt	25,865.0	30,166.0	31,704.0	32,294.0		
Enterprise Value	57,866.5	66,786.5	47,299.2	54,748.7		
Revenue, Adj	75,822.0	69,881.0	43,987.0	44,608.0	62,553.4	66,068.6
Growth %, YoY	13.3	-7.8	-37.1	-31.6	42.2	5.6
Gross Profit, Adj	—	—	—		16,013.7	16,583.2
Margin %	—	—	—		25.6	25.1
EBITDA, Adj	18,232.0	17,226.0	8,213.0	10,091.0	14,222.5	15,624.3
Margin %	24.0	24.7	18.7	22.6	22.7	23.6
Net Income, Adj	4,839.7	3,231.6	-1,490.3	256.3	2,497.6	3,285.1
Margin %	6.4	4.6	-3.4	0.6	4.0	5.0
EPS, Adj	1.35	0.90	-0.42	0.08	0.70	0.94
Growth %, YoY	77.9	-33.4	—	-73.7		34.8

Figure VIII.9 Financial Analysis (FA). Source: Bloomberg®



ENI IM Equity | Export | Settings | Page 3/6 Historical Price Table

Eni SpA — High 10.568 on 04/01/21

Range 05/11/2020 - 05/07/2021 Period Daily — Low 5.885 on 10/28/20

Market Bid Price / Last Price Currency EUR — Average 8.539 8.5403

View Price Table — Net Chg 1.687 19.55%

Date	Bid Price	Last Price	Date	Bid Price	Last Price	Date	Bid Price	Last Price
Fr 01/01/21			Fr 12/11/20	8.808	8.808	Fr 11/20/20	8.125	8.125
Th 12/31/20			Th 12/10/20	8.936	8.936	Th 11/19/20	8.076	8.077
We 12/30/20	8.548	8.548	We 12/09/20	8.842	8.843	We 11/18/20	8.192	8.194
Tu 12/29/20	8.562	8.562	Tu 12/08/20	8.79	8.79	Tu 11/17/20	8.084	8.086
Mo 12/28/20	8.595	8.596	Mo 12/07/20	8.765	8.767	Mo 11/16/20	7.968	7.968
Fr 12/25/20			Fr 12/04/20	8.744	8.744	Fr 11/13/20	7.66	7.66
Th 12/24/20			Th 12/03/20	8.452	8.452	Th 11/12/20	7.617	7.618
We 12/23/20	8.584	8.584	We 12/02/20	8.499	8.499	We 11/11/20	7.591	7.592
Tu 12/22/20	8.355	8.36	Tu 12/01/20	8.414	8.415	Tu 11/10/20	7.678	7.678
Mo 12/21/20	8.243	8.244	Mo 11/30/20	8.303	8.303	Mo 11/09/20	7.376	7.378
Fr 12/18/20	8.621	8.621	Fr 11/27/20	8.581	8.582	Fr 11/06/20	6.542	6.542
Th 12/17/20	8.711	8.711	Th 11/26/20	8.557	8.558	Th 11/05/20	6.595	6.596
We 12/16/20	8.754	8.756	We 11/25/20	8.671	8.672	We 11/04/20	6.597	6.597
Tu 12/15/20	8.751	8.752	Tu 11/24/20	8.62	8.62	Tu 11/03/20	6.504	6.514
Mo 12/14/20	8.674	8.678	Mo 11/23/20	8.33	8.33	Mo 11/02/20	6.359	6.362

Figure VIII.10 Historical Prices (HP). Source: Bloomberg®

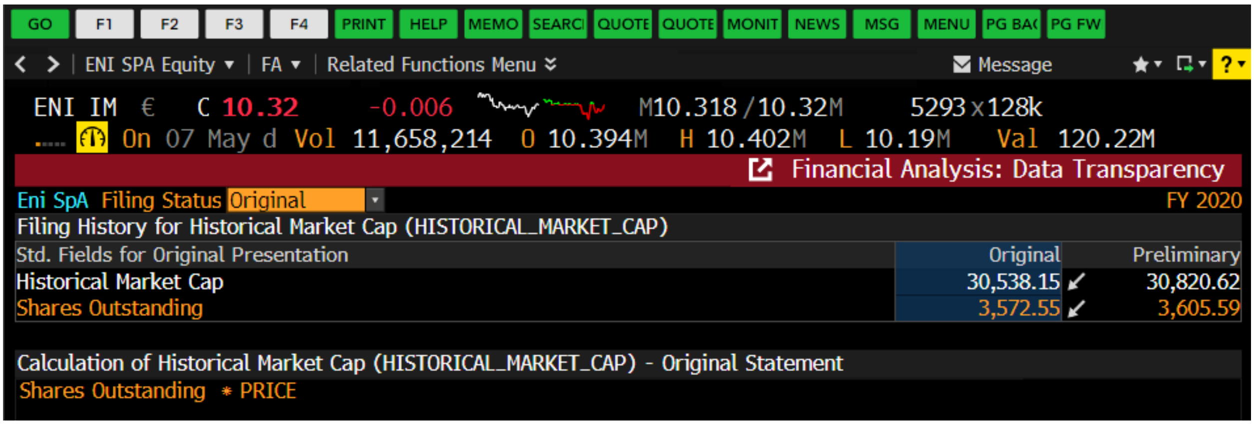


Figure VIII.11 Financial Analysis (FA). Source: Bloomberg®

Once all the necessary inputs are available, the nonlinear system of equations for V_A and σ_A can be numerically solved and consequentially we can compute the 1-year probability of default, PD. In this case, in accordance with the KMV model, the result is about 0.77%. This value is very close to the one provided by the Bloomberg® Default Risk (DRSK) module, only about 1 basis point of gap.

Figure VIII.12 Default Risk (DRSK) module. Source: Bloomberg®

The models discussed in this chapter are able to satisfy the measurements related to counterparty risk and they can also be applied for answering the needs of the credit risk mitigation intended in accordance with the more traditional conception. Credit risk, as per traditional definition, is understood as the risk that a debtor does not fulfill the contractual obligations assumed or that this occurs at different (and usually subsequent) deadlines compared to the pre-established deadlines. This setting of the definition of credit risk has a major limitation: it leads to consider only the negative event (default), while any unexpected changes in the quality of debtors also assume importance in the management of credit risk. The traditional definition tends to transform the decision of lending into a dichotomous choice: to trust or not to trust the applicant.
We will see that in the management of loans and credit risk management, other aspects are also relevant: which technical form, which commercial pricing, which guarantee or form of risk mitigation, which effect of the single loan on the overall portfolio (concentration, diversification,…).

The analysis of loan management requires a broader definition of credit risk that considers both the degree of risk before the decision to lend and the changes in credit quality - and, therefore, in the riskiness of a subject - that occur during the life of the loan, while not leading to the negative event. Based on this concept, it is possible to complete the definition of credit risk and the effects suffered by the bank as a set of three dimensions: financial, economic and income.

The **financial dimension** determines a cost linked to the delayed payment, partial or total, of the sums due, because of the liquidity or treasury costs that the bank incurs, which are not always offset by interests on arrears. Examples in this respect could be the cost of funding not remunerated by any use, or the costs of recourse bank to find liquidity due to non-repayment.

The **economic dimension** causes a cost linked to the failure to repay the due sums. Examples could be the losses suffered by the bank and the loss of profit from an alternative investment.

Lastly, the **income dimension** should be added which causes an (opportunity) cost linked to the unpaid risk variation. At the loan portfolio level, the change in risk translates into a higher loss against which there is an inadequate risk premium and, therefore, into a lower value produced for the bank shareholders.

The definition of credit risk is not yet exhaustive though. It must be defined along its components, which translate what has been said previously into elements that, operationally, the bank must measure and monitor, in order to measure the credit risk and minimize the losses caused by credit activity. It is precisely the translation of credit risk into economic loss that forms the foundation of loan management in a bank. What is the loss that a credit portfolio can suffer? In practical terms, it is customary to distinguish between expected loss and unexpected loss. One component of credit risk is the expected loss, that is the measure of the loss that a bank expects on a loan granted or on a pool of exposures, with reference to a given time horizon, usually one year (holding period).

The expected loss (EL) is calculated as a product of the Probability of Default (PD), the Loss Given Default (LGD) and the Exposure At Default (EAD) as follows:

$$EL = PD \cdot EAD \cdot LGD \quad \textit{(Eq. VIII.41)}$$

The **probability of default** (PD) measures the probability that the debtor will default on the assumed obligations, i.e., that it will default. Default is defined as the state of non-fulfillment of the obligations, which the legislation has set at 90 days from the contractual expiry. The PD depends on the creditworthiness of the borrower and being referred only to the borrower, it is not affected by the technical form of the loan or the existence of guarantees. It can be measured ex ante through the use of rating systems or other assessment methods based on analyst judgments.

Let us analyze the concept through an example. A company that has a PD of 2% means that it has a 2% probability of running into a situation of default during the year (or over a different time horizon). In other words, in a portfolio of 1,000 loans made to 1,000 individuals with PD = 2%, at the end of the period there should be 980 borrowers who will repay the loan and 20 borrowers who will not meet their obligations.

The **loss given default** (i.e. the non-recoverable credit portion) measures the portion of credit which, in the event of default, could not be recovered either by using the credit protections acquired or through the credit recovery process itself. Measuring an LGD of 30% on a loan means that, in the event of insolvency, 30% of the credit will not be protected by any form of guarantee or other instrument that can reduce the effects of the debtor's inability to fulfill his obligations.

For example, in the case of a loan of EUR 100,000 with collateral (for example, a mortgage), if the LGD is valued at 25% it means that in the event of default, the portion of the loan that will not be recovered is equal to EUR 25,000. Unlike the PD which refers to the debtor, the LGD refers only to the transaction in question. It should also consider the time required for the enforcement of the guarantees, the legal and administrative costs that may be incurred and, last but not least, the opportunity cost of non-repayment of the loan. In this sense, the LGD can be calculated by difference with respect to the recovery rate of a credit (RR - Recovery Rate), equal to the value of the flows expected from the recovery action and therefore equal to:

$$RR = 1 - LGD \quad \textit{(Eq. VIII.42)}$$

The estimate of LGD (or recovery rate) is based on internal bank data capable of capturing all the specific elements of the operation, guarantee, geographical area, judicial efficiency, costs and administrative times, the quality of the recovery process of impaired loans and, finally, the socio-economic-cultural characteristics of the environment, which modify the effectiveness of the different forms of risk mitigation.

This methodology must therefore derive estimates from the insolvencies that have actually reached the end of the dispute and for which all the necessary information is available. The variables on which the recovery rate therefore depends are:

- percentage of recovered credit.
- discount rate capable of considering the financial cost of recovery time.
- opportunity costs.
- internal operating costs.
- external legal and administrative costs incurred to manage the recovery process.
- recovery time.

In general, LGD can be calculated as:

$$LGD = 1 - \frac{\frac{RV - AC}{(1+i)^t}}{EAD} \quad \textit{(Eq. VIII.43)}$$

where:

RV is the estimated recovery value to be obtained thanks to the forms of risk mitigation.

AC is the administrative and legal costs necessary for managing the recovery process.

i is the discount rate, which considers the financial cost of time, opportunity costs and the risk of the operation. Certain banks use the internal rate of transfer of funds for the bank.

t is the estimated time for recovery, in years.

EAD is the expected exposure at the time of default.

Let us make an example and assume a loan of EUR 100,000 against which a bank obtains a real guarantee worth EUR 50,000. Against the default of the debtor, the bank has recovered an amount of EUR 50,000 in five years, incurring costs of EUR 20,000. Considering a discount rate of 6%, the LGD is equal to:

$$LGD = 1 - \frac{\frac{50{,}000 - 20{,}000}{(1 + 0.06)^5}}{100{,}000} = 77.582\%$$

The **exposure at default** (EAD) measures the amount of residual capital still to be repaid when the debtor defaults. To affirm that the EAD of a loan of EUR 100,000 is equal to EUR 60,000 means that when default becomes likely, the amount of capital still to be repaid will be EUR 60,000. This information is essential to calculate the expected loss, since it is important to evaluate the amount still to be repaid at the time the insolvency occurs rather than the original amount. The concept of EAD is closely linked to the technical form of the loan. In the case of loans with a predefined amortization plan, this tends to coincide with the residual capital plus the interest due at the time the default occurs.

On the other hand, there are numerous forms of financing in which it is not possible to know ex ante what the exposure will be since it depends on the method of use of the debtor (for example a credit line) or endorsement credits which use is not even known. For all these technical forms, the EAD is typically estimated by adding the amount of available credit to the current utilized, multiplied by an appropriate **credit conversion factor** (CCF). The CCF represents the ratio between the unused part of the credit line that is estimated to be used in the event of default, and the part currently unused and it depends on the right to intervene, block unused sums and withdrawal possessed by the bank in the event of default and/or in the event of a significant change in the debtor's riskiness. It is worth noting that the three components of the expected loss can be referred to three elements that have always been at the center of loan management: the riskiness of the debtor, the existence of guarantees and the technical form of the loan. Using the examples given in the definitions, the expected loss is:

$$EL = PD \cdot EAD \cdot LGD = 2\% \cdot 60{,}000 \cdot 77.582\% = 930.984$$

The Unexpected Loss (UL) is defined as the difference between the loss actually incurred by the bank, measured ex post, and the expected loss measured ex ante. Clearly, when the bank has an effective model for estimating expected losses, the true concept of credit risk concerns precisely the unexpected loss. It is obviously not possible to predict the unexpected loss since any existing information is inserted into the measurement of the components of the expected loss. The expected loss, measured ex ante, is to be considered as a cost component and it is included in the pricing of each loan, like the other costs necessary for the disbursement and management. Given its nature, the expected loss management takes place at the level of the single transaction and of the single applicant, through the selection and pricing process. The expected loss of the loan portfolio is therefore given by the sum of the expected loss of the individual loans disbursed and, therefore, cannot be managed at the aggregate level.

The unexpected loss, on the other hand, is managed at the level of the loan portfolio through the diversification and composition of the loan portfolio, which are actions aimed at minimizing the unexpected loss. In any case, the unexpected loss is covered "of last resort" in the bank's assets. In fact, it is necessary to specify that the reference to own assets must not be interpreted in a static perspective, and referring exclusively to the level of the assets existing in a given period. The credit management process, in fact, monitors and progressively identifies the credits that have a lower recovery value compared to the value recorded in the financial statements (impairment), determining progressive adjustments and provisions (provisioning) equal to the non-recoverable value. This process constitutes a method through which the bank allocates portions of its revenues on an ongoing basis, to cover loans that will not be repaid. In other words, all this means allocating resources, generated by the banking activity, to cover the unexpected loss (and not to the result for the year) and, where these resources are not sufficient, a loss is determined and, therefore, the use of the bank's own capital.

We try to associate a probability to every possible loss value, to obtain a continuous function of all possible eventualities. It is therefore a question of identifying a plausible distribution of the bank's losses in order to quantify the amount of equity that allows to cope with the worst eventualities (unexpected loss).

The concept of unexpected loss is linked to VaR (Value at Risk), i.e. the maximum value of losses that a bank can incur with a certain probability, obviously with reference to a given time horizon and to a given portfolio of loans (or of another nature).

FURTHER READINGS

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Hull J., Predescu M., White A. – "Relationship between Credit Default Swap Spreads, Bond Yields and Credit Rating Announcements" – Journal of Banking and Finance 28:2789-2811 (2003).
Merton R. C. – "On the Pricing of Corporate Debt: the Risk Structure of Interest rates" – Journal of Finance Vol. 29, 449-70 (1974).

VIII.3 PROBABILITY OF DEFAULT

The forecasting activity influences strategic decisions, therefore making credit risk forecasts means, among other purposes, estimating the expected loss, EL. The estimation of the EL passes through the estimation of the risk factors that determine it, namely PD, LGD and EAD.

The moment of classification of the debtor in default plays a fundamental role in defining the risk factors, representing a sort of watershed, changing the relationship between the bank and the debtor: before the default, the exposure is an investment towards a counterparty in operations, while after the default, it is an investment to be recovered. The definition of default should be the same when estimating the risk factors (PD, LGD, EAD), otherwise there is a risk of overlaps which lead to biased EL estimates.

The principles for the definition of default are:

- **certainty**: the credit analyst is interested in knowing whether the debtor is in a state of insolvency without incurring errors of assessment.

- **prudence**: the credit analyst is interested in promptly grasping the symptoms of insolvency to recover the exposure and avoid capital losses.

Depending on the prevailing principle, the possible definitions of default can range over a very broad spectrum:

t_1: exposures past due for more than 60 days.
t_2: relevant overdue exposures for over 90 days (regulatory perspective).
t_3: the bank returning the exposure.
t_4: transfers to losses and specific value adjustments.
t_5: the bank selling the exposure and realizing a significant loss.
t_6: restructuring of the exposure with realization of losses.
t_7: bankruptcy filed by the bank or an equivalent action.
t_8: bankruptcy or similar situation.

The definitions of default can be very different: the definitions that pursue a high level of prudence are those based on the behavior of the debtor in monitoring performance, firmly anchored to the observation of overdue amounts. At the other extreme, we find the definitions based on the judicial assessment of the state of default, and in fact, the bankruptcy laws define the debtor in state of insolvency regardless of the analyst's purposes. As a result, from t_1 to t_8 the level of prudence in the default definition decreases and the level of certainty increases. The regulatory perspective bases the definition of default on past due to evaluate a company. In fact, according to the supervisory regulations of the Bank of Italy (Accounting Matrix, Circular n. 272, 2008), which incorporate Basel 2, past due plays a crucial role in the classification of a loan as non-performing. In addition, starting from the end of 2014, the classification of loans has been revised to take into account the harmonization at European level, promoted by the EBA, of non-performing exposures and those subject to concessions.

The granting of the creditor takes the form of the renunciation to some contractually defined rights, which result in an immediate or deferred benefit for the debtor, who benefits from this renunciation, which may correspond to a possible loss for the creditor. Following the review, impaired loans (non-performing exposures of the EBA standard) are classified into:

- **bad debts:** On-balance sheet and off-balance sheet exposures to a party in a state of insolvency (even if not legally ascertained) or in substantially comparable situations, regardless of any loss forecasts made by the bank. Therefore, regardless of the existence of any guarantees (real or personal) on the exposures. Exposures whose anomalous situation is attributable to profiles relating to country risk are excluded.

- **unlikely to pay:** Credit exposures, for which the bank considers it unlikely that, without recourse to actions such as the enforcement of guarantees, the debtor will fulfill all of its credit obligations, in principal and/or interest.

- **impaired past due** and/or **overdue exposures**: For the purposes of classifying exposure in this category, two qualifying aspects are relevant:

a) the continuity of the past due, since the exposure must have expired continuously for at least 90 days to detect a non-temporary attitude of the debtor.

b) the threshold of relevance, as exposures overdue by more than 90 days are relevant only if they represent at least 5% of the debtor's total exposure.

- **exposures subject to forbearance:** which are divided into:

a) exposures subject to impaired concessions, which correspond to the non-performing exposures with forbearance measures of the EBA standard.

b) other exposures subject to concessions, which correspond to the "Forborne performing exposures" of the EBA standard.

As previously highlighted, the selection of the definition of default depends on the objectives that the analyst intends to achieve. For example, the rating agencies, professional operators in producing risk assessments for the market, do not provide for definitions on automatism in the detection of the past due. Here are the definitions provided by the rating agencies:

Standard and Poor's: "An obligor rated "SD" (selective default) or "D" is in payment default on one or more of its financial obligations (rated or unrated) unless Standard & Poor's believes that such payments will be made within five business days, irrespective of any grace period. The "D" rating will also be used upon the filing of a bankruptcy petition or the taking of similar action if payments on a financial obligation are jeopardized. A "D" rating is assigned when Standard & Poor's believes that the default will be a general default and that the obligor will fail to pay all or substantially all of its obligations as they come due. An "SD" rating is assigned when Standard & Poor's believes that the obligor has selectively defaulted on a specific issue or class of obligations, but it will continue to meet its payment obligations on other issues or classes of obligations in a timely manner. A selective default includes the completion of a distressed exchange offer, whereby one or more financial obligation is either repurchased for an amount of cash or replaced by other instruments having a total value that is less than par".

Moody's: "Issuers assessed as "C" are typically in default, with little prospect for recovery of principal or interest; or these issuers are benefiting from a government or affiliate support but are likely to be liquidated over time; without support there would be little prospect for recovery of principal or interest".

Fitch: "RD: Restricted default "RD" ratings indicate an issuer that in Fitch Ratings' opinion has experienced an uncured payment default on a bond, loan or other material financial obligation but which has not entered into bankruptcy filings, administration, receivership, liquidation or other formal winding-up procedure, and which has not otherwise ceased operating. This would include:

a) the selective payment default on a specific class or currency of debt.

b) the uncured expiry of any applicable grace period, cure period or default forbearance period following a payment default on a bank loan, capital markets security or other material financial obligation.

c) the extension of multiple waivers or forbearance periods upon a payment default on one or more material financial obligations, either in series or in parallel; or

d) the execution of a distressed debt exchange on one or more material financial obligations.

"D" ratings indicate an issuer that in Fitch Ratings' opinion has entered into bankruptcy filings, administration, receivership, liquidation or other formal winding-up procedure, or which has otherwise ceased business".

In addition to the selection of the definition of default, the forecast of the risk factors is influenced by the time horizon chosen to predict the classification of the debtor as insolvent. In fact, the result is different when assessing the risk factors by predicting that the debtor will turn out to be insolvent within one year, 3 years and 10 years. The following approaches can be adopted in the selection of the time horizon:

- **maturity approach**: the time horizon corresponds to the expiry of the exposure. For overruns it will be necessary to consider a very short-term deadline.

- **liquidation approach**: the time horizon corresponds to the deadline for the liquidation of the exposure, i.e. the time needed to recover or realize the exposure. In this perspective, the bank does not break down the expected loss into its components (PD, LGD, EAD) but makes an aggregate forecast.

- **common horizon approach**: the bank selects a common standardized time horizon for all exposures. Generally, banks choose an annual time horizon because:

a) the review of credit lines is carried out annually.

b) key information, such as financial statements, are made available at least annually.

It is relevant to specify that PD can be estimated using quantitative rather than qualitative approaches.

- **Quantitative approaches** require that the PD is estimated using a statistical model that processes information of a standardized nature. With the same type of debtor, any information used in the forecast of default will contribute to the estimate of the PD with the same intensity.

- **Qualitative approaches** envisage that the PD is estimated using information that can vary in type, relevance and level of standardization, with the same debtor.

The quantitative approach is chosen to assess the risk of insolvency arising from exposures of a limited amount (small and medium-sized enterprises, small economic operators and consumers). In fact, this approach, based on automated procedures, is much less expensive and much faster than the qualitative approach. On the other hand, the qualitative approach is chosen for companies that represent a significant exposure for the bank or for innovative projects for which the use of standardized models is inadequate.

Therefore, on an overall level, it can be considered that:

- The probabilities of default estimation related to small economic operators and consumers mainly come from **quantitative** approaches.

- The probabilities of default estimation related to small and medium-sized enterprises mainly come from **quali-quantitative** approaches.

- The probabilities of default estimation related to medium and big-sized enterprises mainly come from **qualitative** approaches.

The use of the quantitative approach has grown significantly in recent years. In the past, the goal was to predict business failure a few years in advance, nowadays the immediate goal is both to use the information available to attribute the PD to customers, and also to provide decision support. The most widely used quantitative-statistical technique provides for the attribution to each evaluated company of a score which represents a synthetic and numerical measure of the debtor's state of health. The essential phases for the construction of a scoring model are represented first of all by the building of a sample of customers who have proven to be reliable in the past and a comparable sample of insolvent counterparties. Subsequently, a combination of variables that is considered to have a discriminating function must be identified and the aggregation function of such data must be defined. Then, the last step is represented by the calculation of the probability of default and/or score to be attributed to the individual customer.

The construction of a sample with reliable and insolvent customers allows to use the bank's internal data and is aimed at making it possible to verify whether there are some variables that have characterized the behavior of reliable companies compared to insolvent ones. The second key moment is represented by the selection of information that can help to predict the default and the selection of the formula for their combination. The information on which the scoring models have found frequent application are the accounting information coming from the financial statements. In fact, one of the best known models that has inspired and still inspires the construction of the following models, such as the recent application for unlisted companies or to predict the entry of Italian manufacturing companies into the procedure of extraordinary administration is the model called Z-score due to Altman. It is an easy and straightforward basic approach based on a linear combination of variables with fixed coefficients.

$$Z = 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.4X_4 + 1.0X_5 \quad \textit{(Eq. VIII.44)}$$

Where:

X_1: **Working capital/Total assets**. This variable expresses the value of the company's liquid assets with respect to the total capitalization. It is clear that a company that is experiencing substantial operating losses will have a strong reduction in current assets in relation to total assets.

X_2 : **Retained profits/Total assets**. This index expresses the ability to reinvest profits. A young company will certainly have a lower index than an older company.

X_3 : **Income before interest expense and taxes/Total assets**. This index measures the true productivity of a company's activities, purified of any financial or fiscal leverage factor.

X_4 : **Equity market value/Book value of liabilities**. This index shows how much the assets of a company can be reduced before the total liabilities exceed the assets and the conditions for bankruptcy are created.

X_5 : **Sales/Total assets**. This index highlights the ability of a company to generate revenues with a certain value of the assets. It measures the entrepreneurial ability to relate to the competitiveness of the company's reference market.

This model typically identifies a threshold value, the so-called **cut off**, equal to 1.81: if the company totals a value greater than 1.81 it is to be considered potentially healthy, otherwise insolvent. As an example, let us consider the balance sheet data of one of the most important Italian companies in the beverage sector. The data are retrieved from the Bloomberg® FA (Financial Analysis) module and are reported in Table VIII.17.

Statement Data 12/31/2020	Value [Euro (Millions)]
Tangible Assets	2201.3
Working Capital	801.5
Retained Earnings	2297.2
Earnings Before Interests and Taxes	231.8
Market Value of Equity	11355.8
Total Liabilities	2558.3
Sales	1772.0
Total Shareholders' Equity	1998.4

Table VIII.17 Altman's Z-Score

$$Z = 1.2 \cdot \frac{801.5}{2201.3} + 1.4 \cdot \frac{2297.2}{2201.3} + 3.3 \cdot \frac{231.8}{2201.3} + 0.6 \cdot \frac{11355.8}{2558.3} + 1.0 \cdot \frac{1772}{2201.3} = 5.71$$

Given that the score is far higher than the threshold, we can conclude that the analyzed company is definitely healthy.

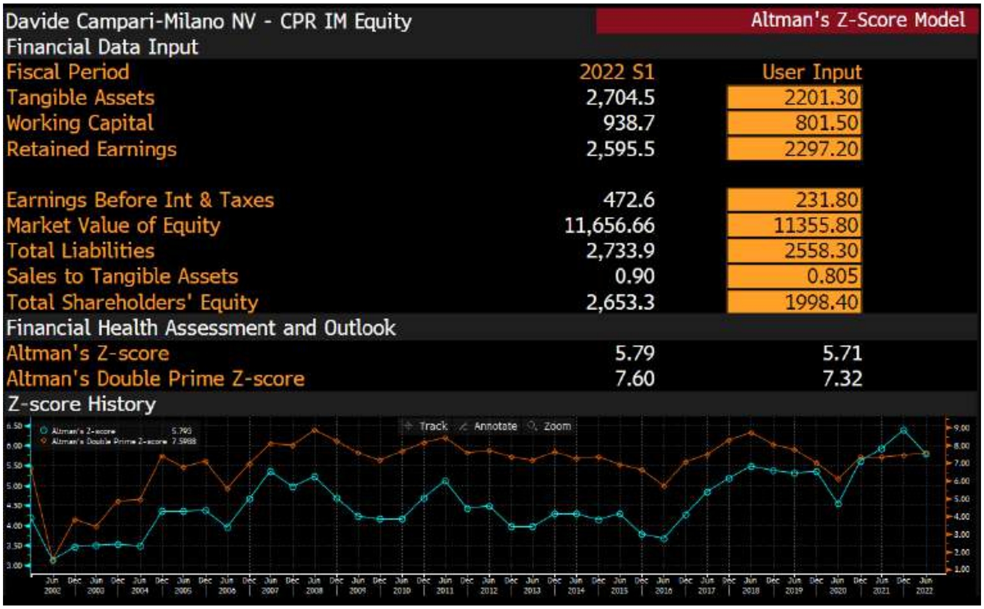

Figure VIII.13 Altman's Z-Score model (AZS). Source: Bloomberg®

Altman's model uses information taken from the balance sheet to forecast insolvency; regarding the formula used, the Z-score is obtained as a linear combination of the variables: since the coefficients are obtained by imposing the constraint that they are able to discriminate between healthy and risky firms, in accordance with the traditional statistical technique called linear discriminant analysis. In fact, two threshold values are often identified and called lower threshold and upper threshold. A company that has a score below the lower threshold is certainly too risky, while a company with a score above the upper threshold is certainly to be considered solvent. The area between the two thresholds is called the gray area: for companies that are located within this area, the model does not provide a unique indication, therefore it is the analyst who has to complete his analysis using qualitative judgement or implementing a more complex statistical model.

Starting from the Altman model, conceived for listed companies, banks have also developed scoring models for unlisted companies and, furthermore, variables that go beyond the balance sheet area have also been taken into consideration: in particular, the scoring models are also applied to the area of information concerning the company's relations with the financial system, using information from the Central Credit Register/private credit bureaux, and the performance relationship with the financial system. A further advancement of the scoring models is represented by the ways in which the solvency of the company is expressed: instead of the score seen in the Altman model, the most recent models, characterized by greater complexity from the point of view of formalization, express the solvency of the company directly in terms of probability. In order to generalize the concept of score, which is a measure summarizing the information contained in factors able to affect the default probability, we need to define the mathematical concepts underlying a credit score model in a more formal way.

As we have discussed, standard scoring models take the most straightforward approach by linearly combining those factors.

Let x denote the factors (their number is K) and b the weights (or coefficients) associated to them, we can represent the score that we obtain in scoring instance i as:

$$\text{Score}_i = b_1x_{i1} + b_2x_{i2} + \cdots + b_Kx_{iK} = \boldsymbol{b}'\boldsymbol{x_i},\ \boldsymbol{x_i} = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \dots \\ x_{iK} \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_K \end{bmatrix} \quad \textit{(Eq. VIII.45)}$$

If the model is to include a constant b_1, we set $x_{i1} = 1$ for each i.

For our example, we use the dataset provided by the book "Credit Risk Modeling" written by Löffler and Posch. The data are stored in a csv file with a total of observations $N = 4{,}000$ having the following fields:

- ID: the identifier associated to a corporate.

- Year: the year corresponding to the values reported in the following fields, ID+Year can be considered a primary key for our database.

- Default: it is the default indicator for year + 1. It is a boolean value that is equal to 1 if the firm has defaulted or 0 if it has not defaulted. In the described model, it is represented by the variable y_i.

- WC/TA: Working Capital/Total Assets. This ratio captures the short-term liquidity of a firm. In the described model it is represented by the variable x_{i1}, $i = 1, \dots, N$.

- RE/TA: Retained Earnings/Total Assets. This ratio measures the historical profitability. In the described model, it is represented by the variable x_{i2}, $i = 1, \dots, N$.

- EBIT/TA: Earnings before interest and taxes/Total Assets. This ratio measures the current profitability. In the described model, it is represented by the variable x_{i3}, $i = 1, \dots, N$.

- ME/TL: Market Value of Equity/Total Liabilities. It is a market-based measure of leverage. In the described model, it is represented by the variable x_{i4}, $i = 1, \dots, N$.

- S/TA: Sales/Total Assets. This ratio is a proxy for the competitive situation of the company. In the described model, it is represented by the variable x_{i5}, $i = 1, \dots, N$.

These $K = 5$ factors constitute the $\boldsymbol{x_i}$. Except for the market value, all these variables are found in the balance sheet and income statement of the company. The market value is given by the number of shares outstanding multiplied by the stock price.

Of course, an analyst could consider other variables as well, such as cash flows over debt service, earnings volatility, stock price volatility, ESG (Environment-Social-Governance) indicators… In this case, the most common method is used, i.e. the widely known Z-score ratios, consistently with the previous example.

The scoring model should predict a high default probability for those observations that defaulted and a low default probability for those that did not. To choose the appropriate weights $\boldsymbol{b}$, we first need to link scores to default probabilities. This can be done by representing default probabilities as a function F of scores:

$$\text{Prob}(\text{Default}_i) = \text{Prob}(y_i = 1) = F(\text{Score}_i) \quad \textit{(Eq. VIII.46)}$$

Like default probabilities, function F should be constrained to the interval from zero to one; it should also yield a default probability for each possible score. Those requirements can be fulfilled by a cumulative probability distribution function, and a distribution often considered for this purpose is the logistic distribution.

The logistic distribution function $\Lambda(z)$ is defined as $\Lambda(z) = \frac{\exp(z)}{1+\exp(z)}$. Applied to the previous equation, we have:

$$\text{Prob}(\text{Default}_i) = \Lambda(\text{Score}_i) = \frac{\exp(\boldsymbol{b}'\boldsymbol{x}_i)}{1+\exp(\boldsymbol{b}'\boldsymbol{x}_i)} = \frac{1}{1+\exp(-\boldsymbol{b}'\boldsymbol{x}_i)} \quad \textit{(Eq. VIII.47)}$$

Models that link information to probabilities using the logistic distribution function are called logit models.

Having collected the factors x and chosen the distribution function F, a natural way of estimating the weights b is the maximum likelihood (ML) method. According to the ML principle, the weights are chosen so that the probability (that is the likelihood) of observing the given default behaviour is maximized.

The first step in maximum likelihood estimation is to set up the likelihood function.

For a borrower that defaulted, the likelihood of observing this is:

$$\text{Prob}(\text{Default}_i) = \text{Prob}(y_i = 1) = \Lambda(\boldsymbol{b}'\boldsymbol{x}_i) \quad \textit{(Eq. VIII.48)}$$

For a borrower that did not default, we obtain the likelihood:

$$\text{Prob}(\text{No Default}_i) = \text{Prob}(y_i = 0) = 1 - \Lambda(\boldsymbol{b}'\boldsymbol{x}_i) \quad \textit{(Eq. VIII.49)}$$

These two formulas can be combined for defining the correct likelihood for an observation $\boldsymbol{i}$, L_i:

$$L_i = [\Lambda(\boldsymbol{b}'\boldsymbol{x}_i)]^{y_i}[1 - \Lambda(\boldsymbol{b}'\boldsymbol{x}_i)]^{1-y_i} \quad \textit{(Eq. VIII.50)}$$

Assuming that the defaults are independent, the likelihood of a set of observations is the product of the individual likelihoods:

$$L = \prod_{i=1}^{N} L_i = \prod_{i=1}^{N} [\Lambda(\boldsymbol{b}'\boldsymbol{x}_i)]^{y_i}[1 - \Lambda(\boldsymbol{b}'\boldsymbol{x}_i)]^{1-y_i} \quad \textit{(Eq. VIII.51)}$$

For the purpose of maximization, it is more convenient to examine the logarithm of likelihood (in fact it does not change the point of maximum).

$$\ln L = \sum_{i=1}^{N} y_i \ln[\Lambda(\boldsymbol{b}'\boldsymbol{x}_i)] + (1 - y_i)\ln[1 - \Lambda(\boldsymbol{b}'\boldsymbol{x}_i)] \quad \textit{(Eq. VIII.52)}$$

It can be maximized by setting its first derivative with respect to $\boldsymbol{b}$ to zero.

$$\frac{\partial L}{\partial \boldsymbol{b}} = \sum_{i=1}^{N} [y_i - \Lambda(\boldsymbol{b}'\boldsymbol{x_i})]\boldsymbol{x_i} \quad \textit{(Eq. VIII.53)}$$

A numerical routine for optimization like the Newton method is able to solve the previous equation with respect to $\boldsymbol{b}$. In order to apply this algorithm, we also need to estimate the second derivative, which is obtained as:

$$\frac{\partial^2 L}{\partial \boldsymbol{b} \partial \boldsymbol{b}'} = -\sum_{i=1}^{N} \Lambda(\boldsymbol{b}'\boldsymbol{x_i})[\boldsymbol{1} - \Lambda(\boldsymbol{b}'\boldsymbol{x_i})]\boldsymbol{x_i}\boldsymbol{x_i}' \quad \textit{(Eq. VIII.54)}$$

The function logit(Dataset,constant,stats) allows to estimate the $\boldsymbol{b}$ in accordance with the previous formulas. To do that, the function requires the following input arguments:

- Dataset is a numpy matrix which can be imported in the Python environment calling the importdata(Filepath) function, where Filepath is the place in which the csv file has been stored. In the first column, the Default indicator for the following year is stored, y, and in the other five columns, the factors (x) are stored.

- constant is a logic value that assumes the value True if the model includes a constant, otherwise it is False.

- stats is a logic value. If it is True, the routine returns the coefficients $\boldsymbol{b}$ as well as the statistics associated to the model, otherwise only $\boldsymbol{b}$.

Substantially, the core of the code lies in the while loop that allows to estimate the coefficients of the model using the ML principle.

The variable Lambda is the prediction, dlnl is the gradient, hesse is the Hessian and lnl is the log likelihood.

Once the gradient and the Hessian have been computed, the Newton rule can be applied. We take the inverse of the Hessian, hinv and multiply it with the gradient hinvg.

$$b_1 = b_0 - \left[\frac{\partial^2 \ln L}{\partial b_0 \partial b_0'}\right]^{-1} \frac{\partial \ln L}{\partial b_0} = b_0 - H(b_0)^{-1}\nabla(b_0) \quad \textit{(Eq. VIII.55)}$$

The logit model has the convenient feature that the log-likelihood function is globally concave, as a result a gradient-based numerical routine is enough for being sure to reach the global optimum. The while loop ends to update the coefficient vector $\boldsymbol{b}$ when the change in the likelihood is sufficiently small or when the maximum number of iterations is reached, then the function returns the coefficients.

The output of the logit routine can be shaped based on the value assumed by stats. If it is False, the function returns a numeric array in which the coefficients of the regression ($\boldsymbol{b}$) are stored, otherwise a numeric matrix having the form reported in the Table VIII.18.

Statistics are fundamental for understanding the goodness of the model. To assess whether a variable helps to explain the default event or not, we can examine a t-ratio for the hypothesis that the variable's coefficient is zero. For the j-th coefficient, such a t-ratio is constructed so that:

$$t_j = \frac{b_j}{SE(b_j)} \quad \textit{(Eq. VIII.56)}$$

Where SE is the estimated standard error of the coefficient.

b_1	b_2	…	b_K
$SE(b_1)$	$SE(b_2)$	…	$SE(b_K)$
$t_1 = b_1/SE(b_1)$	$t_2 = b_2/SE(b_2)$	…	$t_K = b_K/SE(b_K)$
$p-\text{value}(t_1)$	$p-\text{value}(t_2)$	…	$p-\text{value}(t_K)$
$\text{Pseudo}-R^2$	# iterations	0	0
$LR-\text{test}$	$p-\text{value}(LR)$	0	0
Log − likelihood(model)	Log − likelihood(restricted)	0	0

Table VIII.18 Output of the logit function

We take b from the last iteration of the Newton scheme and the standard errors of estimated parameters are derived from the Hessian matrix. Specifically, the variance of the parameter vector is the main diagonal of the negative inverse of the Hessian at the last iteration step. In the logit function, we have already computed the Hessian hinv for the Newton iteration, so we can quickly calculate the standard errors. We set the standard error of the j -th coefficient to sqrt(-hinv[j,j]). The t-ratios are then computed using the previous formula. In the logit model, the t-ratio does not follow a t-distribution as in the classical linear regression. Rather, it is compared to a standard normal distribution. Then, to obtain the p-value of a two-sided test, we exploit the symmetry of the normal distribution:

$$p - \text{value} = 2 \cdot \left(1 - \phi(|t|)\right) \quad \textit{(Eq. VIII.57)}$$

Where ϕ is the cumulative standard normal distribution.

The logit function returns standard errors, t-ratios and p-values in lines two to four of the output if the logical value stats is set to True. In a linear regression, an R^2 is usually reported as a measure of the overall goodness of fit. In nonlinear models estimated with maximum likelihood, the Pseudo- R^2 suggested by McFadden (1974) is typically reported. It is calculated as 1 minus the ratio of the likelihood of the estimated model ($\ln L$) and the one of a restricted model that only has a constant ($\ln L_0$):

$$\text{Pseudo} - R^2 = 1 - \frac{\ln L}{\ln L_0} \quad \textit{(Eq. VIII.58)}$$

Like the standard R^2, this measure is bounded by zero and one. Higher values indicate a better fit. The log-likelihood $\ln L$ is given by the log-likelihood function of the last iteration of the Newton procedure,and is thus already available. The loglikelihood of the restricted model is then left to be determined. With a constant only, the likelihood is maximized if the predicted default probability is equal to the mean default rate, $\bar{y}$. This can be achieved by setting the constant equal to the logit of the default, that is $b_1 = \ln\left[\frac{\hat{y}}{1-\hat{y}}\right]$.
For the restricted log-likelihood, we then obtain:

$$\ln L_0 = \sum_{i=1}^{N} y_i \ln[\Lambda(\boldsymbol{b}'\boldsymbol{x_i})] + (1 - y_i)\ln[1 - \Lambda(\boldsymbol{b}'\boldsymbol{x_i})] =$$

$$= \textstyle\sum_{i=1}^{N} y_i \ln(\bar{y}) + (1 - y_i)\ln(1 - \bar{y}) = \quad \text{(Eq. VIII.59)}$$

$$= N \cdot [\bar{y}\ln(\bar{y}) + (1 - y_i)\ln(1 - \bar{y})]$$

The two likelihoods used for the Pseudo-R^2 can also be used to conduct a statistical test of the entire model, that is, test the null hypothesis that all coefficients except for the constant are zero. The test is structured as a likelihood ratio test: $LR = 2(\ln L - \ln L_0)$.

The more likelihood is lost by imposing the restriction, the larger the LR-statistic will be. The test statistic is distributed asymptotically χ^2 with the degrees of freedom equal to the number of restrictions imposed.

When testing the significance of the entire regression, the number of restrictions equals the number of variables K minus one. The chi2.sf(2*(lnL[Iter]-lnL0),K-1) gives the p-value of the LR test. Both LR and its p-value have been returned by the full output structure. The likelihoods $\ln L$ and $\ln L_0$ are also reported as well as the number of iterations needed to achieve the convergence. Running the function logit(Dataset, constant=True, stats=True), we obtain the results reported in the next figure. The arrangement of the numbers follows exactly the output template shown previously.

Regarding the overall fitting model statistics, we can look at the LR test (160.148) and its p-value (10^{-33}): the logistic regression is highly significant. The hypotheses "the five ratios add nothing to the prediction" can be rejected with high confidence and the regression model helps to explain the default events. Knowing that the model does predict defaults, we would like to know how well it does so.

An analyst usually turns to the R^2 for answering this question, but as in linear regression, setting up general quality standards in terms of a Pseudo- R^2 is difficult.

	0	1	2	3	4	5
0	-2.54348	0.414394	-1.45402	-7.99906	-1.59359	0.619721
1	0.266029	0.572478	0.229486	2.7024	0.323405	0.349199
2	-9.56089	0.723861	-6.33598	-2.95998	-4.92754	1.77469
3	0	0.469151	2.35832e-10	0.0030766	8.32703e-07	0.0759483
4	0.222058	12	0	0	0	0
5	160.148	9.20493e-33	0	0	0	0
6	-280.526	-360.6	0	0	0	0

Figure VIII.14 Full output of the logit function applied on the original credit dataset using all five ratios

A simple but often effective way of assessing this measure is to compare it with ones from other models estimated on similar data sets. From the literature, we know that scoring for listed US companies can achieve a

Pseudo- R^2 of about 35%-40%. This unfortunately indicates that the way we have set up the model may not be ideal given that our Pseudo- R^2 is equal to 22.21%: after this statistical analysis we focus on how to improve the performance of the model.

Turning to the regression coefficients, we can summarize that three out of five ratios have b that are significant on the 1% level or better because their p-value is below 0.01. If we reject the hypothesis that one of these coefficients is zero, we can expect to err with a probability of less than 1%. Each of the three variables (RE/TA, EBIT/TA and ME/TL) has a negative coefficient, meaning that increasing values of the variables reduce default probability. This is coherent, by economic reasoning, as retained earnings, EBIT and market value of equity over liabilities should be inversely related to default probabilities. The constant is also highly significant (p-value ~ 0) and the coefficient on working capital over total assets (WC/TA) and sales over total assets (S/TA), by contrast, exhibit a significance of only 46.92% and 7.59%, respectively. By conventional standards of statistical significance (5% traditionally is the most common choice) we would conclude that these two variables are not or only marginally significant and we would probably consider not using them for prediction. If on the other hand we simultaneously remove two or more variables based on their t-ratios, we should be aware of the possibility that variables might jointly explain defaults even though they are insignificant individually. To statistically test this possibility, we can perform a second regression in which we exclude variables that were insignificant in the first run and then conduct a likelihood ratio test. Running logit(CreditDataset[:,[0,2,3,4]],True,True), we reach the results shown in Figure VIII.15.

The likelihood test for the hypothesis $b_{WC/TA} = b_{S/TA} = 0$ is based on the comparison of the log-likelihoods $\ln L$ of the two models. It is constructed as:

$$LR = 2 \cdot [\ln L\,(\text{model 1}) - \ln L\,(\text{model 2})] \quad \textit{(Eq. VIII.60)}$$

In this case it is referred to a χ^2 distribution with two degrees of freedom because we impose two restrictions. chi2.sf(2*(-280.526-(-282.219)),2) gives 0.1840. This means that if we add the two variables WC/TA and S/TA to model 2, there is a probability of 18.40% that we do not add any explanatory power.

	0	1	2	3
0	-2.31833	-1.41974	-7.17936	-1.61562
1	0.235635	0.228767	2.72537	0.324763
2	-9.83866	-6.20607	-2.63427	-4.97476
3	0	5.43268e-10	0.00843192	6.53297e-07
4	0.217361	12	0	0
5	156.761	9.16409e-34	0	0
6	-282.219	-360.6	0	0

Figure VIII.15 Full output of the logit function applied on the original credit dataset using the most significant financial ratios

The LR test thus confirms the results of the individual tests: both individually and jointly, the two variables would be considered only marginally significant. However, it is important to highlight that a common best practice is to also perform out-of-sample tests of predictive performance before dropping variables from the model. Having specified a scoring model, we want to use it for predicting probabilities of default. In order to do so, we calculate the score and then translate it into a default probability in accordance with:

$$\text{Prob}(\text{Default}_i) = \Lambda(\text{Score}_i) = \frac{\exp(\boldsymbol{b}'\boldsymbol{x}_i)}{1+\exp(\boldsymbol{b}'\boldsymbol{x}_i)} = \frac{1}{1+\exp(-\boldsymbol{b}'\boldsymbol{x}_i)} \quad \textit{(Eq. VIII.61)}$$

The getPDfromLogit(CreditDataset,constant) is able to perform this task for the N observations in the database.

Even the sensitivity to the change of a factor is considered as very important to properly manage the logit model. The getPDsensitivityfromLogit(CreditDataset,constant,bump) absolves this necessity. It takes a further compulsory input argument, that is the bump to be applied to the variable to get the sensitivity of the model compared to the analyzed variable. The sensitivity of the model has been estimated like a Greek for all the available observations, using a two-sided finite difference: $\frac{f(x_i+h)-f(x_i-h)}{2h}$, where h is the bump applied to the factors and $i = 1, \dots, N$.

The output of the function is a matrix made of K columns. In the first column, the sensitivity of the model to a bump applied to all the K factors simultaneously has been reported, the other columns contain the partial sensitivity of the logit model obtained by individually applying the shock to each factor. Hence the second column contains the sensitivity of the model to the first factor (i.e. WC/TA) leaving the other variables unchanged; the third column contains the sensitivity of the model to the second factor (i.e. RE/TA) leaving the other $K-1$ variables unchanged and so on. The final output is a matrix of a dimension equal to $N \times K + 1$.

	0	1	2	3	4	5	6
0	0.00210391	-0.00239384	0.000927692	0.00119954	-0.000261078	-0.000499787	0
1	0.000604251	-0.00218426	0.000812542	0.0011971	0.000305775	-0.000418571	0
2	0.00688436	-0.00307632	0.000970709	0.00105519	-0.00180328	-0.000518927	0
3	0.0438242	-0.00532576	0.00210638	0.00301124	-0.0196708	-0.0013346	0
4	0.00681597	-0.00303925	0.000933568	0.00127319	-0.00192584	-0.000580051	0

Figure VIII.16 Model sensitivity

Having set the scoring models, the Probability of Default and their sensitivities respect to factors, it is worth wondering if the global performance of the regression can be improved in terms of Pseudo- R^2.

In general terms, it can be considered a good result to obtain an overall performance of the model which can be compared to the literature benchmarks for the US listed firms. One of the most popular techniques is to treat outliers in input variables. Explanatory variables in scoring models often contain a few extreme values. They can reflect truly exceptional situations of borrowers, but they can also be due to data errors, conceptual problems in defining a variable or accounting discretion. In any case, extreme values can have a large influence on coefficient estimates, which could impair the overall quality of the scoring model.

A first step in approaching the problem is to examine the distribution of the variables. With this aim, the function getDatasetDescriptiveStats(Dataset) allows to compute the main descriptive statistics for the five analyzed ratios. The output is an object of the DescriptiveStatistics class which contains the fields reported in the first column of Table VIII.19, together with the values assumed for each factor x_i.

Field	WC/TA	RE/TA	EBIT/TA	ME/TL	S/TA
Average	0.14	0.21	0.05	1.95	0.30
Median	0.12	0.22	0.05	1.14	0.26
Standard Deviation	0.17	0.33	0.03	2.99	0.21
Skewness	-1.01	-2.55	-4.84	7.75	4.48
Kurtosis	17.68	17.44	86.00	103.13	71.22
Minimum	-2.24	-3.31	-0.59	0.02	0.04
Percentiles[0]=0.50	-0.33	-1.72	-0.05	0.05	0.06
Percentiles[1]=1	-0.17	-0.92	-0.02	0.08	0.07
Percentiles[2]=5	-0.06	-0.25	0.02	0.22	0.10
Percentiles[3]=95	0.44	0.65	0.09	5.60	0.68
Percentiles[4]=99	0.58	0.90	0.12	14.44	1.05
Percentiles[5]=99.5	0.63	0.94	0.13	18.94	1.13
Maximum	0.77	1.64	0.20	60.61	5.01

Table VIII.19 Descriptive Statistics

A common benchmark for assessing an empirical distribution is the normal distribution. The reason is not that there is a priori a reason why the variables should follow a gaussian distribution, but rather that the normal serves as a good reference point because it describes a distribution in which extreme events have been averaged out.

We remind that the relevant theorem from statistics is the central limit theorem, which says that if we sample from any probability distribution with finite mean and finite variance, the sample mean will tend to the normal distribution as we increase the number of observations to infinity. A good indicator for the existence of outliers is the excess kurtosis (that is defined as kurtosis minus 3). The normal distribution has excess kurtosis of zero, but the variables used in the example have very high values ranging from 17.4 to 103.1.

In this context, a positive excess kurtosis indicates that there are relatively more observations far away from the mean, compared to the normal.

The variables are also skewed, meaning that extreme observations are concentrated on the left (if the skewness is negative) or on the right (if skewness is positive) of the distribution. In addition, we can look at the percentiles.

For example, a normal distribution has the property that 99% of all observations are within ± 2.58 standard deviations of the mean. For the variable ME/TL, this would lead to the interval [-5.77, 9.68]. The empirical 99% confidence interval, however, is [0.05, 18.94] that is wider and shifted to the right, confirming the information we acquired by observing the skewness and kurtosis of ME/TL.

Considering WC/TA, we see that 99% of all values are in the interval [-0.33, 0.63], which is roughly in line with what we would expect under a normal distribution, namely [-0.30, 0.58]. In the case of WC/TA, the outlier problem is thus confined to a small subset of observations. This is most evident by looking at the minimum value of WC/TA: it is -2.24, which is very far away from the bulk of the observations, as it is 14 standard deviations away from the mean, and 11.2 standard deviations away from the 0.5% percentile.

Having identified the existence of extreme observations, a rigorous inspection of the data is advisable because it can lead to the discovery of correctable data errors. In many applications, however, this will not lead to a complete elimination of outliers; even data sets that are 100% correct can exhibit bizarre distributions. Accordingly, it is useful to have a procedure that controls the influence of outliers in an automated and objective way. A commonly used technique applied for this purpose is the so-called **winsorization**, which means that extreme values are pulled to less extreme ones.

Saying that the analyst specifies a certain winsorization level α means that values below the α -percentile of the variable distribution are set equal to the α -percentile, values above the $1-\alpha$ a percentile are set equal to $1-\alpha$. Common values for α are 0.5%, 1%, 2% or 5%. The winsorization level can be set separately for each variable in accordance with its distributional characteristics, providing a flexible and easy way of dealing with outliers without discarding observations. Running the function for $\alpha = 1\%$ for each x_i we obtain the third and fourth moments shown in Table VIII.20:

Measure	WC/TA	RE/TA	EBIT/TA	ME/TL	S/TA
Skewness	0.63	-0.95	0.14	3.30	1.68
Kurtosis	0.01	3.20	1.10	13.48	3.42

Table VIII.20 Skewness and Kurtosis

Both skewness and kurtosis are now much closer to zero. Let us note that both statistical characteristics are still unusually high for ME/TL. This might explain a higher winsorization level for this factor, but there is a smarter alternative. Given that ME/TL has many extreme values to the right of the distribution, a good idea can be to take the logarithm with the aim of pulling them to the left without blurring the differences between those beyond a certain threshold, as we do applying the Winsor method. The logarithm of ME/TL (after winsorization at the 1% level) has a skewness of -0.11 and a kurtosis of 0.18, suggesting that the logarithmic transformation works for ME/TL in terms of outliers.

This transformation leads to an important benefit in terms of Pseudo- R^2: running the logit(WinsorDataset, True, True) we obtain a value of 25.48% and launching the same command with the log of ME/TL we reach 33.97%. Figure VIII.17 shows the full output of the logit function applied on the original credit dataset using winsorized variables at 1%:

	0	1	2	3	4	5
0	-2.4745	0.376492	-2.53848	-22.978	-1.16084	1.409
1	0.318889	0.829262	0.335097	5.48897	0.298466	0.567607
2	-7.75977	0.454008	-7.57534	-4.18622	-3.88935	2.48236
3	8.43769e-15	0.649823	3.57492e-14	2.83644e-05	0.000100513	0.0130517
4	0.254793	12	0	0	0	0
5	183.757	8.43311e-38	0	0	0	0
6	-268.721	-360.6	0	0	0	0

Figure VIII.17 Full output of the logit function applied on the original credit dataset using winsorized variables at 1%

Figure VIII.18 shows the full output of the logit function applied on the original credit dataset using winsorized variables at 1% and the $\ln ME/TL$:

	0	1	2	3	4	5
0	-4.70936	0.909294	-1.67887	-17.0034	-1.40481	1.07475
1	0.377638	0.852232	0.379533	5.81077	0.167202	0.595173
2	-12.4706	1.06696	-4.42352	-2.9262	-8.40187	1.80578
3	0	0.285991	9.71075e-06	0.00343135	0	0.0709521
4	0.339705	10	0	0	0	0
5	244.995	6.51378e-51	0	0	0	0
6	-238.102	-360.6	0	0	0	0

Figure VIII.18 Full output of the logit function applied on the original credit dataset using winsorized variables at 1% and the ln ME/TL

The examination of univariate relationships between default rates and explanatory variables can give valuable hints as to which transformation is appropriate. In the case of ME/TL, it supports the logarithmic one, but in many other cases, it may support a polynomial representation like in the case of sales growth. Often, however, which transformation to choose may not be clear, and an automatic procedure can be very useful especially when there is a huge number of factors. To such end, we can employ the procedure coded in the XTransform function. The main idea is to use the default rate of the range to which they are assigned instead of entering the original values of the variable into logit analysis. In this way we use a data-driven, nonparametric transformation of the input data. Running the function with a number of range equal to 20, we obtain a Pseudo-R^2 very close to that reported in the literature (46.001%). In this example we have described the estimation of a scoring model with logit.

A common alternative is the **probit model**, which replaces the logistic distribution in $\Lambda(\text{Score})$ with the standard normal distribution. Experience and literature suggest that the choice of the distribution is not so crucial in most settings; predicted default probabilities and performance are fairly close.

probit(Dataset, constant, stats) implements this alternative and the Pseudo-R^2 is very close to the previous model (45.1%).

	0	1	2	3	4	5
0	5.15291	0.459452	0.329845	0.187741	0.323704	0.344958
1	2.41525	0.347555	0.175824	0.207495	0.254786	0.268809
2	2.13349	1.32195	1.87599	0.904797	1.27049	1.28328
3	0.0328844	0.186183	0.060656	0.365573	0.203908	0.199393
4	0.450994	48	0	0	0	0
5	325.257	3.70297e-68	0	0	0	0
6	-197.971	-360.6	0	0	0	0

Figure VIII.19 Full output of the probit function

Scoring models have advantages and limitations. Among the first ones we can identify:

- **objectivity**: the choice of variables and the weights to be attributed to each of them depend on a statistical model and on the database used to calibrate the model.

- **strength**: once the model has been defined, the bank is able to implement the investigation by minimizing the time and costs of the procedure and can also implement adequate monitoring procedures.

- **uniformity**: the adoption of a standardized model for the evaluation of the counterparty allows the calculation of average scores by geographical area and branch.

Among the limits, we can identify the following:

- **instability**: structural changes in the economic cycle, in the bank's internal procedures or in the regulations may affect the validity of the model.

- **costs of incorrect classification**: in the construction of the model, the intermediary must choose the optimal solution in the trade-off between precision of the model and costs for its creation and maintenance.

- **quality of the input data**: a reliable valuation model must include balance sheet data, performance data and internal data updated with respect to the time horizon of the valuation.

In addition to the credit scoring techniques which, from time to time, provide for the identification of the relevant indicators, the estimate of the PD through the quantitative approach, has brought the creation of models based on the use of market data as input for the estimation of the risk of debtors whose capital is listed on the financial markets.

Among the different families of models, the structural models are particularly interesting. They estimate the PD based on structural characteristics of the firm: the value of the assets, the value of the debt and the variability of the asset value over time. One of the most common estimation methods is the option-theoretic Merton approach. It is similar to the counterparty risk, and we have already discussed it in the dedicated module. Therefore we will now discuss a widely used alternative structural modeling approach called **CreditGrades** (CG).

In this model, default occurs if the asset value of the firm drops below a random barrier. The randomness of the barrier is the main difference to the KMV model. It captures the fact that typically, the level of a firm's liabilities is not known until the firm defaults. Balance sheet information is only available quarterly and unluckily it does not always represent the current company situation so accurately. In addition, the CG model assumes that default can occur at any time, whereas default in the classical Merton model occurs only at maturity.

Let us express the relevant firm variables on a per-share basis. We thus indicate today's asset value per-share with A. We assume that asset values follow a lognormal distribution. The random default barrier is:

$$\text{Default Barrier} = \Lambda D \quad \textit{(Eq. VIII.62)}$$

Where D is today's debt-per-share. Λ represents the uncertainty in recovery values and is a random variable, which is assumed to follow a lognormal distribution with mean $\overline{\Lambda}$ and standard deviation λ, independent of the asset value process. As in the classical Merton model, we need to estimate today's asset value, its volatility and its drift rate. In the CG model, they are determined as follows: the drift rate is set to zero. This is justified if firms tend to maintain a constant leverage over time. Assets may rise at a certain rate but if debt rises at the same rate, the distance to the default barrier remains constant and we describe the situation by assuming a zero drift for both assets and debt.

For the initial asset value, the **CG model** suggests the approximation $A = S + \overline{\Lambda}D$ where S is the firm's stock price. This choice is based on an inspection of boundary cases.

The asset volatility is set to:

$$\sigma = \sigma_E \frac{S}{S+\bar{\Lambda}D} \text{ (Eq. VIII.63)}$$

Where σ_E is the equity volatility. This expression is an approximation of the theoretical relationship between asset volatility and equity volatility. Using these assumptions, the approximation for the probability that the firm survives at time t, as seen from today ($t = 0$) is:

$$\text{Prob(Survival at } t) = \Phi\left(-\frac{\alpha_t}{2} + \frac{\ln(d)}{\alpha_t}\right) - d\Phi\left(-\frac{\alpha_t}{2} - \frac{\ln(d)}{\alpha_t}\right) \text{ (Eq. VIII.64)}$$

With $d = \frac{S+\bar{\Lambda}D}{\bar{\Lambda}D}\exp(\lambda^2)$ and $\alpha_t^2 = \left(\sigma_E \frac{S}{S+\bar{\Lambda}D}\right) t + \lambda^2$.

Date	Adj Share Price	Debt-per-Share	Hist.90D Volatility	CG 1 yr [%]
Q4-2003	36.11	69.96	25.26	0.0014
Q1-2004	43.36	78.28	24.74	0.0006
Q2-2004	37.83	81.23	27.92	0.0941
Q3-2004	36.95	94.93	25.11	0.0108
Q4-2004	41.89	94.40	22.01	0.0010
Q1-2005	45.59	110.30	18.82	0.0004
Q2-2005	46.10	113.76	23.69	0.0047
Q3-2005	52.83	121.73	20.17	0.0005
Q4-2005	63.00	106.06	25.22	0.0004
Q1-2006	72.98	126.86	23.03	0.0001
Q2-2006	66.61	156.97	29.04	0.0244
Q3-2006	63.81	121.61	27.90	0.0043
Q4-2006	73.67	137.89	29.01	0.0062
Q1-2007	73.31	179.41	30.82	0.0514
Q2-2007	73.38	194.81	27.60	0.0304
Q3-2007	54.83	235.01	34.89	1.0364
Q4-2007	62.63	239.90	54.48	5.239
Q1-2008	50.99	219.77	78.40	17.429
Q2-2008	36.81	127.80	99.93	27.409

Table VIII.21 CreditGrades Probability of Default model – Case study on Lehman Brothers

Now let us introduce an example based on the calculation of a time series of CreditGrade survival probabilities for the well-known Lehman Brothers case. The dataset used comes from Bloomberg® and from the "Credit risk modeling" milestone written by Löffler and Posch. Clearly, the necessary adjustments when dealing with financial companies are not standard. The estimation of the debt-per share variable should be adjusted in order not to overweight short-term borrowings. Furthermore, the volatility of the barrier is usually set to a lower level i.e. 10% as proposed by the literature.

Here we show that these adjustments are enough to perform a timely reasonable credit assessment using the previous Equation. Table VIII.21 summarizes the main inputs for the CG approach from the quarterly report from Q4 2003 to Q2 2008. The debt-per-share ratios (third column) can be computed from the company statements or implied from CDS data, as will be shown later. Using end-of-quarter stock prices (second column) and historical volatility over 90 days (fourth column), we are able to estimate the one-year probabilities of default (fifth column) using one minus the results obtained from the previous equation of the survival volatility. These results are computed setting the volatility of the barrier $\lambda = 10\%$ and the global recovery $\overline{\Lambda} = 50\%$.

In accordance with the CreditGrades theory, the CDS premium and the debt-to-ratio are closely linked. The objective is to estimate the implied CDS premium, s, from D through a formula.
Let us start by considering the pricing formula of a European put option under the CG framework (Finger et al., 2002):

$$P(S,t,B) = X \cdot \exp\big(-r(T-t)\big)\Phi(a_1,a_2) - S \cdot \Phi(a_5) + I(B,\sigma,S,X) \quad \textit{(Eq. VIII.65)}$$

where X is the strike, T is the maturity of the option, B the random default barrier, $\Phi(x,y)$ the integral over the normal distribution density from x to y, and $\Phi(x)$ is the corresponding integral over $(-\infty,x)$ and r the risk-free rate.

$$\begin{aligned} I(B,\sigma,S,X) &= -X \cdot \exp\big(-r(T-t)\Phi(a_3,a_4)\big) + S\big(1-\Phi(a_4)\big) + \\ &+B \cdot \exp(rt)\,\big(\Phi(a_2)-\Phi(a_4)-\Phi(a_5)+\Phi(a_6)\big) + \\ &-\frac{S}{B} \cdot X \cdot \exp(-rt)\,\Phi(a_3,a_4) + 2 \cdot X \cdot \exp\left(\frac{z}{2} - r(T-t)\right) \cdot \\ &\cdot \int_{\frac{z}{\sqrt{2(T-t)}}}^{\infty} \sqrt{2\pi}^{-1} \exp\left(-\frac{s^2}{2}\right) \exp\left(-\frac{z^2}{8s^2}\right) ds \quad \textit{(Eq. VIII.66)} \end{aligned}$$

Where:

$$a_{1,3} = \pm\frac{\frac{1}{2}\sigma^2(T-t) - \sigma\eta}{\sigma\sqrt{T-t}},\ a_{2,5} = -\frac{\sigma(\eta-\eta_X) \mp \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},\ a_{4,6} = -\frac{\sigma(\eta+\eta_X) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

Which depend on the distance to default measures:

$$\eta = \frac{1}{\sigma}\ln\left(1+\frac{S}{B\cdot\exp(rt)}\right),\ \eta_X = \frac{1}{\sigma}\ln\left(1+\frac{X}{B\cdot\exp(rt)}\right) \quad \textit{(Eq. VIII.67)}$$

With two at-the-money put prices, $P_{1,2}$, we can use the expression $P(S,t,B)$ to solve for the implied asset volatility and the implied barrier.

Converting the survival probability into a credit spread we specify R as the recovery rate of the underlying credit and obtain:

$$s = r(1-R)\frac{1-PS_0+\exp\left(r\frac{\sigma^2\Lambda}{\sigma^2}\right)\left(G\left(T+\frac{\sigma^2\Lambda}{\sigma^2}\right)-G\left(\frac{\sigma^2\Lambda}{\sigma^2}\right)\right)}{PS_0-PS_T\exp(-rT)-\exp\left(r\frac{\sigma^2\Lambda}{\sigma^2}\right)\left(G\left(T+\frac{\sigma^2\Lambda}{\sigma^2}\right)-G\left(\frac{\sigma^2\Lambda}{\sigma^2}\right)\right)} \quad \text{(Eq. VIII.68)}$$

Where the function G is given as:

$$G(x) = d^{\left(z+\frac{1}{2}\right)}\Phi\left(-\frac{\ln(D)}{\sigma\sqrt{x}} - z\sigma\sqrt{x}\right) + d^{\left(-z+\frac{1}{2}\right)}\Phi\left(-\frac{\ln(D)}{\sigma\sqrt{x}} + z\sigma\sqrt{x}\right) \quad \text{(Eq. VIII.69)}$$

With $z = \sqrt{\frac{1}{4}+\frac{2r}{\sigma^2}}$ and PS_t is the survival probability.

The following functions implement all the equations needed to find the relationship between the debt-per-share ratio (D) and the spread s:

CreditGrades_PS(S,sigma_E,D,Lambda,sigma_B,t) computes PS_t implementing the $\text{Prob(Survival at } t)$ equation.

CreditGrades_G(x,d1,r,sigma) computes $G(x)$.

CreditGrades_CDS(S,sigma_E,d,Lambda,sigma_B,t,Rec,r) computes s.

As an example, we can run CreditGrades_CDS using the market and statement data of Lehman Brothers on Q3 2007: $S = 54.83$, $\sigma_E = 34.89\%$, $D = 235.01$, $\Lambda = 50\%$, $\sigma_B = 10\%$, $T = 1$, $R = 40\%$ (the standard percentage for an officially listed CDS) and $r = 4.90125\%$ (USD Libor with 1-year tenor). We reach a theoretical implied 1-year spread of 61.87 bps which is quite close to the one observed on the market ($= \sim 70$ bps).

The green dotted line shows the term structure of the listed spreads at the early beginning of the problems (30th September 2007 - Q3 2007). One year senior spread is about 70 basis points. The yellow dotted line shows the term structure of the listed senior spreads at the end of June 2008 (30th June 2008 - Q2 2008). One year spread is about 400 basis points. On the morning of the default, the one year senior CDS was over 1,300 basis points.

The use of a qualitative approach for the prediction of the PD is based on the determination of a score, like the quantitative one. However, in this case, the information used and the weight attributed to them is not determined in an automated manner but it is left to the credit analyst. This approach is implemented by the **rating agencies** in the evaluation of issuers: in fact, the agencies themselves underline the nature of the rating opinion and the relevance of the qualitative-quantitative dimension.

Figure VIII.20 Lehman Brothers CDS premium on Q3 2007, Q2 2008 and on the day of bankruptcy

Here are a few examples of statements and dislosures from the main and well-known rating agencies:

- "A credit rating is Standard & Poor's opinion of the general creditworthiness of an obligor, or the creditworthiness of an obligor with respect to a particular debt security or other financial obligation, based on relevant risk factors" (**Standard and Poor's**, 2012).

- "Fitch's corporate ratings make use of both qualitative and quantitative analyses to assess the business and financial risks of fixed-income issuers. Ratings are an assessment of the issuer's ability to service debt in a timely manner and are intended to be comparable across industry groups and countries" (**Fitch**, 2013).

- "Issuer ratings are opinions of the ability of entities to honor senior unsecured financial obligations and contracts.

Because it involves a look into the future, credit rating is by nature subjective. Moreover, because long-term credit judgments involve so many factors unique to particular industries, issuers, and countries, we believe that any attempt to reduce credit rating to a formulaic methodology would be misleading and would lead to serious mistakes. That is why Moody's uses a multidisciplinary or 'universal' approach to risk analysis, which aims to bring an understanding of all relevant risk factors and viewpoints to every rating analysis" (**Moody's**, 2009).

Banks can also adopt a qualitative approach to predicting the risk of default; this solution is adopted to evaluate large corporate, and to evaluate the contribution of aspects that are difficult to standardize and evaluate within a quantitative approach. In the latter case, the qualitative approach is particularly suitable for investigating: the context in which the company operates, the distinctive features of the company, the relations with the environment, the markets, the human resources, as well as processes and production. Within the area of the distinctive characteristics of the company, internal processes and human resources, it is important to evaluate the control structures, organization and quality of the documentation. As part of the assessment of the context in which the company operates, relations with the external environment and the markets, the bank evaluates the characteristics of the sector and the competitive positioning. A weight will thus be assigned to each of the evaluated elements that allows to determine its score. In recent times, much attention has been paid to the integration of qualitative variables of an ordinal character (i.e. the categories that can be ordered from best to worst such as, for example, the intensity of competition in the sector) with quantitative variables in the scope of models for the attribution of the score. The scientific literature and the indications of the supervisory authorities however indicate that the direction to be taken is to integrate the use of quantitative and qualitative information, improving the culture of risk management.

FURTHER READINGS

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VIII.4 LOSS GIVEN DEFAULT

The **Loss Given Default** (LGD) is the percentage of the exposure classified in default that the bank is unable to recover at the end of the recovery process. The Basel 2 framework has introduced a definition of LGD that takes into account a plurality of factors: "The definition of loss to be applied to the LGD estimate is that of economic loss, to measure which among all relevant factors must be taken into account.

These include significant discounts on the nominal value and the relevant direct and indirect costs associated with credit recovery" (Basel Committee on Banking Supervision, 2006).

First of all, to estimate the LGD, the bank must define the source of the data to be used. The approaches that can be adopted by the bank can be distinguished in relation to the use of internal data compared to the use of external data. If the bank uses external data, the LGD is estimated using:

- the price of corporate bonds after the declaration of insolvency. This solution can be considered suitable for estimating the LGD for credit exposures to large corporates, banks, sovereign states. It requires verifying the consistency with the definition of the default state chosen.

- implicit credit spreads in yields on corporate bonds and/or credit derivatives.

The most frequent approach used by banks is the one based on internal data. As part of this approach, the LGD is estimated according to the following formula:

$$LGD = 1 - \frac{\sum_{t=1}^{N} \frac{RF_t}{(1+i)^{\tau_t}} - \sum_{t=1}^{N} \frac{EF_t}{(1+i)^{\tau_t}}}{EAD} \quad \text{(Eq. VIII.70)}$$

Where:

- RF_t is the recovery cash flow at time τ_t.

- EF_t is the cost flow at time τ_t.

- i is the discount rate to determine the present value at the time of default.

Therefore, to estimate the LGD it is necessary to define the state of default, the incoming cash flows, and the discount rate. The definition of the state of default coincides with that introduced for the estimate of the PD: certainly, the more prudent the definition of default, the greater is the number of debtors classified in state of insolvency, but the higher are the amounts recovered.

Technically, debtors classified in a state of default who are performing or for which no losses are recorded are defined as "treated debtors": banks must take into account the cure rate, i.e. the incidence rate of treated debtors compared to those classified in default. A high incidence of the cure rate shows that the definition of default adopted is unsuitable for capturing the risk of insolvency and can lead to distortion in the development of models. Moving on to the definition of financial cash flows, they are distinguished in incoming and outgoing.

The cash inflows are determined by:

- the enforcement of collateral.
- the enforcement of personal guarantees.
- any debt restructuring agreements.
- the liquidation of the debtor's assets.

The contribution of guarantees in mitigating the LGD differs between personal guarantees (function of the guarantor's PD) compared to collateral. Within the collateral, it is possible to identify the contribution of the different types. Table VIII.22 reports the average LGDs by type of collateral in accordance with the studies of Carty et al. (1998) and Araten and Jacobs (2004).

Type of Guarantee	Carty et al.	Arten & Jacobs
Money and traded securities	10.13%	35.80%
Non-tradable securities		44.90%
Commercial credits		35.10%
Stock		40.90%
Commercial credits and stocks		41.60%
Fixed Capital	14.57%	42.30%
Real Estate		39.40%
General guarantee on the debtor's property	-	47.20%
Other	11.22%	41.80%

Table VIII.22 Average LGDs by type of collateral

The second column of Table VIII.22 shows the statistics of Carty based on 200 large corporate loans using LGDs on external data, and the third column shows the statistics studied by Araten and Jacobs based on 3,761 retail loans using LGD on internal data. It should be noted that higher-value loans are, on average, characterized by a lessened LGD: collateral, money and similar guarantees and real estate allow to mitigate the LGD.

The financial outflows are mainly determined by the costs of the recovery process

	In-house Costs	Outsourced Costs
Specific Costs	Problematic credit management Back Office processes Out-of-court recovery procedures	External lawyers Debt collection agencies
Common Costs	Legal department costs	-

Table VIII.23 Costs of the recovery process

Table VIII.24 shows the results of the credit recovery survey carried out by the Bank of Italy in 2001 within the Italian banking system. A more recent analysis shows that in 2014 the incidence of recovery costs for the system was on average 2.8%, therefore an increase compared to the previous survey characterized by an average value of 2.3%.

Geographic Area	Incidence of the recovery activity [% operating costs]
Noth-West	1.80%
North-East	1.80%
Center	2.70%
South	5.30%
Islands	3.10%
Italy	2.30%

Table VIII.24 Average costs of bankruptcy procedures in Italy

The credit recovery processes can last for a long time, therefore, banks should discount the financial flows, both incoming and outgoing. In theory and in practice, a wide debate has developed on the most suitable **type of discount rate**. In fact, the viable solutions are represented by:

- **the contractual rate applied to the debtor before classification in the state of default**. The approach provides that the flows recovered by the intermediary after the display of the state of insolvency are discounted at the contractual rate defined at the beginning of the relationship or at the last contractual rate renegotiated with the customer. The discounting to the contractual rate can be considered reasonable only if it is considered that the opportunity cost of the failure to recover the amounts at the expiry of the contract is correctly identified by this rate and it is therefore assumed that the intermediary has the possibility to choose between investment alternatives that guarantee the same return.

- **the historical or the implied risk-free rate**. The difficulties related to the identification of the possible return on the investment of the available resources released in the event of the debtor's fulfillment of the obligations assumed can lead to the choice of using the minimum opportunity cost for the temporal deferral of revenue, i.e., the risk free rate. The applicability of the approach is, therefore, subject only to the identification of the reference market and the best proxy (approximation) available for the return of the risk-free asset.

- **the risk-adjusted rate of the recovery process**. The hypothesis of using a risk-free rate can lead to underestimating the loss in the event of insolvency, since it is unlikely that the investments made by a financial intermediary guarantee a return not exceeding the risk-free rate, being characterized by a non-zero risk of loss. Since the investment risk is different compared to the time before the default, as the bank's capital is now invested in the activities that can generate revenue in the recovery process, this solution is more robust than the contractual rate.

The LGDestimation(T,i,EAD,RF,EF) function implements the previous LGD formula.

In the proposed example, let us suppose that, after the default, the time necessary for the bankruptcy procedure is 24 months. Every month the credit institution has to pay a fixed amount of money equal to EUR 1,000 (EF_t) for legal fees. The Exposure at default (EAD) is EUR 100,000 and there is a positive expectation for the bank that it will be able to recover a total amount of EUR 60,000 divided into tranches: 10,000 at the end of the 21st month and 50,000 at the end of the second year. Changing variable i in the script, the code can be run under different scenarios of the rates adopted for discounting.

- Setting i=RiskFree, the risk-free interest rate is used for discounting the future cash flows. The market interest rates term structure, which we have already used in the counterparty module, section "PD via bond spread", was implemented. In this case the LGD is equal to 63.58%.

- Setting i=AdjRate, a spread of 350 bps has been added over the risk free curve. In this second scenario the LGD is equal to 66.69%.

- Running the code with a hypothetical contractual rate of 4% used for the discounting process, we obtain an LGD equal to 67.48%.

The output of the code is a tuple of two elements: the first is the estimated LGD and the second is a Numpy array with the details of the future cash flows. The Tableau automatically generated from the LGDestimation function has in the first column the year fractions (t), in the second column the discount rates (i), the third column contains the recovery cash flows (RF_t), cost flows are displayed in the fourth column (EF_t) and the discounted cash flows are reported in the last two columns. So far we have discussed the issues related to the estimation of LGD. Now let us proceed and consider the most important factors for performing predictions. With this aim, it is useful to classify the variables used for LGD forecasting into four categories:

- **Instrument-related variables**. Debt instruments can differ in several aspects that are relevant for LGD estimation. In the event of default, not all borrowers are equal. They are divided in function of the priority of repayment in a liquidation or restructuring (i.e. the seniority). Furthermore, some debt instruments are secured, giving the debt holders the right to claim specific assets connected with the instrument. For instance in a mortgage loan, in case of default, the bank has the right to seize the building and use the sale proceeds to partially recover the debt. There may also be differences between the LGD of bank loans and public bonds because the quality of the security is typically higher for bank loans than for bonds.

Debt Type	Average LGD
Bank loans	0.262
Senior secured bonds	0.428
Senior unsecured bonds	0.57
Senior subordinated bonds	0.717
Subordinated bonds	0.806

Table VIII.25 Average LGD, 1987-2009 (Source: Standard & Poor's, 2010)

In a regression model, we can capture such average differences in two alternative ways:

a) Create one variable that contains the historical average LGD of the respective debt type. For a senior secured bond, this variable would record the average LGD of a senior secured bond; for a subordinated bond, the same variable would record the average LGD of subordinated bonds, and so forth.

b) Introduce dummy variables for debt types. The senior secured bond dummy, for example, would take the value 1 for senior secured bonds, and 0 otherwise.

In the second approach, we estimate average LGD differences across debt types with our data. Any differences not explained by other variables will be reflected in the regression coefficients of the dummy variables. With the first approach, we can bring in information from other sources. This can help to increase the precision of the estimates, particularly if the size of the data set is small. When using the dummy variable approach to model differences among K debt types, we would only include dummy variables for $K-1$ debt types. The coefficients of the dummy variables then reflect how LGDs differ from the LGD of the type that is not represented in the regression. Unluckily, modeling average differences across debt types is typically not sufficient to capture the effects of seniority and security. Let us consider a firm whose outstanding debt consists only of subordinated bonds. In this case, the subordinated are the most senior bonds. We can capture such effects by ordering the claims according to their seniority and security. We can summarize the priority standing of an instrument in different ways. One of the most well-established techniques is to define it through the ratio:

$$\frac{Face\ value\ of\ debt\ with\ higher\ priority - Face\ value\ of\ debt\ with\ lower\ priority}{Total\ face\ value\ of\ debt} \quad \text{(Eq. VIII.71)}$$

The higher this measure, the higher is the expected LGD.

- **Firm-specific variables**. The overall losses incurred by creditors will equal the value of liabilities minus the value of the firm's assets after bankruptcy costs. We can therefore hope to increase predictive accuracy if we find variables that contain information about the post-default asset value or bankruptcy costs. Among the most considered variables in the literature are the following:

<u>Tangibility</u>: For several reasons, tangible assets (usually defined as property, plant and equipment) could on average lead to lower LGD. For instance, they can be used to generate revenue during restructuring, or they may tend to lose less value than intangible assets.

$$\frac{Property, Plant, Equipment}{Total\ assets} \quad \text{(Eq. VIII.72)}$$

<u>Market-to-book ratio:</u> For firms with traded equity, the market-to-book ratio, which is often referred to as Tobin's Q, is usually computed as:

$$\frac{Market\ value\ of\ equity + book\ value\ of\ liabilities}{Total\ assets} \quad \text{(Eq. VIII.73)}$$

where the total of assets is the book value of assets. The higher the ratio, the higher is the market valuation of the firm's assets. If the valuation continues to be relatively high after default, the LGD will be relatively low. On the other hand, firms with a high market-to-book ratio are typically growth firms, which are highly valued because they promise high profits in the future. As such promises can dissipate quickly in the case of default, a high market-to-book ratio could also indicate high LGDs.

Leverage: It is typically measured as:

$$\frac{Total\ debt}{Total\ assets}\ (Eq.\ VIII.74)$$

This ratio is usually expected to be positively correlated with the LGD. One possible reason for such relationship is that debt structure is typically more complex for highly leveraged firms. This can complicate bankruptcy procedures, leading to larger bankruptcy costs and a higher LGD.

- **Macroeconomic variables**. Macroeconomic conditions can be useful for predicting LGD because they can help to estimate the value of the assets in case of default. In many cases, we would use figures for the country in which the borrower is domiciled or where he generates the largest part of its income. In other cases, regional or global aggregates can be more suitable. Variables that could be considered include the following:

Capacity utilization: If capacity utilization is low, demand for the firm's assets will tend to be low because competitors do not need additional production capacities. This depresses prices and leads to a high LGD.

GDP growth: The explanation is similar to the capacity utilization. A negative economic environment is expected to go along with high LGDs.

Corporate bond spreads: Generally, the value of an asset is obtained as the present value of cash flows that can be generated with the asset. Ceteris paribus, the asset value will therefore fall with increasing discount rates. To capture variation in discount rates we can use a yield spread, for instance defined through: Corporate bonds yield or Treasury bonds yield. The higher the spread, the higher is the expected LGD.

Default rates: High default rates indicate a negative economic environment. The mechanism leading to high LGDs would therefore be similar to the ones described above. Default rates may measure the relevant valuation factors in a more specific way. High default rates imply a high supply of defaulted debt, which tends to depress prices of such debt.

- **Industry-specific variables**. It can also be a good idea to replace the suggested variables above with variables defined on the level of the industry to which the observed firm belongs. For instance:

Capacity utilization: At a given point in time, the economic environment can differ substantially between industries. Using industry capacity utilization instead of (or in addition to) economy-wide capacity utilization can help to better capture the economic environment that is relevant for the valuation of a bankrupt firm's assets.

Market-to-book ratio: If a firm does not have traded equity, we cannot compute the firm-specific market-to-book ratio. In such cases, we can use the average market-to-book ratio of traded firms in an industry.

Let us now examine an example of LGD prediction based on a database provided in the "Credit Risk Modeling" book by Löffler & Posch. In this yearly dataset, we have three explanatory variables:

a) the historical average LGD of the respective debt type, computed with data ending the year before default.

b) the industry default rate in the year before the default.

c) the leverage.

The LGD is the response variable of the model. Let us start the analysis by applying an ordinary least squares (OLS) regression to the data. Thus, the equation for the model is:

$$LGD_{ijt} = b_1 + b_2 LEV_{i,t-1} + b_3 LGD_{A_{j,t-1}} + b_4 I_{DEF_{i,t-1}} + u_{ijt} \text{ (Eq. VIII.75)}$$

where LGD_{ijt} is the LGD on instrument j of firm i, observed in year t.

$LGD_{A_{j,t-1}}$ is the average LGD of instruments with the same type as j, computed with data ending in $t-1$. $I_{DEF_{i,t-1}}$ is the default rate that was observed in year $t-1$ for the industry to which borrower i belongs.

$LEV_{i,t-1}$ is firm i's leverage in $t-1$.

Unlike a traditional Ordinary Least Square (OLS), it should be noted in the database that one firm can contribute several observations to a data set because the study is at instrument-level. In our data, the firm with ID 1, for instance, enters with LGD values for both senior unsecured and subordinated bonds. This is likely to lead to correlations in the error terms u. If the assets of a defaulted firm have lost significantly in value or if its bankruptcy costs are high, the instruments issued by the firm will have a relatively high LGD.

If the reasons behind such a high loss rate are not adequately captured through our explanatory variables, the error terms will be correlated. In the presence of such correlations, OLS coefficient estimates are still reliable, but the standard errors are no longer so. Depending on the nature of the correlation, OLS standard errors can be too low or too high, leading to inflated or deflated t-statistics. For this reason, we should not use the standard built-in Python function for the OLS regression, but we should instead code a version able to take into account these biases caused by clusters. The structure of the OLS cluster-robust estimator is represented by the following regression relationship:

$$y_{ij} = \beta_1 + \beta_2 x_{ij2} + \beta_3 x_{ij3} + \cdots + \beta_K x_{ijK} + \epsilon_{ij} \text{ (Eq. VIII.76)}$$

where observations with the same i belong to the same cluster. Let the overall number of observations be denoted by N. It is more convenient to formulate the regression in a vectorized form:

$$\boldsymbol{y} = \boldsymbol{X}\beta + \epsilon \text{ (Eq. VIII.77)}$$

where $\boldsymbol{y}$ collects the values of dependent variable in a $N \times 1$ column vector; $\boldsymbol{X}$ collects the values of the explanatory variables in a $N \times K$ matrix. The coefficient vector β is $K \times 1$, while ϵ is $N \times 1$.

Let us denote the vector containing the OLS coefficient estimates by $\boldsymbol{b} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$.

The estimates are unbiased: $E[\boldsymbol{b}] = \beta$. For the variance of the coefficient estimates, it then follows that:

$$\text{Var}(b) = E[(\boldsymbol{b}-\beta)(\boldsymbol{b}-\beta)'] = E[((\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}-\beta)((\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}-\beta)'] =$$

$$= E[(\beta + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\epsilon - \beta)(\beta + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\epsilon - \beta)'] = E\left[\left((\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\epsilon\epsilon'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\right)\right] =$$

$$= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'E[\epsilon\epsilon']\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} \quad \textit{(Eq. VIII.78)}$$

In standard OLS, it can be assumed that the covariance of error terms, $E[\epsilon\epsilon']$, takes a simple form: all off-diagonal elements are zero (i.e. errors are independent) and elements on the diagonal take an identical value, σ^2. In this situation, unluckily, this assumption is likely to be inappropriate. However, we can hope to estimate $\boldsymbol{X}'E[\epsilon\epsilon']\boldsymbol{X}$. This is the intuition behind the cluster-robust estimator. Each cluster gives an estimate of $\boldsymbol{X}'E[\epsilon\epsilon']\boldsymbol{X}$ and by averaging across the clusters, we are likely to reach a reliable estimator. To implement this new approach, we estimate the error terms through the regression residuals u_{ij}, obtained through $y_{ij} - (\beta_1 + \beta_2 x_{ij2} + \beta_3 x_{ij3} + \cdots + \beta_K x_{ijK})$. By collecting the x-values pertaining to a cluster i in the matrix X_i and collecting the residuals of cluster i in a vector u_i, we obtain:

$$X_i = \begin{bmatrix} 1 & x_{i11} & \dots & x_{i1K} \\ 1 & x_{i21} & \dots & x_{i2K} \\ \dots & \dots & \dots & \dots \\ 1 & x_{iJ1} & \dots & x_{iJK} \end{bmatrix}, \ u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{ij} \end{bmatrix} \quad \textit{(Eq. VIII.79)}$$

where J is the number of observations in cluster i, meaning that J can differ across the clusters.

We then determine:

$$\Theta = \sum_{i=1}^{M} \boldsymbol{X}_i' u_i u_i' \boldsymbol{X}_i \quad \textit{(Eq. VIII.80)}$$

Where M is the number of clusters.

The estimate of the variance-covariance matrix of coefficients is then obtained through:

$$\frac{N-1}{N-K}\frac{M}{M-1}(X'X)^{-1}\Theta(X'X)^{-1} \quad \textit{(Eq. VIII.81)}$$

Where $\frac{N-1}{N-K}\frac{M}{M-1}$ is the small-sample adjustment that is commonly made.

The LGDRegressionAnalysis function implements this estimator of the coefficients' standard errors that is robust to correlation within cluster on the LGDDataset provided by the Löffler & Posch book.

The output of the LGDRegressionAnalysis function is an object of the LGDRegressionOutput class. It contains all the statistics associated to the model:

- The OLS Coefficients (attribute: coeff).
- The cluster-robust standard errors (attribute: SEcoeff).
- The t-statistics (attribute: tstat).
- The R^2 of the model (attribute: R2).
- The in-sample $RMSE$ (root mean squared error) of the model (attribute: RMSE).
- The out-of-sample RMSE of the model if the input for the out-of-sample batch has been explicitly setted (attribute: outofsampleRMSE), otherwise it is NaN.

Coefficients are correctly estimated on the entire sample, and they are $b_1 = 0.195879$, $b_2 = 0.182673$, $b_3 = 0.505693$ and $b_4 = 0.028936$. The in-sample RMSE is equal to 0.277 but unluckily if we test the model on an out-of-sample, this measure increases to 0.4. One of the possible problems to this lack of performance in the model is that the relationship between variables in the regression cannot be explained using a linear model. In this case it is rather challenging to guess about a possible function that is able to link the independent variables (i.e. LEV, LGD_A, I_{DEF}) with the response (LGD). Given that it is hard to define an a-priori regression model, Machine Learning can help to capture the non-linearities between the variables.

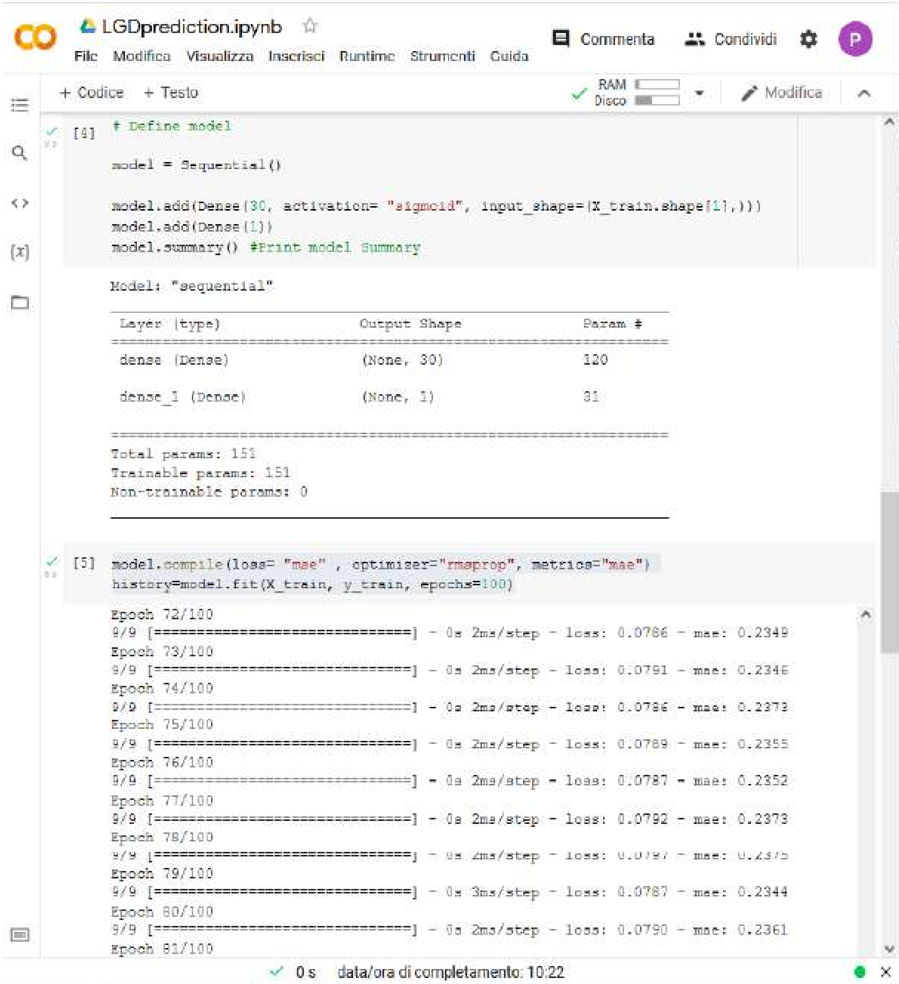

Figure VIII.21 Google Colab platform

A feed-forward shallow neural network with 30 neurons is able to noticeably reduce the out-of-sample error.

We train a feed-forward neural network with a hidden layer of 30 neurons on a data set constituted by 90% of the available data using sigmoids as activation functions. 10% of the sample will be used for testing the out-of-sample performance of the Machine Learning model.

Without using a deep-learning network (i.e. an architecture with more than one hidden layers or with a single layer but with specific features) we are able to reach performances on the test sample which are aligned with the in-sample training data set. The in-sample RMSE (Root Mean Squared Error) is 0.24 and the error remains constant even in the test on the out-of-sample data set.

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VIII.5 EXPOSURE AT DEFAULT

The **Exposure at Default** (EAD) represents the bank's exposure at the time of default. There can be certain or uncertain amount exposures. In the context of exposures with an uncertain amount, the EAD forecast differs according to the different technical forms. From a regulatory point of view, banks generally estimate this quantity applying this formula:

$$EAD = U_0 + CCF \cdot M_0 \quad \textit{(Eq. VIII.82)}$$

Where U_0 is the current usage, M_0 is the currently available margin, i.e. the difference between the agreed and the current usage; CCF (Credit Conversion Factor) is the rate according to which M_0 can be transformed into cash exposure.

Considering the above formula, it is clear that CCF plays a crucial role in forecasting the EAD. Araten and Jacobs (2001) show that the CCF is a decreasing function of the time between the reference date and the time to default in years (see Table VIII.26).

In other words, reading Table VIII.26 from right to left, as the default approaches, it is more likely to observe a reduction in exposure rather than an increase; this translates into a value of the CCF equal to 72% five years before default, which is reduced by up to 32% one year before default. This evidence is to be interpreted in the light of the increase in the use of available margins upon approaching the default. The two authors also identify an inverse relationship between the value of the CCF and the creditworthiness of the transaction (facility risk grade): a worse rating is generally associated with a lower CCF value and vice versa. This result could be interpreted with a view to reducing credit lines or imposing stricter covenants. Finally, the authors conclude that certain indicators such as the type and extent of the transaction or the sector to which the debtor belongs do not have any relationship with the CCF.

According to the EAD equation and using the historical data in Table VIII.26, the EAD for an exposure with a credit line of 20,000, an available margin of 5,000, a rating equal to B- one year before default would be:

$$EAD = U_0 + CCF \cdot M_0 = 15{,}000 + 26.5\% \cdot 5{,}000 = 16{,}325$$

The EADprediction(Credit_Line,CCF,Available_Margin) automatically performs this calculation taking into account the proper interpolation of the values in the historical Table VIII.26. Once the characteristics of the CCF have been defined for forecasting purposes, it is necessary to question how it can be estimated. As required by Basel 2, the database must be segmented by size, rating and technical form. In the case of a current account credit overdraft, the CCF can be estimated as follows:

$$CCF = \frac{U_t - U_0}{M_0} \quad \textit{(Eq. VIII.83)}$$

Where U_t is the use at the time of default, U_0 is the current usage and M_0 is the available current margin.

Risk Class	1Y	2Y	3Y	4Y	5/6 Y
1. AAA/AA-		12.10%			
2. A+/A-	78.70%	75.50%	84.00%		
3. BBB+/BBB	93.90%	47.20%	41.70%	100%	
4.BBB/BBB-	54.80%	52.10%	41.50%	37.50%	100%
5. BB	32.00%	44.90%	62.10%	76.00%	68.30%
6. BB-/B+	39.60%	49.80%	62.10%	62.60%	100%
7. B/B-	26.50%	39.70%	37.30%	97.80%	
8. CCC	24.50%	26.70%	9.40%		

Table VIII.26 Average of CCFs for revolving credit transactions

From a mathematical perspective, it is not an easy task to estimate the expected exposure of a generic asset at a future time. The example proposed above is based on cash amounts of a revolving credit, as a result, the NPV is the available quantity at the moment of default, which can be estimated using the described methodology. A more complicated case can be represented by a loan with fixed or floating installments. In this case the asset can be modelized like a bond and, consequently, an analyst should compute its NPV at the time of default. For the case of a straight bullet bond, the future cash flows are pre-determined, but the discount rates seen in today's markets are not the same as the zero rates at time of default t. A Monte Carlo method that numerically integrates an SDE (Stochastic Differential Equation) has thus to be implemented. A motion is usually chosen which embeds a mean-reverting effect to this aim like Vasicek or Hull-White dynamics depending on available data. Obviously, the same reasoning can be applied for floaters: in this case we use the information of the projected rates not only for discounting purposes, but also for the determination of the expected future cash flows through the forward rates. This kind of approach can be used in the case of optionality such as cap/floor instruments on the loan using a numeric technique (Monte Carlo with the direct application of the pay-off on the paths) or the closed formulas, being careful to project all the market input data at the time of default.

The problem of the Expected Exposure estimation is also present in the **Credit Valuation Adjustment** (CVA). According to financial literature, the CVA is the difference between the risk free and its price including the risk of default. In other words, it is the cost that must be incurred to cover the risk of counterparty bankruptcy: it is necessary for derivatives whose fair value is positive. It is considered a financial risk measure and it is mainly applied to derivatives which are not collateralized. From a conceptual point of view, the CVA can be considered as the cost of hedging the counterparty risk. Observing the mathematical formula that defines this credit valuation adjustment, the direct relationship with the modern definition of credit risk is clear.

$$CVA = (1 - R) \int_0^T EE(t) dPD(0, t) \quad \textit{(Eq. VIII.84)}$$

Where R is the recovery rate, EE is the expected exposure and PD is the probability of default. The recovery rate is out of the integral because it is constant in financial markets and usually set equal to 40% in accordance with the standardized CDS premium. This choice is reasonable especially if the PD is estimated from Credit

Default Swaps. Surely, the most difficult part is the estimation of EE. Considering that a financial portfolio can be made of complex derivatives, a general approach to this problem is required, able to generalize the methodology discussed previously in the pure credit risk field. In this context, let us examine the main procedural steps for the CVA engine implemented by the Financial Engineering Office in Banca CARIGE (now merged into the BPER Group) focusing on the Expected Exposure term. The fundamental steps for the implementation of the CVA measurement can be summarized as follows:

- Definition of the IT architecture able to host the calculation.
- Bootstrap of market data aimed at a correct determination of the fair value of derivatives.
- Implementation of the most appropriate stochastic differential equation (SDE) for the representation of the dynamics followed by the input market data.
- SDE parameter calibration using listed market quotes.
- Probability of default estimation based on a market approach.
- CVA computation in accordance with the previous equation.

Following an accurate analysis, based both on the size of the non-collateralized derivatives, and on the required computational time, it was decided to develop the pricing engine in-house using Matlab. These libraries have been compiled in order to be first tested and validated in an Excel-VBA environment (bas modules) and then released in production as dynamic libraries for DOT NET (dll file). Using an integrated interface, developed in C#, the computation engine can be iteratively called from the official pricing system of the Bank, also providing the CVA or DVA measurement (if $NPV \leq 0$) in addition to the fair-value of the financial instrument. The role of the IT system is crucial at this level, in fact, depending on the number of financial instruments to be processed, the computational time for an engine based on the Monte Carlo technique may require the use of parallel computing and a server dedicated exclusively to this task. The next step was to develop the set of mathematical models needed to the valuation of all kinds of derivatives subjected to this adjustment. These codes were tested and integrated in production in a short time. Once the pricing functions have been validated, both theoretically by comparison with the technical-scientific literature, and experimentally with the Bloomberg® modules, the goal is to estimate the reasonable evolution of the current price over time. In accordance with financial best practice, the same pricing engine has to be used at different future dates and the input market data has to be simulated using a suitable Monte Carlo engine.
Table VIII.27 shows the stochastic dynamics implemented for the simulation of the term structures.

Financial Instruments	Model	Dynamics
Interest Rate Swap	Hull & White	$dr = [\theta(t) - \alpha r]dt + \sigma dW_t$
Interest Rate Cap/Floor/Collar	Hull & White	$dr = [\theta(t) - \alpha r]dt + \sigma dW_t$
Currency Option	Ho & Lee	$dr = \theta(t)dt + \sigma dW_t$
Inflation Indexed Swap	Hull & White	$dr = [\theta(t) - \alpha r]dt + \sigma dW_t$
Inflation Cap/Floor/Collar	Hull & White	$dr = [\theta(t) - \alpha r]dt + \sigma dW_t$
Exchange Rate Derivatives	Rendleman	$dr = \mu r dt + \sigma r dW_t$

Table VIII.27 Stochastic Differential Equations implemented for the simulation of term structures

r is the rate of the term structure to be projected.

$\mu = r - r_f$ is the spread between the domestic and the foreign rate.

α is the speed of mean reversion.

σ is the volatility of the underlyings.

$\theta(t)$ is the starting slope of the term structure.

dW_t is the stochastic Wiener process (mean zero and unitary variance).

Depending on the model for the projection of $r(t)$, different calibration techniques have been used and they are summarized in Table VIII.28. For the Ho & Lee model and the Rendleman & Bartter model, the only parameter to be calibrated is volatility, which can be observed directly from the market, because it is directly contributed or implicitly derived from the premiums of the options actively traded on the trading platforms.

Financial Instruments	Model	MKT Data for calibration
Interest Rate Swap	Hull & White	ATM Interest Rate Cap Vol.
Interest Rate Cap/Floor/Collar	Hull & White	ATM Interest Rate Option Vol.
Currency Option	Ho & Lee	ATM Currency Vol.
Inflation Indexed Swap	Hull & White	ATM Inflation Floor Premium
Inflation Cap/Floor/Collar	Hull & White	ATM Inflation Option Premium
Exchange Rate Derivatives	Rendleman	ATM Forex Vol.

Table VIII.28 Calibration techniques for the simulation of the term structures

In the Hull & White model there is an additional parameter which is α: in fact, the dynamics of the underlying is represented by the Stochastic Differential Equation (SDE):

$$dr = [\theta(t) - \alpha r]dt + \sigma dW_t \quad \textit{(Eq. VIII.85)}$$

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial T} + \alpha f^M(0,t) + \frac{\sigma^2}{2\alpha}[1 - \exp(-2\alpha t)] \quad \textit{(Eq. VIII.86)}$$

Where $f^M(0,t)$ is the instantaneous forward rate at time 0 and maturity t.

This SDE can be discretized using a trinomial stochastic Hull-White tree, which is characterized by the same parameters as the Monte Carlo method. The main advantage of using this last model for evaluating α and σ to be used in the SDE integration is that it is deterministic. This is an extremely nice feature for setting a minimization problem over all the known market quotes able to solve our estimation problem:

$$\min_{\boldsymbol{x}} f[Model\,Value\,(\boldsymbol{x}) - Market\,Value] \quad \textit{(Eq. VIII.87)}$$

With $\boldsymbol{x}_L < \boldsymbol{x} < \boldsymbol{x}_U$ and $\boldsymbol{x} = [\alpha, \sigma]$.

$\boldsymbol{x}$ is the array of the parameters to be estimated, $\boldsymbol{x}_L$ and $\boldsymbol{x}_U$ are respectively the lower-bound and upper-bound that define the search domain of the feasible solutions and f normally assumes a quadratic form (in analogy to the traditional minimization of the sum of the squared errors, SSE). Therefore, in order to compute the optimal values of mean-reversion and volatility, i.e. those values which minimize the discrepancy between the model and the market value, it is necessary to implement an optimization algorithm, such as a quasi-newton method, like L-BFGS or a direct search methodology such as the Nelder-Mead simplex. This routine will iteratively recall the trinomial stochastic tree, compare the theoretical value obtained from it, $Model\,Value\,(\boldsymbol{x})$, with the target, $Market\,Value$, until the desired accuracy has been reached.

It is worth to say that the plain-vanilla Caps and Floors are characterized by an analytical tractability in the Hull & White pricing framework. The existence of a pricing formula is definitely an interesting property, as it allows a greater convergence speed of the optimization problem. Besides, the routine does not require the time-consuming recalculation of the entire tree structure, but only the evaluation of the following cap/floor formulas:

$$\text{Cap}(t, \tau, N, X) = N \sum_{i=1}^{n} P(t, t_{i-1}) \Phi\left(-h_i + \sigma_P^i\right) - (1 + X\tau_i) P(t, t_i) \Phi(-h_i) \quad \textit{(Eq. VIII.88)}$$

$$\text{Floor}(t, \tau, N, X) = N \sum_{i=1}^{n} (1 + X\tau_i) P(t, t_i) \Phi(h_i) - P(t, t_{i-1}) \Phi\left(h_i - \sigma_P^i\right) \quad \textit{(Eq. VIII.89)}$$

With

$$\sigma_P^i = \sigma \sqrt{\frac{1 - \exp[-2\alpha(t_{i-1} - t)]}{2\alpha}} \cdot \frac{1 - \exp[-\alpha(t_i - t_{i-1})]}{\alpha}; \; h_i = \frac{1}{\sigma_P^i} \ln \frac{P(t, t_i)(1 + X\tau_i)}{P(t, t_{i-1})} + \frac{\sigma_P^i}{2}$$

X is the strike price of the option, N is the notional of the option, τ is the year fraction between two subsequent caplets (/floorlets), $P(t, t_i)$ is the discount factor, $\Phi(\cdot)$ is the cumulative standard normal distribution. α and σ are the mean reversion and volatility of the Hull & White tree. Once the parameters of the simulators have been tuned in accordance with the observed market data, $Nruns$ simulations are performed of all the inputs necessary for pricing the derivative in the future dates. For each simulation date, $Tsim$, three curves computed in correspondence of the percentiles 40% (r_{Tsim}^{DOWN}), 50% (r_{Tsim}^{MID}) and 60% (r_{Tsim}^{UP}) are stored in a database. The percentile values are parameterized and therefore customizable by the user, who is able to estimate the statistical reliability of the simulation conducted, by comparing r_{Tsim}^{MID} with the initial term structure, $r(t = 0)$, in fact, for $Nruns \to \infty$ it has to converge to these expected values. We now have all terms for solving the initial equation and estimating the Credit Valuation Adjustment:

$$CVA = (1 - R) \int_0^T EE(t) dPD(0, t) \quad \textit{(Eq. VIII.90)}$$

The described procedures have been published on professional magazines (ASSIOM FOREX Letters and

MathWorks User-Story Journal) and presented in several academic and professional conferences in 2014-2015. Readers interested in this topic can find further details in the reference section.

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VIII.6 RATING SYSTEMS

A **credit rating system** uses a limited number of rating grades to rank borrowers according to their probability of default. Rating assignments can be based on a qualitative process or on default probabilities estimated with a scoring model, a structural model or other means. To translate default probability estimates into ratings, an analyst defines a set of rating grades and their rules: grade AAA is assigned to borrowers with a probability of default lower than 0.02%, grade AA is assigned to borrowers with a probability of default between 0.02% and 0.05% and so on. The most relevant example of these structured analysis applications is given by **rating agencies**. Their aim is to run a systemic survey on all determinants of default risk. There are several national and international rating agencies operating in all developed countries. The rating agencies' approach is very interesting because model-based and judgmental-based analyses are integrated, and it can be considered a state-of-the-art approach in this field. They thus have the possibility to overcome the information asymmetry problem through a direct expert valuation, supported by information which is not accessible to other external valuators. Rating agencies' revenues derive for the most part from counterparties' fees; only a small amount is derived from the direct selling of economic information to investors and market participants. This business model is apparently very peculiar because of the obvious conflict of interests between the two parties. If the cost of the rating assignment is charged to companies that have the most benefit from it, how can it possibly be fair?

Nevertheless, this business model is considered as solid, as the Nobel Laureate George Ackerloff and the "lemon principle" can help to understand. If there is a collective conviction among market participants and exchanged goods are generally of bad quality, the seller of better quality goods will encounter many difficulties in selling them, because he will have trouble in convincing people of the quality of his offer. In such circumstances, the seller of better quality goods: either tries to adapt, and switch to low quality goods in order to be aligned with market judgement or he has to find a third party, a highly reputable expert, that could try to convince market participants that the offer is of really good quality and it is worth paying a higher price. In the first case, the market will experience a suboptimal situation, because part of the potential offer (good quality products) will not be traded. In the second case, the market will benefit from the reliable external judgement, because of the opportunity to segment the demand to gain a wider number of negotiated goods. Generally speaking, when there is information asymmetry among market participants (i.e. inability for market participants to have a complete and transparent evaluation of the quality of the offered goods) only high reputation external appraisers can assure the quality of goods, overcoming the so-called "lemon" problem.

Traders, investors, and buyers can lean on the expert judgment. Therefore, issuers are interested in demonstrating the credit quality of their issues, and rating agencies are interested in maintaining their reputation. The disruption in the evaluator's reputation is something that could induce a much wider market disruption, and this observation is very important in the light of the recent financial crisis, where rating agencies' structured products judgements have been strongly criticized. Consequently, the possibility to obtain privileged information on the counterparty's management visions, strategies and budgeting is essential to a reliable rating agencies' business model; as a result, the structure of the rating process becomes a key part of the rating assignment process because it determines the possibility to reach independent, objective and sufficient insider information.

The decision making process for rating assignment at Standard & Poor's can be summarized in eight steps:

a. Ratings request from issuer.
b. Initial evaluation.
c. Meeting with issuer management.
d. Analysis.
e. Rating committee review and vote.
f. Notification to issuer.
g. Publication and Dissemination of rating opinion.
h. Surveillance of rated issuers and issues - and back to step e).

Rating agencies' assignment methodologies are differentiated according to the counterparty's nature (corporations, countries, public entities, …) and/or according to the nature of products (structured finance, bonds, …). Focusing on corporate borrowers, the final rating derives from two analytical areas:

- Business risk which depends on Country risk, Industry characteristics, Company position, Profitability and peer group comparison.

- Financial risk which is impacted by Accounting, Governance, Cash flow adequacy, Capital structure and Liquidity.

The main financial ratios used by S&P's rating agency are:

- profitability ratios from historical and projected operations, gross and net of taxes.

- coverage ratios such as cash flow from operations divided into interest and principal to be paid.

- quick and current liquidity ratios.

Typically, the larger the cash flow margins from operations, the safer the financial structure; and, therefore, the better the borrower's credit rating. This general rule is integrated with considerations regarding the country of incorporation (the so-called "sovereign risk"), the industry profile, the competitive environment and the business sector. Other traditional analytical areas include: management reputation, reliability, experience, and past performance; coherence and consistency in the firm's strategy; organization adequacy to competitive needs; diversification in profit and cash flow sources; firm's resilience to business volatility and uncertainty.
Recently, new analytical areas were introduced to take new sources of risk into account. The new analytical areas can be summarized as follows:

- internal governance quality (competence and integrity of board members and management, distribution and concentration of internal decision powers and layers, . . .).

- environmental risks, technology and production processes, compliance and sustainability.

- potential exposure to legal or institutional risks, and to main political events.

Over time, aspects like internal governance, environmental compliance and liquidity have become crucial. Despite the effort in creating an objective basis for rating assignment, the rating agency, in the end, gives an opinion that is expressed using a categorical variable (i.e. the credit risk rating).

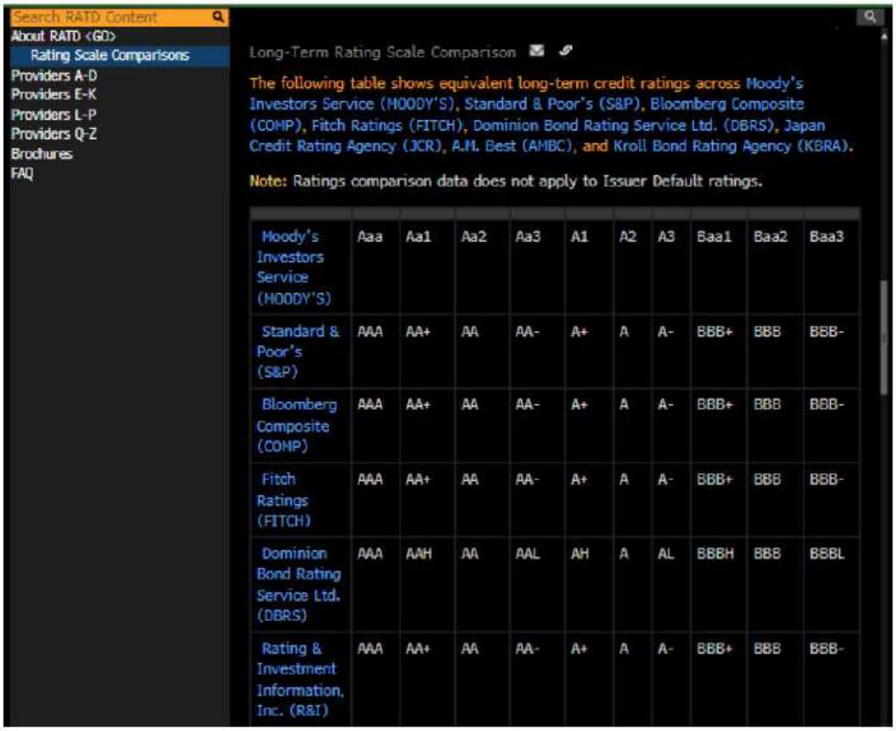
It is worth to highlight that the creation of a rating process is very complex and is typically structured as follows: a preliminary analysis, meetings with the counterparty under scrutiny, the preparation of a rating dossier submitted by the Analytical Team to the Rating Committee (usually composed of 5-7 voting members), a new detailed analysis if needed, the final approval by Rating Committee, the official communication to and subsequent meeting with the counterparty, and, if necessary, another approval process and rating submission to the Rating Committee.
Moreover, the rating is not directly determined by ratios; for instance, the more favourable the business risk, the higher the financial leverage compatible with a given rating class.

	AAA	AA	A	BBB	BB
Excellent	30	40	50	60	70
Above Average	20	25	40	50	60
Average		15	30	40	55
Below Average			25	35	45
Vulnerable				25	33

Table VIII.29 Financial leverage (debt/capital, in percentage), business risk levels and ratings

In general, favorable positions in certain areas could be counterbalanced by less favorable positions in others, with some transformation criteria: financial ratios are not intended to be hurdle rates or prerequisites that should be achieved to attain specific debt rating. Average ratios per rating class are ex-post observations and not ex ante guidelines for rating assignment.
Over time, the rating industry has changed, mostly due to consolidation processes that have left only three big international players. It is worth noting that the three competitors have different rating definitions. Moody's releases mainly issues ratings and far less issuer's ratings. On the contrary, S&P focuses on providing a credit quality valuation referred to the issuer, despite the fact that the counterparty could be selectively insolvent on public listed bonds or on private liabilities. Lastly, Fitch adopts an intermediate solution, offering an issuer rating, limited to the potential insolvency on publicly listed bonds, without considering the counterparty's private and commercial bank borrowing.
Therefore, ratings released by the three international rating agencies are not directly comparable. This was clearly seen when British Railways defaulted in the United Kingdom and was privatized, while the outstanding debt was immediately covered by state guarantee. British Railways issues were set in "selective default" by S&P while (coherently) having remained "investment grade" for Moody's and "speculative grade" for Fitch. In recent years, nonetheless, market pressure urged agencies to produce more comparable ratings, increasingly built on quantitative analyses, beyond qualitative ones, adopting a wider range of criteria. Particularly, after the "Corporate America scandals" (ENRON is probably the most renowned), new criteria were introduced, such

as the so called "Core earnings methodology on treatment of stock options, multi annual revenues, derivatives and off-balance sheet exposure and so on". Liquidity profiles were also adopted to assess the short term liquidity position of firms, as well as the possibility to dismantle certain investments or activities in case of severe recession, and so forth. New corporate governance rules were also established with reference to conflict of interests, transparency, the quality of board members, investor's relations, minorities' rights protection and so on. Monitoring was enhanced and market signals (such as market prices on listed bonds and stocks) were taken into further consideration.



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Long-Term Rating Scale Comparison

The following table shows equivalent long-term credit ratings across Moody's Investors Service (MOODY'S), Standard & Poor's (S&P), Bloomberg Composite (COMP), Fitch Ratings (FITCH), Dominion Bond Rating Service Ltd. (DBRS), Japan Credit Rating Agency (JCR), A.M. Best (AMBC), and Kroll Bond Rating Agency (KBRA).

Note: Ratings comparison data does not apply to Issuer Default ratings.

Moody's Investors Service (MOODY'S)	Aaa	Aa1	Aa2	Aa3	A1	A2	A3	Baa1	Baa2	Baa3
Standard & Poor's (S&P)	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-
Bloomberg Composite (COMP)	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-
Fitch Ratings (FITCH)	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-
Dominion Bond Rating Service Ltd. (DBRS)	AAA	AAH	AA	AAL	AH	A	AL	BBBH	BBB	BBBL
Rating & Investment Information, Inc. (R&I)	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-

Figure VIII.22 Long-Term Rating Scale Comparison. Source: Bloomberg® RATD function

Thanks to the increased comparability, an Info-provider like Bloomberg® provides dedicated functions like RATD (Ratings definitions and unique individual scales for available agencies on the Terminal) to enable traders and analysts to simplify their analyses (see Figure VIII.22).

Let us conclude this part dedicated to Rating Agencies quoting a definition found in a Standard & Poor's report:

"The rating experience is as much an art as it is a science".

In this section, a few methods for answering fundamental questions are provided, such as "With what probability will the credit risk rating of a borrower decrease by a given level?". In credit risk management, the probabilities of rating transition or rating migration are usually presented in transition matrices.
Let us consider a rating system with three classes A, B, C and a default category D. The transition matrix for this rating system is a table listing the probabilities that a borrower rated A at the start of a period, has rating A, B, C or D at the end of the same period; likewise for B-rated and C-rated companies.
Table VIII.30 illustrates this matrix representation for this simple rating system.

Rating	A	B	C	D
A	$\text{Prob}(A \to A)$	$\text{Prob}(A \to B)$	$\text{Prob}(A \to C)$	$\text{Prob}(A \to D)$
B	$\text{Prob}(B \to A)$	$\text{Prob}(B \to B)$	$\text{Prob}(B \to C)$	$\text{Prob}(B \to D)$
C	$\text{Prob}(C \to A)$	$\text{Prob}(C \to B)$	$\text{Prob}(C \to C)$	$\text{Prob}(C \to D)$

Table VIII.30 Structure of a transition matrix

Row headers give the rating at the beginning of the time period, column headers the rating at the end of the period. The period length is often set to one year, but other choices are possible as well. The default category does not have a row of its own because only performing positions are analyzed in statistical samples. It should also be specified that the default is treated as an absorbing category i.e. probabilities of migrating from D to A, B and C are set to zero. A borrower that moves from B to D and back to B within the analyzed period is counted as a defaulter in this model. The data are usually estimated from observed historical rating transitions. For agency ratings, there is practically no alternative to using historical transitions because agencies do not associate their grades with probabilities of default or transition.
In this context, let us examine two estimation procedures built on historical transitions: the **cohort approach** and the **hazard approach**. The former approach is the traditional technique, which estimates transition probabilities through historical transition frequency. Though widely established, the cohort approach does not make full use of the available data. The estimates are not affected by the timing and the sequencing of transitions within a year. One consequence hereof is that transition rates to low grades are often zero for high-quality issuers. Such events are so rare that they are seldom observed empirically. Still, there is indirect evidence that they can nevertheless happen. A fact that an analyst does observe is that high-grades issuers are downgraded within a year, say to BBB and that BBB issuers can default within a few months. An approach that circumvents such problems and makes efficient use of the data would be to estimate transition rates using the hazard rate approach. It is a similar technique adopted by both industrial engineers in plants maintenance and by medical scientists for their survival studies.
A cohort comprises all obligors holding a given rating at the start of a given period. In the **cohort approach**, the transition matrix is filled with empirical transition frequencies that are computed as follows.
Let $N_{i,t}$ denote the number of obligors in category i at the beginning of period t ($N_{i,t}$ is therefore the size of the cohort i, t). Let $N_{ij,t}$ denote the number of obligors from the cohort i, j that have obtained grade j at the end of period t.
The transition frequencies in period t are computed as:

$$\hat{p}_{ij,t} = \frac{N_{ij,t}}{N_{i,t}} \text{ (Eq. VIII.91)}$$

Usually, a transition matrix is estimated with data from several periods. A common way of averaging the period transition frequencies is the obligor-weighted average, which uses the number of obligors in a cohort as weights:

$$\hat{p}_{ij} = \frac{\sum_t N_{i,t}\hat{p}_{ij,t}}{\sum_t N_{i,t}} \text{ (Eq. VIII.92)}$$

Inserting the definition of $\hat{p}_{ij,t}$ in the previous equation, we obtain:

$$\hat{p}_{ij} = \frac{\sum_t N_{i,t}\hat{p}_{ij,t}}{\sum_t N_{i,t}} = \frac{\sum_t N_{i,t}\frac{N_{ij,t}}{N_{i,t}}}{\sum_t N_{i,t}} = \frac{\sum_t N_{ij,t}}{\sum_t N_{i,t}} = \frac{N_{ij}}{N_i} \text{ (Eq. VIII.93)}$$

Therefore, the obligor-weighted average can be directly obtained by dividing the overall sum of transitions from i to j by the overall number of obligors that were in grade i at the start of the considered periods. The rating transition dataset "RatingTransitionsDataset.csv" considered for the analysis has three fields: the corporate ID, the date in which the transition has been recorded and the grade. All three components have a numeric mode: the ID is a progressive counter, the date has been expressed using the Excel serial date number identification (for instance: 36676 corresponds to 30th May 2000) and the grade has been codified using categorical variables: 1 - AAA; 2 - AA; 3 - A; 4 - BBB; 5 - BB; 6 - B; 7 - CCC, CC and C; 8 - D. A rating withdrawal, i.e. NR (not rated), has been coded with 0.

The dataset used for the analyses comes from the "Credit risk modeling" book written by Löffler and Posch. It has been provided by the two authors only for illustrative purposes (the data inside are not real, but they emulate a market case). Furthermore, the data contained in the data set are arranged using the standard metric adopted by the main info providers.

In the Figure, a screenshot is shown from the Bloomberg® Credit profile - CRPR module and the transition ratings associated to a firm have been reported exactly in the same format as our example.

The cohort(RatingDataset,periods=1) function implements the single period cohort approach described above. The compulsory input is the Rating dataset to be expressed in the standard format.

Figure VIII.23 Example of Fitch rating transition. Source: Bloomberg® CRPR function

The output returned by the routine is a bi-dimensional numpy array in which the data has been conformed to the way in which Rating Agencies publish transition matrices. Transition from default and not rated are not shown while transitions to not-rated are shown in the column on the right. The output of the function applied to the "Rating-TransitionsDataset.csv" is shown in the Figure.
From left to right and from top to bottom, the grades are reported from the best to the lowest one (from 1 to 7). The latest two columns are reserved to default (8) and not-rated (0) probabilities, as described previously.

	1	2	3	4	5	6	7	8	0
1	90.625	1.04167	0	0	1.04167	0	0	0	7.29167
2	1.53203	85.376	8.6351	0.139276	0	0.139276	0	0	4.17827
3	0.139276	2.99443	86.5599	5.71031	0.348189	0.139276	0	0.0696379	4.039
4	0	0	3.67475	85.1446	6.09851	1.01642	0.0781861	0.312744	3.67475
5	0	0	0.657895	7.56579	71.3816	10.6908	1.64474	0.986842	7.07237
6	0	0.192308	0.384615	0.769231	7.30769	75.3846	8.07692	1.73077	6.15385
7	0	0	0	0	1.63934	7.10383	61.2022	10.3825	19.6721

Figure VIII.24 One-year transition matrix with the Cohort approach

The matrix mirrors the empirical findings common to matrices published by rating agencies. First, on diagonal entries are the highest values; they are in the range of 61% to over 90%. This means that the rating system is relatively stable. Second, default frequencies for the two best classes are zero. Since the possibility of an obligor defaulting cannot be ruled out, we would expect the true default probability of the best grades to be nonzero, albeit very small. But with a very small default probability, the default events are so rare that it is typical to observe no defaults.
For example, for a rating class with 100 obligors and a default probability of 0.01%, the expected number of defaults over 20 years is 0.2. If we want to estimate probabilities for transitions over a single period (i.e. one year in the example), we can do it specifying an integer value for the parameter periods. Assuming that transitions are independent across the years, a T-period transition matrix can be obtained by multiplying the one-period matrix with itself $(T-1)$ times.
Let P_T denote the transition matrix over T periods, then:

$$P_T = P_1^T = \underbrace{P_1 P_1 \dots P_1}_{T\ times} \text{ (Eq. VIII.94)}$$

The problem here is that the matrices should be symmetric, but the output has a size of 7×9. To fix this problem, the trick is to add two rows for the default and not-rated category. For the default category, which we assumed to be absorbing, the natural way of filling the row is to put zeros off-diagonal and ones on-diagonal. For the NR category, we could have estimated the transition rates. In the previous section we did not include migrations to NR in our calculations. We could thus perform an NR-adjustment and work with the NR-adjusted matrix. In this context we assume that the NR status is absorbing as well. Supposing our aim is to generate a three-year matrix, we run the function using cohort(RatingDataset,periods=3). Before returning the transition probabilities, the function removes the two added auxiliary rows necessary for the multiplication, given that they are not important and this helps to improve the readability of the table.

	1	2	3	4	5	6	7	8	0
1	74.472	2.42403	0.256588	0.204965	2.07405	0.270179	0.0474162	0.030983	20.2198
2	3.59611	62.9392	19.2024	1.57971	0.18826	0.316668	0.0277575	0.0274162	12.1224
3	0.449129	6.65845	66.0773	12.7591	1.53899	0.55594	0.0620256	0.253719	11.6453
4	0.0152032	0.290338	8.27067	63.4143	11.5343	3.53337	0.58264	1.06636	11.2929
5	0.00333886	0.105601	2.02343	14.3363	39.2243	17.8984	4.01805	3.21653	19.1741
6	0.00906523	0.407156	1.00937	2.85515	12.333	45.8459	11.7067	6.22777	19.6059
7	0.000262369	0.0318009	0.0964389	0.434031	3.27883	10.4399	24.1345	21.0254	40.5588

Figure VIII.25 The three-years transition matrix with the Cohort approach

Like any estimate, numbers are affected by sampling errors. It is thus necessary to also provide confidence intervals for having a trustable outcome. We show how to use the binomial distribution for obtaining confidence bounds within the cohort approach.

Let PD_i denote the true probability of default for rating class i. The estimated probability is:

$$\hat{p}_{ik} = \frac{N_{i,k}}{N_i} \text{ (Eq. VIII.95)}$$

Assuming that defaults are independent across time and across obligors (and unluckily it is a rather strong hypothesis), the number of defaults is binomially distributed with N_i successes and success probability PD_i.
If we are seeking a two-sided, $1-\alpha$ confidence interval where α is a value such as 5%, the lower bound $PD_i^{\min}$ must be such that the probability of observing N_i defaults or more is $\alpha/2$.
$PD_i^{\min}$ therefore solves the condition:

$$1 - \text{BINOM}\left(N_{ik} - 1, N_i, PD_i^{\min}\right) = \alpha/2 \text{ (Eq. VIII.96)}$$

Where $\text{BINOM}(x, N, q)$ denotes the cumulative binomial distribution for observing x or less successes out of N trials with success probability q.

The upper bound $PD_i^{\max}$ must be such that the probability of observing N_i or less defaults is $\alpha/2$:

$$\text{BINOM}(N_{ik}, N_i, PD_i^{\max}) = \alpha/2 \text{ (Eq. VIII.97)}$$

CohortConfidenceIntervals(RatingDataset,alpha=0.05) allows to estimate the confidence interval for the Rating data set in accordance with the model previously described.

If the estimated default probability is zero, it is not necessary to code a solver for the upper bound that can be obtained solving the equation $(1-PD_i)^{N_i} = \alpha$.
Obviously, the lower bound for the probability is zero. On the contrary, if the estimated PD is a non-null value, the goal seeking problem can be numerically solved through a minimization problem which involves a lambda expression. The code for the estimation of the Lower Bound for a non-degenerate case has been shown below:
minimize(lambda PDmin:BinomialLowerBound(PDmin,Ni,Prob,alpha)

The BinomialLowerBound is the cost function defined as:

```
def BinomialLowerBound(PDmin,Ni,Prob,alpha):
    if ((PDmin<0) | (PDmin>1)):
        return +99999999
    else:
        return ((1-binom.cdf(Ni*Prob-1,Ni,PDmin)-(alpha/2)))**2
```

Before returning the output, a check on the feasibility of the independent variable PDmin has been done ($PD \in [0,1]$). A similar optimization problem has been set for the Upper Bound case. It is worth to say that in both routines, a direct search heuristic such as Nelder-Mead simplex worked better than a quasi-newton method such as the L-BFGS. The confidence sets for the transition probabilities to move from a grade i to the default has been displayed in Table VIII.31:

Grade i → Default	1	2	3	4	5	6	7
Lower Bound	0	0	0.0016	0.0875	0.3625	0.7937	6.3687
Estimation	0	0	0.0696	0.3127	0.9868	1.7308	10.3825
Upper Bound	3.0724	0.4164	0.3875	0.8000	2.1375	3.2625	15.7375

Table VIII.31 Confidence sets for PD using the cohort approach

The resulting confidence bounds are relatively wide. In most cases, they overlap with the ones of adjacent rating classes. What may seem surprising is that the upper bound for the best rating category 1 is higher than the ones for rating 2 to 5. The reason is that the number of observations in class 1 is relatively low (96), which increases the confidence intervals. The cohort approach does not make full use of the available data. Specifically, the estimates of the cohort approach are not affected by the timing and sequencing of transitions within the period. As an example, an obligor that moves in the same year from AA to A and in the following one returns to AA, is not tracked by the previous methodology: for the statistics produced with the cohort approach it remains stable across the two considered years. An alternative approach that captures transitions within a period is called the duration or hazard rate approach.
This methodology is more sophisticated than the previous one and it relies on the Markov chain theory. The first step is to estimate the so-called generator matrix Λ, which provides a general description of the transition behavior. The off-diagonal entries of Λ estimated over the period $[t_{0,t}]$ are given as:

$$\lambda_{ij} = \frac{N_{ij}}{\int_{t_0}^{t} Y_i(s)ds} \text{ for } i \neq j \text{ (Eq. VIII.98)}$$

Where N_{ij} is the observed number of transitions from i to j during the time period considered in the analysis, and $Y_i(s)$ is the number of firms rated i at time s. The denominator therefore contains the number of obligor-years spent in rating class i. Let us note the similarity to the cohort approach. In both cases, we divide the number of transitions by a measure of how many obligors are at risk of experiencing the transition. In the cohort approach though, we count the obligors at discrete points in time, while in the hazard approach we count the obligors at any point in time, thanks to the integral operator.
The on-diagonal entries are constructed as the negative value of the sum of the λ_{ij} per row:

$$\lambda_{ij} = -\sum_{i \neq j} \lambda_{ij} \text{ (Eq. VIII.99)}$$

This new formulation is able to take into account cases similar to the ones described above and, consequently, it improves the outcome statistics. From Markov chain mechanics, a T-year transition matrix $P(T)$ is the one derived from the generator matrix, as follows:

$$P(T) = \exp(\Lambda T) = \sum_{k=0}^{\infty} \frac{\Lambda^k T^k}{k!} \text{ (Eq. VIII.100)}$$

where ΛT is the generator matrix multiplied by the scalar T and $\exp(\cdot)$ is the matrix exponential function.

HazardRate(RatingDataset , periods = 1) is able to compute the Λ matrix and perform the exponential function to the generator. It is worth to note that the exponentiation of a matrix is not an easy task in geometry. Let us assume for a moment that we only have four categories including default and NR. The matrix exponential $\exp(\Lambda T)$ would then be the following:

$$\exp(\Lambda T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + T \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{bmatrix} + \frac{T^2}{2!} \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{bmatrix}^2 + \sum_{k=2}^{\infty} \frac{(\Lambda T)^k}{k!} \text{ (Eq. VIII.101)}$$

This can approximately be evaluated by truncating the infinite sum at some suitable point.

Rating	Watch	Effective
BBB		12/17/2021
BBB-		05/12/2020
BBB		04/27/2017
BBB+		03/18/2013
A-		02/06/2012
A	*-	12/20/2011
A		10/11/2011
AA-		07/28/2006
A+		12/20/2002

Figure VIII.26 Credit Profile (CRPR) module. Source: Bloomberg®

In the auxiliary functions MexpGenerator, the stop criteria have been given by an error threshold of 10^{-320}

or the reaching of 170 iterations of k. T is defined by default equal to 1 by the input periods, but the user can change this technical specification, setting the desired time horizon.

	1	2	3	4	5	6	7	8	0
1	93.0211	1.32872	0.719502	0.039667	0.0188322	0.0185046	0.0116341	0.0106472	4.83136
2	1.20166	88.3403	6.49199	0.366565	0.0282466	0.0174665	0.00903829	0.0078147	3.53687
3	0.107236	2.32584	88.649	4.77794	0.324992	0.105095	0.0206197	0.00968441	3.6796
4	0.00195654	0.0501148	3.37071	86.0381	5.22449	1.51734	0.323492	0.0482758	3.42554
5	0.000370539	0.0226639	0.571482	7.60861	73.6792	10.5395	1.71777	0.445513	5.4148
6	0.000946785	0.129047	0.17805	1.13318	7.16064	75.5459	7.70092	2.2419	5.90946
7	0.000183453	0.0337981	0.0860625	1.09608	2.14453	8.9343	60.1943	10.3296	17.1812
8	0	0	0	0	0	0	0	100	0
0	0.00208947	0.275397	0.562865	0.791649	0.694805	0.719315	0.441214	0.437073	96.0756

Figure VIII.27 One-year transition with the hazard approach

	0	1	2	3	4	5	6	7	8
0	80.5372	3.364	2.09924	0.320157	0.151975	0.151011	0.0832377	0.0996171	13.1935
1	2.98625	69.4139	15.3882	1.74581	0.247172	0.158289	0.0705692	0.0759186	9.91385
2	0.341966	5.51191	70.5619	11.1442	1.36777	0.587262	0.158675	0.104162	10.2221
3	0.0174623	0.358244	7.88902	65.2391	10.4586	4.45633	1.0728	0.440404	10.068
4	0.00488203	0.171194	1.92507	15.2169	42.841	18.5116	4.17673	2.19965	14.953
5	0.00748513	0.334618	0.715299	3.89316	12.7063	46.4766	11.1439	7.30017	17.4224
6	0.00363974	0.212142	0.564084	2.78767	4.70312	13.3651	23.4481	21.0766	33.8395
7	0	0	0	0	0	0	0	100	0
8	0.0167704	0.745011	1.58315	2.2289	1.78622	1.92046	0.997399	1.44217	89.2799

Figure VIII.28 Three-years transition with the hazard approach

-0.0724422	0.0144884	0.00724422	0	0	0	0	0	0.0507095
0.013221	-0.125091	0.0732237	0.00203399	0	0	0	0	0.0366119
0.00100917	0.0262385	-0.122614	0.0544953	0.00201834	0.000504586	0	0	0.0383486
0	0	0.0383747	-0.154628	0.0648984	0.0141084	0.00282167	0	0.0344244
0	0	0.00491773	0.0946663	-0.315964	0.140155	0.0172121	0.00245886	0.0565539
0	0.00145706	0.00145706	0.00874238	0.0947091	-0.294327	0.113651	0.0189418	0.0553684
0	0	0	0.0122044	0.0244087	0.13018	-0.516651	0.13018	0.219679
0	0	0	0	0	0	0	0	0
0	0.00290727	0.00581454	0.00814035	0.0075589	0.0075589	0.00523308	0.00407018	-0.0412832

Figure VIII.29 The generator matrix obtained from Rating data set

The matrices have a size of 9×9 because they include default and not-rated classes. The matrix generator Λ used by the Markov chain has also been reported in the Figure VIII.29.
At this point, we wonder: when it is not possible to have detailed information about the historical rating transitions (dates and grades), is it possible to obtain a valid generator Λ? The answer is positive, it is possible, but not all transition matrices have a generator and for those that do, the generator may not be unique. Conditions for a valid generator include the underlying Markov chain to be stochastically monotonic.
To construct an approximation of the generator, we can assume that there is only one transition per obligor and period.
Let p_{ij} denote the entries of the transition matrix P, then the generator is given by:

$$\lambda_{ij} = \ln(p_{ij}) \quad \textit{(Eq. VIII.102)}$$

$$\lambda_{ij} = p_{ij} \frac{\lambda_{ii}}{(p_{ii}-1)} \; with \; i \neq j \quad \textit{(Eq. VIII.103)}$$

This approximated (but useful) methodology for synthesizing generator, Λ_{APPROX} has been coded in a function called Transition2Generator(ProbTransitionMatrix) . This function can convert a probability transition matrix into a generator for Markov chains. In order to understand the magnitude of this approximation in the considered data set, we can run the function passing the one-year transition matrix. The output is the approximated generator, Λ_{APPROX} and we can call the MexpGenerator to perform the inverse operation. The gap can easily be measured by the difference between the recomputed synthesized transition matrix and the original one. The approximated error is reported in the Figure below.

0.00857331	-0.0646123	0.0156726	0.0376227	0.0173114	0.0174767	0.00838633	0.0125573	-0.052988
-0.0393339	0.0885887	-0.358524	0.143963	0.0313343	0.0204959	0.00803239	0.0102128	0.0952306
0.0105194	-0.132453	0.169238	-0.309119	0.0913426	0.0578867	0.0186575	0.0149366	0.0789923
0.00235191	0.043398	-0.165851	0.312969	-0.651821	0.108311	0.0389158	0.0657588	0.245968
0.000823936	0.0246682	0.129791	-0.437413	0.616621	-1.21654	0.0197777	0.268305	0.593969
0.00107322	0.00693234	0.0629365	0.284925	-0.870615	0.772162	-1.63828	0.49441	0.886454
0.000706869	0.031841	0.0871153	0.148675	0.107341	-1.01393	0.39079	0.182394	0.0650625
0	0	0	0	0	0	0	-2.84217e-14	0
0.00194415	-0.00882942	-0.00671974	-0.00751362	-0.0441927	-0.0280508	-0.0622945	0.0376305	0.118026

Figure VIII.30 Gap between the transition matrix computed using the whole Rating transition data set and the one estimated using the approximated Markov chain generator Λ_{APPROX}

Comparing this approximate generator to the transition matrix, we notice that they are similar but not identical. In our data, the assumption that there is only one transition per year is not fulfilled, leading to a discrepancy between the approximate generator and the one estimated with the detailed data.

Regarding the confidence intervals, it is no longer a reasonable choice to use a binomial distribution because

there is no direct counterpart to the N_i of the cohort approach. As a result, we employ **bootstrap simulations**. In a bootstrap analysis, the statistician re-samples from the data used for estimation and re-estimates the statistics with the re-sampled data. Having done this many times, the analyst can then derive a distribution of the statistic of interest.
HazardConfidenceIntervals(RatingDataset,M=100,toclass=8,alpha=0.05) implements the following logical steps which help to perform the statistical bootstrap:

1. Randomly draw with replacement an obligor's complete rating history. Repeat as many times as the obligors in the original rating data set.
2. Calculate the generator Λ and the 1-year transition matrix $\exp(\Lambda)$ for the sample generated in step 1.
3. Repeat steps 1 and 2 M times.
4. Determine percentiles of the transition probabilities from step 3.

The choice made in step 1 is not the only possible one. In a simple setting with N independent observations, we would re-sample N times with replacement to maintain the size of the original data set. Our rating data set, by contrast, has several dimensions: the number of obligors, the number of rating actions, the number of obligor-years for which data is available, the calendar-time spanned by the first and last rating action, and several more. Among the one-dimensional bootstrap strategies, drawing obligors appears to be the most "natural" one. The function returns an array with two columns, where each row corresponds to a rating class and the columns contain the lower and upper confidence bounds. We run the function on our rating data set with the default input parameter. In this way, we draw $M = 100$ bootstrap samples and we calculate the confidence for the probability of default (toclass=8) with a 5% confidence. The results are summarized in Table VIII.32.

Bound	1	2	3	4	5	6	7	8	NR
Lower	0.0032	0.0042	0.0054	0.0302	0.1938	1.5379	7.0288	100	0.16
Upper	0.0149	0.0156	0.0255	0.0705	0.8247	3.3252	12.9404	100	0.76

Table VIII.32 Statistical bootstrapped confidence bounds for default probabilities from the hazard approach

The smaller confidence bands for the top rating classes present a striking difference to the binomial confidence bounds obtained for the cohort estimates. The intuition is that the hazard estimate of this grade's PD is not only based on the behaviour of the few obligors within this grade, but also on the behaviour of obligors in other grades.
Default rates are essential to pricing for risk management. For example, based on a forecast for next year's default rate, a bank can set appropriate loan rates for short-term loans. In the previous part, we showed how to estimate average transaction rates based on data extending over several years. If such rates are used to estimate next year's transition rates, we would then implicitly assume that next year is a typical or average year. Although this may be an appropriate assumption in certain situations, in others we may have good reason to believe that the following year will be relatively good or bad for credits. If the economy is moving into a recession, we should expect default rates to be relatively high, as a result of this discontinuity compared to the past. In this section, we show how to use available information to predict default rates for corporates rated by

a major rating agency.

It is crucial to know that agencies do not aim at assigning ratings in such a way that the one-year default probability of a rating category is constant across time. By contrast, ratings are meant to be relative assessments of credit quality. If overall economic conditions have deteriorated, affecting all borrowers in a similar way, the previous relative ordering would still be correct, even though the default probability of a given rating category may substantially deviate from its past average. In the example studied in this section, we predict default rates for calendar years, that is from the end of year t to the end of year $t + 1$. Therefore, we need information that is already known at the end of year t. Let us consider four different traditional factors, each of which is captured by one empirical variable.

Macroeconomic conditions: liquidity and profits of corporates are affected by overall economic conditions. We could capture them by a measure of current activity, such as GDP growth over the preceding year. However, we can hope to do better if we use a forecast of future economic activity instead of current activity, and if we use a measure of activity that is closely associated with corporate conditions. We therefore use forecasts of one-year changes in corporate profits. To control for effects of inflation, we also deflate the forecasts. We denote this variable as PRF_t:

$$PRF_t = \frac{1+\text{forecasted change in corporate profits(in } t \text{ for } t,\ t+1)}{1+\text{forecasted change in GDP deflator(in } t \text{ for } t,\ t+1)} - 1 \quad \textit{(Eq. VIII.104)}$$

Corporate bond spreads: yields of corporate bonds should be set so that the expected return from holding a bond is at least as high as the return from holding a risk-free government bond. Otherwise, there would be little incentive to buy risky corporate bonds. Roughly speaking, the expected return on a corporate bond is its yield minus the loss rate. The corporate bond spread, which is the difference between the yield of a corporate bond and a comparable government bond, should therefore vary with the loss rates expected by the market. The variable SPR is defined as:

$$SPR_t = \text{yield of corporate bonds(in } t) - \text{yield of sovereign bonds(in } t) \quad \textit{(Eq. VIII.105)}$$

Aging effect: it has been documented in the literature that issuers who first entered the bond market three to four years ago are relatively likely to default. This empirical phenomenon is called the aging effect. There are several possible explanations, one being that the debt issue provides firms with cash - enough cash to survive for several years even if the business plan envisaged at the time of the bond issue did not work out. So, if new issuers run into difficulties, liquidity problems will only appear with a certain delay.

Let us define the variable AGE as the fraction of current issuers that had their first-time rating three to four years ago:

$$AGE_t = \frac{\#\text{newly rated issuers(from } t-4 \text{ to } t-3)}{\#\text{rated issuers(in } t)} \quad \textit{(Eq. VIII.106)}$$

Average risk: when analyzing average default rates of a group comprising several rating categories, we should take into account the fact that the composition of the group can change over time. It is considered a good practice to capture differences in average risk through the percentage of current investment-grade issuers that are rated BBB:

$$BBB_t = \frac{\#\text{BBB rated issuers(in } t)}{\#\text{Investment grade issuers(in } t)} \quad \textit{(Eq. VIII.107)}$$

The first step for analyzing data and performing a prediction is to set up a regression model. For this aim, we do not use a linear regression for the well-known drawbacks: default rate predictions could be negative. In addition, linear regression does not take into account that the realized default rate will vary less around the expected default probability if the number of issuers is large. With the Independence assumption, the number of defaults observed in a given year follows a binomial distribution. In this context, it is more convenient to use the Poisson distribution instead. If the number of issuers is large and the default probability is small, the Poisson provides a very good approximation to the binomial. The density function of the Poisson, which specifies the probability that the number of defaults is equal to an observed number D_t, is:

$$\text{Prob}(\#\text{defaults}_t = D_t) = \frac{\exp(-\lambda_t)\lambda_t^{D_t}}{D_t!} \quad \textit{(Eq. VIII.108)}$$

Where $D_t!$ denotes the factorial of D_t.

The standard way to model the variation of default rates across time using a Poisson model is to assume that the expected number of defaults varies in the following way with explanatory variables x:

$$\lambda_t = \exp[\beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \cdots + \beta_K x_{Kt}] \quad \textit{(Eq. VIII.109)}$$

The exponential function makes sure that the expected number of defaults is always non-negative. Equivalently:

$$\ln(\lambda_t) = [\beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \cdots + \beta_K x_{Kt}] \quad \textit{(Eq. VIII.110)}$$

In vector notation, with $\beta^T = [\beta_1, \beta_2, \beta_3, \dots, \beta_K]$ and $x^T = [1, x_{2t}, x_{3t}, \dots, x_{Kt}]$, this can be written as follows: $\ln(\lambda_t) = \beta^T x_t$.

The goal of the estimation is to determine the weights β that describe the impact of the variable on the default occurrence. To apply the maximum likelihood principle, we need the likelihood L, which is the probability of observing an entire sample. From the independence assumption we obtain:

$$L = \text{Prob}(\#\text{defaults}_1 = D_1) \cdot \text{Prob}(\#\text{defaults}_2 = D_2) \cdot \ldots \cdot \text{Prob}(\#\text{defaults}_T = D_T) =$$
$$= \frac{\exp(-\lambda_1)\lambda_1^{D_1}}{D_1!} \cdot \frac{\exp(-\lambda_2)\lambda_2^{D_2}}{D_2!} \cdot \ldots \cdot \frac{\exp(-\lambda_T)\lambda_T^{D_T}}{D_T!} \quad \textit{(Eq. VIII.111)}$$

Considering the logarithms, we obtain:

$$\ln(L) = \sum_{t=1}^{T} [-\lambda_t + D_t \ln(\lambda_t) - \ln(D_t!)] \quad \textit{(Eq. VIII.112)}$$

Inserting equation $\ln(\lambda_t) = \beta^T x_t$, this can be written as:

$$\ln(L) = \sum_{t=1}^{T} [-\lambda_t + D_t \beta^T x_t - \ln(D_t!)] \quad \textit{(Eq. VIII.113)}$$

This last equation can be maximized using the Newton method. The test data set DefaultRatesVarsDataset has been provided by the "Credit Risk Modeling" book written by Löffler & Posch, as usual, and it contains seven fields:

Year: a descriptive variable, not used in the regression.
D: the number of investment grade defaulters, i.e., the response variable.
LNN: the log number of investment-grade issuers at the start of the year. It captures the effect that the expected number of defaults increases with the number of issuers (explanatory variable 1).
PRF: the forecasted change in corporate profits (explanatory variable 2).
AGE: the fraction of new issuers (explanatory variable 3).
BBB: the fraction of BBB-rated issuers (explanatory variable 4).
SPR: the spread on corporate bonds (explanatory variable 5).

PoissonRegression(DefaultDataset) performs the Poisson regression on the considered data set: it implements the mathematical formulas described above and returns the coefficients of the Poisson regression and some other useful model statistics such as t-statistics and p-values. For a non-linear model such as the Poisson regression, we cannot compute an R^2 as we do in a linear model.
A Pseudo- R^2 that is often reported in the literature is defined by relating the log-likelihood of the model to the log-likelihood of a model that only has a constant in it:

$$\text{Pseudo} - R^2 = \frac{\ln L(\text{model})}{\ln L(\text{model with all } \beta \text{ except } \beta_1 \text{ set to 0})} \quad \textit{(Eq. VIII.114)}$$

The output of the function has been arranged in a bi-dimensional array with the statistics of the Poisson regression organized as displayed:

b_1	b_2	...	b_K
$SE(b_1)$	$SE(b_2)$	...	$SE(b_K)$
$t_1 = b_1/SE(b_1)$	$t_2 = b_2/SE(b_2)$	...	$t_K = b_K/SE(b_K)$
$p - \text{value}(t_1)$	$p - \text{value}(t_2)$	...	$p - \text{value}(t_K)$
$\text{Pseudo} - R^2$	ln Likelihood	0	0

Table VIII.33 Output of the logit function with stats=True

Running PoissonRegression(DefaultDataset) on the whole dataset, we obtain the results reported in the below Figure. In the first column, the statistics to the constant of the model are reported then followed by the ones related to the other explanatory variables considered in the model from 1 to 5. The profit forecast PFR and the aging variable AGE are highly significant in the general model 1 because their t statistics are well above 1.96 in absolute terms. The other variables show little significance. Excluding the spread SPR and the fraction of BBB rated issuers, we come to model 2.
The β for such model are: $\beta_{CONST} = -18.24$, $\beta_{LNN} = +2.18$, $\beta_{PRF} = -0.20$ and $\beta_{AGE} = +0.29$.
Predictions of the default rate can be based on λ, which is obtained applying $\lambda_t = \exp[\beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} +$

$\cdots + \beta_K x_{Kt}]$. Dividing λ by the number of issuers, N yields the expected default rate.

The PoissonTrend(DefaultDataset) function provides the prediction. Substantially it calls the previous function, takes the regression coefficients and applies the λ_t equation. By doing so, the following year investment grade default rate has been estimated equal to 0.051% using the first model or 0.065% using the restricted version with only the three explanatory variables (plus the constant term).

	0	1	2	3	4	5
0	-14.1008	1.45971	-0.180828	0.29753	0.0126831	0.41712
1	11.8858	1.80416	0.0455875	0.108676	0.0853939	0.356483
2	-1.18637	0.809079	-3.96661	2.73776	0.148524	1.1701
3	0.235478	0.41847	7.29026e-05	0.00618585	0.881929	0.241962
4	0.493429	-28.3149	0	0	0	0

Figure VIII.31 Poisson regression statistics for the prediction of investment-grade default rates – Model 1

The default rates we have analyzed are also recorded in transition matrices, together with transition rates. The latter are average frequencies with which ratings migrate from one rating to another. Like default rates, transition rates vary over time, and to model this fact, we can consider two equivalent statements:
- the probability of a migration from A to B is 2.5%.
- the rating migrates from A to B whenever a standard variable ends up between 1.645 and 1.96.

Both statements are equivalent because the probability that a standard normal variable ends up between 1.645 and 1.96 is 2.5%. It is easy to verify norm.cdf(1.96)-norm.cdf(1.645). In fact, we can describe the entire transition matrix by the concept that transitions are driven by a standard normally distributed variable x without losing any information.

Instead of describing transition behavior through transition rates, we can also describe it through a set of thresholds: 1.645 and 1.960 would be a pair of thresholds that describes a bin. This approach helps to perform simulations: we can shift the transition matrix into bad or good year. We can illustrate the binning procedure for transition from A, considering the transition rates shown in the Table VIII.34.

We can start to define the bins at any of the two extreme transitions, transitions to AAA or transition to default. Choosing the second option, if the probability of a migration from A to D is 0.042%, we can define the D bin as $[\Phi^{-1}(0.00042), -\infty]$, where Φ^{-1} denotes the inverse cumulative standard normal distribution function.

The upper threshold for this D bin becomes the lower threshold for the CCC/C bin. The latter can be achieved by setting the upper threshold to $\Phi^{-1}(0.00042 + 0.00031) = -3.18$.

	AAA	AA	A	BBB	BB	B	CCC/C	D
AAA	91.386	7.947	0.508	0.093	0.062	0.001	0.001	0.001
AA	0.603	90.65	7.936	0.603	0.062	0.114	0.021	0.010
A	0.052	1.991	91.427	5.858	0.44	0.157	0.031	0.042
BBB	0.021	0.171	4.112	89.508	4.561	0.812	0.182	0.288
BB	0.033	0.044	0.276	5.799	83.508	8.114	0.992	1.235
B	0.001	0.057	0.215	0.351	6.249	82.27	4.766	6.091
CCC/C	0.001	0.001	0.322	0.472	1.426	12.56	54.139	31.079

Table VIII.34 Average Transition rates from 1981. Source: "Credit Risk Modeling" by Löffler & Posch

We can continue in this way. Though we have eight bins, we only need to compute seven thresholds.

	AAA	AA	A	BBB	BB	B	CCC/C	D
A	0.052	1.991	91.427	5.858	0.440	0.157	0.031	0.042
Bin	(+inf,3.28]	(3.28,2.04]	(2.04,-1.51]	(-1.51,-2.47]	(-2.47,-2.83]	(-2.83,-3.18]	(-3.18,-3.34]	(-3.34,-inf)

Table VIII.35 Binning procedures for transition from A grade

The getThresholdsMatrix(TransitionDataset) function allows to compute all the thresholds for the bins of the distribution for all the rating grades. The input argument is the Transition matrix. The output of the function has been reported in the below Figure.

The usual representation of the grades is applied: from top to down and from right to left, the ratings become worse and worse.

-1.36498	-2.4751	-2.95173	-3.21598	-4.01281	-4.10748	-4.26489
2.5098	-1.35656	-2.40438	-2.86729	-2.97814	-3.42271	-3.71902
3.26881	2.04454	-1.5119	-2.47296	-2.83379	-3.18252	-3.33927
3.54008	2.89266	1.71656	-1.56809	-2.23162	-2.59715	-2.76114
3.41407	3.17084	2.69495	1.54222	-1.26236	-2.00897	-2.24606
4.26489	3.24854	2.77856	2.49827	1.48532	-1.23417	-1.54718
4.26489	4.10748	2.72245	2.41074	2.00992	1.04583	-0.493612

Figure VIII.32 Upper thresholds of the bins

As stated above, this representation is more convenient to perform simulations on the transition rates. Let us

imagine that the normal density is shifted to the left, i.e. it assumes a negative mean rather than zero. The probability of a transition is the probability of ending up in the associated bin. This probability is equal to the area enclosed by the boundaries of the bin and the density function. Therefore, a shift to the left would increase the probabilities of downgrades as well as the probability of default.
Importantly, we still have fully specified transition probabilities, albeit ones that are different from those we used for the threshold determination. Likewise, we could reduce the probabilities of downgrade and default by shifting the distribution to the right.

Transition probabilities that result after the shift can be computed calling the getAdjustedTransitionMatrix function which takes as input the transition matrix and the magnitude of the shift that is called "**credit index**". A negative number of the credit index means that the distribution function is shifted to the left, thus increasing the probabilities of downgrade and default. The probability that a normal variable with mean m and standard deviation 1 ends up to the left of a threshold is given by $\Phi(threshold - m)$. In order to achieve the probability of ending up in a bin, we use this formula to obtain the probability of ending up below the upper threshold of the bin, and then subtract the probability of ending up below the lower threshold of the bin. We could compute the latter with the normal distribution, but we can also sum over the cells in the same row that are located to the right of the bin we are in.

For the AAA bins (first column of the tables), we exploit the fact that transition probabilities sum up to 1. Using a credit index equal to -0.25, we obtain the adjusted transitions reported in the Figure below. This is a useful tool in order to perform what-if analysis on the transition rates or adjusting the historical transitions if an unlikely event has occurred, which can cause a discontinuity in the economic scenario, like for example the covid-19 pandemic.

0	1	2	3	4	5	6	7	8
1	86.757	11.9393	0.958837	0.194039	0.142459	0.0026726	0.00275444	0.00297367
2	0.289185	86.2867	11.8636	1.11737	0.124692	0.242948	0.0493931	0.0261184
3	0.0216742	1.06629	88.5627	9.03836	0.822322	0.320499	0.0677865	0.100326
4	0.00752982	0.0761815	2.378	88.1646	6.99758	1.43021	0.344164	0.60171
5	0.0124119	0.0188024	0.130291	3.49338	80.7768	11.6392	1.63275	2.29636
6	0.000316747	0.0230741	0.0994692	0.17669	3.83469	79.6141	6.52315	9.72847
7	0.000316747	0.000341097	0.147059	0.242124	0.80148	8.5604	49.8717	40.3766

Figure VIII.33 Transitions rates using a credit index equal to -0.25

FURTHER READINGS

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VIII.7 CREDIT PORTFOLIO RISK

A **credit portfolio risk** model produces a probability distribution of losses that can arise from holding a portfolio of risky instruments. A financial institution can use such models to compute percentiles on the loss distribution (such as Var) and other more meaningful risk measures such as Expected Shortfall and Unexpected Loss. On this topic, starting from 1997, four approaches have been published: CreditMetrics, CreditRisk+, CreditPortfolioView and KMV PortfolioManager. These models are similar to the underlying structure, as a result only one approach is covered here: the asset value approach exemplified by **CreditMetrics**.
In this approach, the portfolio loss distribution is obtained through a Monte Carlo simulation. To keep focused on the credit issue, we propose a simplified framework in which we only consider losses from default (but not from changes in market value). It is a good practice to split portfolio credit risk modeling into four main steps. Here after, we describe those steps for a general model and for a specific approach: a default-mode model in which we only consider losses from default:

1. Specify probabilities of individual credit events. Default mode: only specify probabilities of default (PDs) because other events (changes in credit quality) are ignored in the modeling.
2. Specify value effects of individual credit events. Default mode: specify the loss given default (LGD), which is a percentage of the exposure at default (EAD) that is lost in the case of default.
3. Specify correlations of individual credit events and value effects. Default mode: specify default correlations and (possibly) correlations of LGDs.
4. Based on steps 1 to 3, obtain the portfolio value distribution (via simulations or analytically).

In previous sections, we explored different ways of obtaining default probabilities: logit/probit scores, structural models, historical default rates per rating category or a financial market approach to determine the PD as required in step 1. LGD can be measured using historical averages or multivariate prediction models (step 2). Since the previous sections focus on the measurement of individual credit events from a portfolio perspective, we now model the **default correlations**.
With the aim of formalizing default correlation, we use the standard definition of the correlation coefficient of two random variables X_1 and X_2:

$$\rho_{X_1,X_2} = \frac{\text{cov}(X_1,X_2)}{\sigma(X_1)\sigma(X_2)} \quad \textit{(Eq. VIII.115)}$$

Where $\text{cov}(\cdot)$, denotes the covariance, and σ is the standard deviation. In our case, the random variable is a default indicator y_i that takes the value 1 if obligor i defaults and 0 otherwise.
The default correlation we are searching is therefore:

$$\rho_{ij} = \frac{\text{cov}(y_i,y_j)}{\sigma(y_i)\sigma(y_j)} \quad \textit{(Eq. VIII.116)}$$

We then focus on the denominator, and we apply the standard definition of variance:

$$\sigma^2(y_i) = \text{Prob}(y_i = 1)\big(1 - E(y_i)\big)^2 + \text{Prob}(y_i = 0)\big(0 - E(y_i)\big)^2 \quad \textit{(Eq. VIII.117)}$$

Denoting the default probability $\text{Prob}(y_i = 1)$ with p_i and exploiting the fact that $\text{Prob}(y_i = 1)$ is the same as $E(y_i)$ we obtain:

$$\sigma^2(y_i) = p_i(1 - p_i)^2 + (1 - p_i)(0 - p_i)^2 = p_i(1 - p_i)^2 + p_i^2(1 - p_i) = p_i(1 - p_i) \quad \textit{(Eq. VIII.118)}$$

which is the familiar result for the variance of a Bernoulli variable with a success probability equal to p_i.
To express the covariance in terms of default probabilities, we utilize the general result $\text{cov}(X_1, X_2) = E(X_1, X_2) - E(X_1)E(X_2)$:

$$\text{cov}(y_i, y_j) = E(y_i, y_j) - E(y_i)E(y_j) = p_{ij} - p_i p_j \quad \textit{(Eq. VIII.119)}$$

where p_{ij} denotes the joint default probability $\text{Prob}(y_i = 1, y_j = 1)$. Thus, the default correlation is completely specified by the individual and joint default probabilities:

$$\rho_{ij} = \frac{p_{ij}}{\sqrt{p_i(1-p_i)p_j(1-p_j)}} = \frac{p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}} \quad \textit{(Eq. VIII.120)}$$

Even though the default correlation can be expressed with two intuitive measures - individual and joint default probabilities - it would be a daunting task to build a portfolio risk analysis on estimated pair-wise default correlations.

In a portfolio with $1{,}000$ obligors, there are $\frac{(1000^2-1000)}{2} = 499{,}500$ correlations – far too many to specify.
In practical applications, an analyst sets up a simplifying structure that reduces the number of parameters to be estimated. Instead of directly forcing the structure on default correlations themselves, it is more convenient to first represent defaults as a function of continuous variables and then set the structure on these variables.
Let us name these variables A_i, $i = 1, \ldots, N$. The default indicator can then be represented as:

$$\text{Default}_i \Leftrightarrow y_i = 1 \Leftrightarrow A_i \le d_i \quad \textit{(Eq. VIII.121)}$$

$$\text{No Default}_i \Leftrightarrow y_i = 0 \Leftrightarrow A_i > d_i \quad \textit{(Eq. VIII.122)}$$

where d_i is the critical value that marks the default of borrower i if variable A_i falls below it. The joint default probability between two obligors then is:

$$\text{Prob}(y_i = 1, y_j = 1) = \text{Prob}(A_i \le d_i, A_j = d_j) \quad \textit{(Eq. VIII.123)}$$

From an econometrician's perspective, the variables A_i are latent, i.e. unobservable variables that determine an observed, discrete outcome. In the credit risk literature, the latent variables are usually interpreted as the firms asset values. This goes back to the option-theoretic approach of Merton, in which a firm defaults if its asset value falls below a critical threshold associated with the value of liabilities.
In the following, the mechanics of the approach are described for the simplest but widely used case in which the asset values are assumed to be normally distributed with correlations that go back to a single common factor.

Commonly, borrower i's asset value A_i depends on the common factor Z and an idiosyncratic factor ϵ_i:

$$A_i = \omega_i Z + \sqrt{1-\omega_i^2}\epsilon_i,\ \text{cov}(\epsilon_i, \epsilon_j) = 0,\ i \neq j,\ \text{cov}(Z, \epsilon_i) = 0,\ \forall_i \textit{ (Eq. VIII.124)}$$

where Z and ϵ_i are standard normal variables. By construction, A_i is also standard normal. The asset correlation is completely determined by the factor sensitivities ω:

$$\rho_{ij}^{\text{asset}} = \frac{\text{cov}(A_i, A_j)}{\sigma(A_i)\sigma(A_j)} = \frac{\text{cov}\left(\omega_i Z + \sqrt{1-\omega_i^2}\epsilon_i, \omega_j Z + \sqrt{1-\omega_j^2}\epsilon_j\right)}{1\times 1} = \text{cov}(\omega_i Z, \omega_j Z) =$$

$$= \omega_i \omega_j \text{var}(Z) = \omega_i \omega_j \textit{ (Eq. VIII.125)}$$

Which is the ensuing default correlation? As seen above, we first need the default probability, which is given by:

$$\text{Prob}(A_i \leq d_i) = p_i = \Phi(d_i) \textit{ (Eq. VIII.126)}$$

Where Φ denotes the cumulative standard normal distribution function. The joint default probability is:

$$\text{Prob}(A_i \leq d_i, A_j \leq d_j) = p_{ij} = \Phi_2(d_i, d_j, \rho_{ij}^{\text{asset}}) \textit{ (Eq. VIII.127)}$$

Where Φ_2 denotes the cumulative bivariate standard normal distribution function with correlation ρ.

There are several ways of parameterizing the asset correlation model, i.e. choosing ds and the ωs. We can set the default triggers d so that they result in the default probabilities that we have estimated with a default prediction model like the logit/probit model, a structural model or from an analysis of default rates.

It is common practice to choose the factor sensitivities such that they are in line with observed default behaviour.

Let us assume that we have collected default information for a group of obligors over several years. Let D_t denote the number of obligors that defaulted in period t, and N_t the number of obligors that belonged to the group at the start of period t. We assume that one period corresponds to one year. Data is observed over T years. The essential information for our purpose is the default probability and the joint default probability.
The average default probability can be estimated by averaging the annual default rates:

$$\hat{p} = \frac{1}{T}\sum_{i=1}^{T}\frac{D_t}{N_t} \textit{ (Eq. VIII.128)}$$

In the absence of other information, we assume that all obligors have the same default probability, i.e. we set $p_i = p_j = p$; our default threshold is then $d_i = d_j = d = \Phi^{-1}(p)$.
We can estimate the joint default probability in a similar way. In the last equation, we relate the number of observed defaults to the possible number of defaults; now we relate the number of observed joint defaults to the possible number of joint defaults. If there are D_t defaults, the number of pairs of defaulters that we can form follows from combinatorial analysis:

$$\binom{D_t}{2} = \frac{D_t(D_t-1)}{2} \text{ (Eq. VIII.129)}$$

In fact, the binomial coefficient $\binom{n}{k}$ yields the number of subsets with k elements that can be formed out of a set with n elements. It is given by $\frac{n!}{k!(n-k)!}$. If all obligors defaulted, we would obtain the maximum number of pairs of defaulters, which is:

$$\binom{N_t}{2} = \frac{N_t(N_t-1)}{2} \text{ (Eq. VIII.130)}$$

The joint default rate in year t is the number of default pairs $\binom{D_t}{2}$ divided by the maximum number of default pairs $\binom{N_t}{2}$:

$$\hat{p}_{2t} = \frac{\frac{D_t(D_t-1)}{2}}{\frac{N_t(N_t-1)}{2}} = \frac{D_t(D_t-1)}{N_t(N_t-1)} \text{ (Eq. VIII.131)}$$

Using the information from T years, the estimator for the joint default probability takes the average from the observed annual joint default rates:

$$\hat{p}_2 = \frac{1}{T}\sum_{t=1}^{T} \hat{p}_{2t} = \frac{1}{T}\sum_{t=1}^{T} \frac{D_t(D_t-1)}{N_t(N_t-1)} \text{ (Eq. VIII.132)}$$

Again, we would assume that the joint default probability is equal for all borrowers. The asset correlation follows suit. From $\text{Prob}(A_i \leq d_i, A_j \leq d_j)$ we know that:

$$p_{ij} = \Phi_2\left(d_i, d_j, \rho_{ij}^{\text{asset}}\right) \text{ (Eq. VIII.133)}$$

We can estimate ρ_{ij}, d_i and d_j from the previous relations, so it is an equation with only one unknown: the asset correlation. It is not a problem that can be solved analytically but a numerical solver is needed. Specifying the default thresholds and the asset correlation in this way, it becomes an application of the method of moments. In this method, an analyst calibrates unknown parameters such that the model results match empirical estimates of moments. The two moments used in this context are $E(y_i) = p_i$ and $E(y_i y_j) = p_{ij}$.
As an example, let us consider the Investment Grade Default data set provided with the "Credit Risk Management" book written by Löffler and Posch. The table has three fields: the year (t), the number of defaults (D_t) and the number of obligors (N_t). The getAssetCorrelationMomentMatching function allows to estimate the asset correlation using the method of moments (p_{ij} or ω^2) as well as the default probability $\hat{p}$ and the factor sensitivity (ω). The only compulsory input is the Default data set and the output structure is a tuple of three scalars. The goal seeking has been performed using the direct search heuristic of the Nelder-Mead simplex through the specification of the objective function in goal_seeking.
The optimal parameters estimated are $p = 0.1144\%$, $\omega = 22.094\%$ and $\omega^2 = 4.881\%$.
If there are several groups of obligors (for instance investment grade and speculative grade issuers) and we want

to calibrate the asset value model, we could do it separately for the individual group. There is a drawback to this though. We are implicitly assuming that the defaults are independent across groups.
In reality it is a rather strong assumption and generalizing the method of moments in an intra-group context is not so straightforward. The approach presented in the next paragraph is not only more flexible, but it also makes better use of available information. When applied to the asset value approach, the maximum likelihood principle can be particularized in this way: determining default probabilities and factor sensitivities so that the probability (i.e. the likelihood) of observing the historical default data is maximized.
First, we need to describe default behaviour through an appropriate distribution function. To derive this distribution function, we have to start with the concept of conditional default probability:

$$p_i(Z) = \text{Prob}(A_i \leq \Phi^{-1}(p_i)|Z) \quad \textit{(Eq. VIII.134)}$$

Inserting $A_i = \omega_i Z + \sqrt{1-\omega_i^2}\epsilon_i$ into the last equation, we obtain:

$$p_i(Z) = \text{Prob}\left(\omega_i Z + \sqrt{1-\omega_i^2}\epsilon_i \leq \Phi^{-1}(p_i)\right) =$$

$$= \text{Prob}\left(\epsilon_i \leq \frac{\Phi^{-1}(p_i)-\omega_i Z}{\sqrt{1-\omega_i^2}}\right) = \Phi\left[\frac{\Phi^{-1}(p_i)-\omega_i Z}{\sqrt{1-\omega_i^2}}\right] \quad \textit{(Eq. VIII.135)}$$

If the factor realization is not good (i.e. -2), the conditional default probability is relatively high and there will be many defaults. The crucial insight for the following is that once we know Z, the default of borrower i provides no information on the likely default of another borrower. To understand this, we have to acknowledge that once we have fixed the value of Z, the randomness in the last equation is entirely due to ϵ_i but we assume that ϵ_i and ϵ_j are independent for $i \neq j$. Conditional on a factor realization, defaults are thus independent; knowing whether borrower i has defaulted or not does not help us predict whether borrower j defaults or not. Each default variable y_i can then be seen as a 0-1 random variable with success probability $p_i(Z)$. If the conditional default probability is uniform across issuers at $p(Z)$, the total number of defaults D follows a binomial distribution with success probability $p(Z)$ and N trials. Let us now remember that the binomial density for x successes out of n trials with success probability q is $\binom{n}{x} q^x (1-q)^{(n-x)}$. Applying this formula to our problem leads to the following likelihood for the number of defaults within sector k in a given year t:

$$L_{kt} = \int_{-\infty}^{+\infty} \binom{N_{kt}}{D_{kt}} p_k(Z)^{D_{kt}} \left(1-p_k(Z)\right)^{N_{kt}-D_{kt}} d\Phi(Z) \quad \textit{(Eq. VIII.136)}$$

We integrate over factor Z because we do not know which factor has materialized. If we have default data for sector k that spreads over T years, we assume that defaults are independent across time and reach the following likelihood:

$$L_k = \prod_{t=1}^{T} \int_{-\infty}^{+\infty} \binom{N_{kt}}{D_{kt}} p_k(Z)^{D_{kt}} \left(1-p_k(Z)\right)^{N_{kt}-D_{kt}} d\Phi(Z) \quad \textit{(Eq. VIII.137)}$$

If we were to apply the maximum likelihood approach to the data of only one sector - for instance to the investment grade default as in the previous paragraph – we would maximize the last equation to obtain parameters p_k and ω_k. If there are more sectors $k = 1, \dots, K$ we have to model the joint distribution of defaults. Clearly, we want to allow for the dependence, and the simplest way is to assume that there is only one systematic factor that affects each sector. For a single year t, the likelihood can be written as:

$$L_t = \int_{-\infty}^{+\infty} \prod_{k=1}^{K} \left(\binom{N_{kt}}{D_{kt}} p_k(Z)^{D_{kt}} \left(1 - p_k(Z)\right)^{N_{kt}-D_{kt}} \right) d\Phi(Z) \text{ (Eq. VIII.138)}$$

For T years, this leads to:

$$L = \prod_{t=1}^{T} \int_{-\infty}^{+\infty} \prod_{k=1}^{K} \left(\binom{N_{kt}}{D_{kt}} p_k(Z)^{D_{kt}} \left(1 - p_k(Z)\right)^{N_{kt}-D_{kt}} \right) d\Phi(Z) \text{ (Eq. VIII.139)}$$

Unfortunately, likelihoods which take a similar form to the one in the previous equation, are very difficult to maximize. A traditional analytical procedure consisting in setting the first derivative equal to zero is not feasible. We need to implement a numerical technique that evaluates the integral in the likelihood function. We thus implement the Gauss-Hermite procedure that consists in approximating the integral with a weighted sum.
The integral is evaluated at a discrete number of points, the abscissas; these values are then weighted with a specific weighting function. To integrate over a function $f(x)$, Gauss-Hermite uses:

$$\int_{-\infty}^{+\infty} f(x)dx \approx \sum_{i=1}^{n} \omega(x_i) \exp\left(x_i^2\right) f(x_i) \text{ (Eq. VIII.140)}$$

The abscissa x_i and the weights are obtained using the canonical Gauss-Hermite procedure: the abscissas for a chosen order n are the roots of the n-th Hermite polynomial. The Hermite polynomials are given by:

$$H_0 = 1, H_1 = 2x, H_{n+1} = 2xH_n - 2nH_{n-1} \text{ (Eq. VIII.141)}$$

The higher the order n, the higher the number of abscissas in the discrete sum, the more precise the estimated integral is. Simulation studies in literature suggest that a good trade-off between precision and computational effort is to set the order equal to $n = 32$.
In order to get the 32 sample points (x) with the related weights (ω) we can use the numpy function np.polynomial.hermite.hermgauss(order). This function returns a tuple of two arrays that contains the values for approaching the Hermite-Gauss quadrature. Since we will integrate over the standard normal distribution, the function $f(x)$ in the previous equation will always be of the form $g(x)\phi(x)$, where ϕ is the density of the standard normal distribution function. To avoid code repetition and time-consuming evaluations of ϕ we can directly compute $\omega(x) \exp(x^2)\, \phi(x)$ and then weight $g(x)$ with this expression.

```
order=32
hermgauss_quadrature=np.polynomial.hermite.hermgauss(order)
x=hermgauss_quadrature[0] #sample points
omega_x=hermgauss_quadrature[1] #weights
hermgauss_approxPhi=omega_x*np.exp(x**2)*norm.pdf(x)
```

Once we have set the Hermite-Gaussian grid with 32 knots, we can handle the maximization problem of our likelihood function for $k = I$, i.e. the investment grade. As usual practice in the econometric field, we consider the log-likelihood function, that is:

$$\ln L_k = \sum_{t=1}^{T} \ln \int_{-\infty}^{+\infty} \binom{N_{kt}}{D_{kt}} p_k(Z)^{D_{kt}} \left(1 - p_k(Z)\right)^{N_{kt}-D_{kt}} d\Phi(Z) \quad \textit{(Eq. VIII.142)}$$

getAssetCorrelationLnL allows the user to compute the tuple containing the three variables of interest: the probability of default p, the factor sensitivity ω and the asset correlation ω^2.

In the data set considered here these parameters are equal to: $p = 0.121\%$, $\omega = 27.646\%$ and $\omega^2 = 7.643\%$. The Python function LnLMaximization(params) calls the minimization routine which optimizes the $-\ln L$.

In this case we have to face a bi-dimensional problem in the two independent variables p and ω. As a result we have to unpackage the two scalars inside the objective function using pistar,wistar=params.

Conceptually the other steps of implementation are similar to the previous one-dimensional case. Again, the Nelder-Mead heuristic works well in this context, iteratively generating simplexes in the reference $k = 2$-dimensional vectorial space. As we remember from geometry notions, a k -simplex is a k -dimensional polytope which is the convex hull of its $k + 1$ vertices, and it becomes a tetrahedron in 2-D space.

One of the reasons for the Maximum Likelihood procedure was that it allows estimation of correlations with data from several segments. So, the next step is to also add the information of speculative grade defaults to the dataset, allowing different factor sensitivities among groups. We maintain the same structure as the previous csv: the year (t), the number of defaults (D_t) and the number of obligors (N_t). In this context, we have to change the calculation of the binomial densities. Since we assume that both grades are driven by the same factor, we evaluate the joint probability of observing investment and speculative grade defaults for a given knot Z. The joint probability of independent events is the product of individual probabilities, so we multiply binomial densities of investment and speculative defaults for a given Z. Although it is a quite straightforward generalization from a theoretical mathematical perspective, the main difficulties come from the numerical perspective. The first problem lies in the estimation of the number of combinations of N taken k at a time. For high values, it can go in overflow. We can work around this difficulty by doing the calculations for logarithms, and by using an approximation for the factorial that is then used to calculate the binomial coefficient. The Stirling numerical approximation of the factorial was used when the standard np.comb function arises an exception (see lncombin function). The second kind of problem has its roots in operations research. Now the maximization problem of the Log-likelihood function has four independent variables to be found: the default probabilities and asset correlations of both investment and speculative grades ($p_i, p_s, \omega_i, \omega_j$).

Calling the routine LnLMaximization2, the analyst is able to find these four optimized values using the Nelder-Mead search (eventually) with a grid of starting points for the heuristic. A good choice for the initial guesses in the parameters is to set p_i and ω_i equal to the previous computed values using the bidimensional solver. The other values belonging to the speculative grades double them. A good solution could also be to code a nature-inspired algorithm based on a population of agents (such as genetic algorithm or particle swarm optimization) in case the routine were not able to reach a satisfactory stop criteria. The values that allow to maximize $\ln L$ are

$p_i = 0.115\%, p_s = 3.962\%, \omega_i = 21.950\%, \omega_j = 27.590\%$ with an objective function equal to 185.79. Recalling the four steps for measuring a portfolio credit risk reported at the beginning of this section, we are now able to set the Monte Carlo engine for our credit portfolio. In particular, the default correlations are modelled by linking defaults to a continuous variable: the asset value A. Borrower i defaults if its asset value falls below a certain threshold d_1 chosen to match the specified PD_1 in accordance with:

$$\text{Default}_i \Leftrightarrow y_i = 1 \Leftrightarrow A_i \leq d_i \quad \textit{(Eq. VIII.143)}$$

$$\text{No Default}_i \Leftrightarrow y_i = 0 \Leftrightarrow A_i > d_i \quad \textit{(Eq. VIII.144)}$$

If the asset values are assumed to be standard normally distributed, we would set $d_i = \Phi^{-1}(PD_i)$, where Φ denotes the cumulative standard normal distribution function

Correlation in asset values can be modeled through factor models. We start with a simple one containing only one systematic factor Z:

$$A_i = \omega_i Z + \sqrt{1-\omega_i^2}\epsilon_i,\ \text{cov}(\epsilon_i, \epsilon_j) = 0,\ i \neq j,\ \text{cov}(Z, \epsilon_i) = 0,\ \forall_i \quad \textit{(Eq. VIII.145)}$$

$$Z \sim N(0,1),\ \epsilon_i N(0,1), \forall_i \quad \textit{(Eq. VIII.146)}$$

In other words, we assume that systematic (Z) and idiosyncratic (ϵ) shocks are independent. In the asset value approach, the standard way of obtaining the portfolio distribution (i.e., the fourth step in the logical flow reported at the beginning of this chapter) is to run a Monte Carlo simulation, which typically has the following structure:

1. Randomly draw asset values for each obligor in the portfolio.
2. For each obligor, check whether it defaulted; if yes, determine the individual loss $LGD_i \times EAD_i$.
3. Aggregate the individual losses into a portfolio loss.
4. Repeat steps 1-3 sufficiently to reach a distribution of credit portfolio losses.

In the proposed example, we have already computed the PD_i, LGD_i, EAD_i and asset correlations using a model discussed previously. We use a one-factor model with normally distributed asset values so the correlations are fully specified once we have specified the factor sensitivities ω_i. The data set used has been retrieved from the "Credit Risk modeling" by Löffler and Posch and it is imported in the IDE through a csv file which consists of five fields: the portfolio Identification number (ID), the Probability of Default (PD), the Loss Given Default (LGD), the Exposure at Default (EAD) and the asset correlation (W).

Starting from this information, the logical steps (1-4) have been implemented in the Python function MonteCarloSim. This function takes four input arguments: the data set, the number of simulations, the confidence at which the risk measures are computed and a Boolean variable which allows to plot the histogram of the portfolio losses. The output structure is a tuple with three elements:

- a $(1 \times NSim)$ array with the simulated credit portfolio losses.
- the Value at Risk at a given confidence (common choices are 95% or 99%). Mathematically, VaR_α is the $(1-\alpha)$ quantile of the loss distribution.

- the Expected Shortfall at a given confidence (common choices are 95% or 99%). Mathematically, the expected shortfall at $\alpha\%$ level is the expected return on the portfolio in the worst $\alpha\%$ of cases.
Graphics is a Boolean variable and if it is true, the function also returns the empirical distribution of the simulated losses on the analysed credit portfolio in the graphical device of the Python IDE.
Running the function with 100.000 simulations, 95% of confidence on the Credit Portfolio Data set, we obtain a VaR = 300 and ES = 381.365. The empirical distribution of losses is shown in the Figure below.

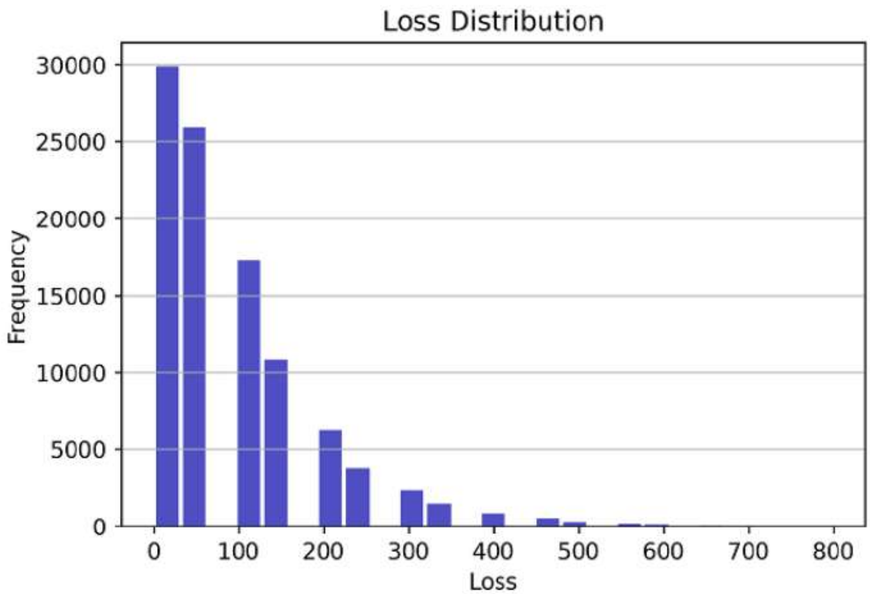

Figure VIII.34 Statistical distribution of the simulated losses on the credit portfolio ($NSim = 100{,}000$)

FURTHER READINGS

Düllmann K., Scheicher M., Schmieder C. – "Asset correlations and credit portfolio risk: an empirical analysis" – Discussion paper series 2: Banking and Financial studies from Deutsche Bundesbank (2007).
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Gupton G. M., Finger C. C., Bhatia M. – "CreditMetrics – Technical Document" – RiskMetrics Group (1997).
Löffler G., Posch P. N. – "Credit risk modeling" – Wiley (2011).
Wilson T. – "Portfolio credit risk I" – Risk Vol.10, n. 9, 111-117 (1997).
Wilson T. – "Portfolio credit risk II" – Risk Vol.10, n. 9, 56-61 (1997).

VIII.8 MODEL VALIDATION

Having set up a model as a rating system, it is natural to test its quality. Traditionally there are two dimensions along which ratings are commonly assessed: **discrimination** and **calibration**. When checking the first feature we wonder: how well does a rating system rank borrowers according to their true probability of default (PD)? When examining calibration we wonder: how well do estimated PDs match true PDs?
Let us consider the following example that illustrates the two dimensions:

Borrower	Rating system 1 (Associated PD)	PD System 2 [%]	True PD [%]
B1	A(1%)	2.01	1.50
B2	B(5%)	2.00	2.00
B3	C(20%)	1.99	2.50

Table VIII.36 Discrimination VS Calibration in a rating system

Rating 1 might represent an agency rating system, with A being the best rating. An agency rating itself is not a PD but can be associated with PDs based on average historical default rates per rating class. Rating system 2 might be based on a statistical credit scoring model which directly produces PD estimates. The rank ordering of system 1 is perfect, but the PDs differ dramatically from the true ones. By contrast, the average PD of system 2 exactly matches the average true PD, and individual deviations from the average PD are small. However, it does not discriminate at all because the system's PDs are inversely related to the true PDs.
In the first section, methods are introduced for evaluating discriminatory power (cumulative accuracy profiles and receiver operating characteristics), both discrimination and calibration (Brier score) or only calibration (Binomial test and a test allowing for default correlation).
Contrary to what was assumed in the example given above, true default probabilities cannot be observed in practice. The presented evaluation methods therefore rest on a comparison of predicted default risk with actual, observed default occurrence. The second section focuses on the validation of credit portfolio. Portfolio credit risk models produce a probability distribution for portfolio credit losses. To validate the quality of a given model, we can examine whether observed losses are consistent with the model's predictions. Many procedures exist for testing the quality of a distribution and here we introduce the Berkowitz test, which is a powerful test that has been examined both for credit risk and for market risk models.
The **Cumulative Accuracy Profile** (CAP) provides a way of visualizing discriminatory power. The key idea is the following: if a rating system discriminates well, defaults should occur mainly among borrowers with a low rating. To graph a CAP, the analyst needs historical data on ratings and default behaviour. The latter would, for example, record whether a borrower defaulted in the year subsequent to having received a certain rating. Observations belonging to a rating category that contains borrowers already in default would be excluded.
The CAP is constructed by plotting the fraction of all defaults that occurred among borrowers rated x or worse

against the fraction of all borrowers that are rated x or worse. Table VIII.37 below shows how to compute the points of the CAP curve.

Observation	1	2	3	4	5	6	7	8	9	10
Rating	A	A	A	B	B	B	C	C	C	C
Has default occurred?	0	0	0	0	1	0	0	1	1	1

Table VIII.37 Illustration of the Cumulative Accuracy Profile - CAP

Let us start with the worst rating, C, wondering what the fraction of all defaults that we cover when we include all borrowers rated C is. 40% of all observations are rated C, the three defaults that occurred among C-rated borrowers making up 75% of all defaults. This gives the first point of the CAP curve (0.4,0.75). Similarly, 70% of all observations are rated B or worse, while borrowers with a rating of B or worse cover 100% of all defaulters. This yields the second point (0.7,1.0). The final point is always (1,1) because if we look at all observations, by construction, we include all observations and all defaults. Obviously, the starting point is the origin.

The **Accuracy Ratio** (AR) condenses all the information contained in CAP curves into a single number. It can be obtained by relating the area under the CAP curve but above the diagonal to the maximum area the CAP can enclose above the diagonal. Thus, the maximum accuracy ratio is 1. The analysis is restricted to the area above the diagonal because the latter gives the expected CAP curve of an uninformative rating system which does not discriminate at all between low and high risks. Theoretically, accuracy ratios can lie in the range [-1,+1]. For a rating system, the accuracy ratio should be above zero, because otherwise the rating system can be substituted with a random generator of grades.
If a rating system perfectly ranks debtors according to their true default probability, it will nevertheless fail to achieve an accuracy ratio of 1 in extremely rare cases. The CAP(Dataset,graphics) function allows the user to automatize the CAP curve building. It takes a data set organized in the same way as reported in Table VIII.37 and returns the points which constitute the curve, as well as the computation of the Accuracy Ratio. If the user enables the graphics flag, the script also produces a useful plot depicting the CAP curve.
As an example, if we call the function with the analyzed data, we obtain an Accuracy Ratio of 70.84%

Another analytical tool that is closely related to the CAP is the **Receiver Operating Characteristic** (ROC).
The ROC curve can be obtained by plotting the fraction of defaulters ranked x or worse against the fraction of non-defaulters ranked x or worse. The two graphs thus differ in the abscissa definition. A common summary statistic of a ROC analysis is the area under the ROC curve (AUC). Reflecting the fact that the CAP is very similar to the ROC, there is an exact linear relationship between AR and the AUC:

$$\text{Accuracy Ratio} = 2 \times \text{Area Under Curve} - 1 \quad \textit{(Eq. VIII.147)}$$

The choice between CAP and ROC is therefore largely a matter of habits. Both convey the same information in a slightly different way of representing the output.
ROC(Dataset,graphics) works exactly in the same way as the previous Python function. The different portions

of code are related to the formula of the area and the definition of the x-points. Running the script, we obtain the plot depicted in the Figure and an AUC equal to 85.42%. We can quickly check the reliability of the estimation verifying the relationship:

$$\text{Accuracy Ratio} = 2 \times \text{Area Under Curve} - 1 = 2 \times 0.8542 = 0.7084$$

Typical accuracy ratios of rating systems used in practice lie between 50% and 90%, but apart from this, little can be said about the accuracy ratio that a "good" system should achieve.
The reason is that the measure is strictly dependent on the portfolio structure, in particular on the heterogeneity of a portfolio compared to default probabilities. The interpretation of CAP/ROC curves can be easier if different rating methodologies are tested on the same data set.

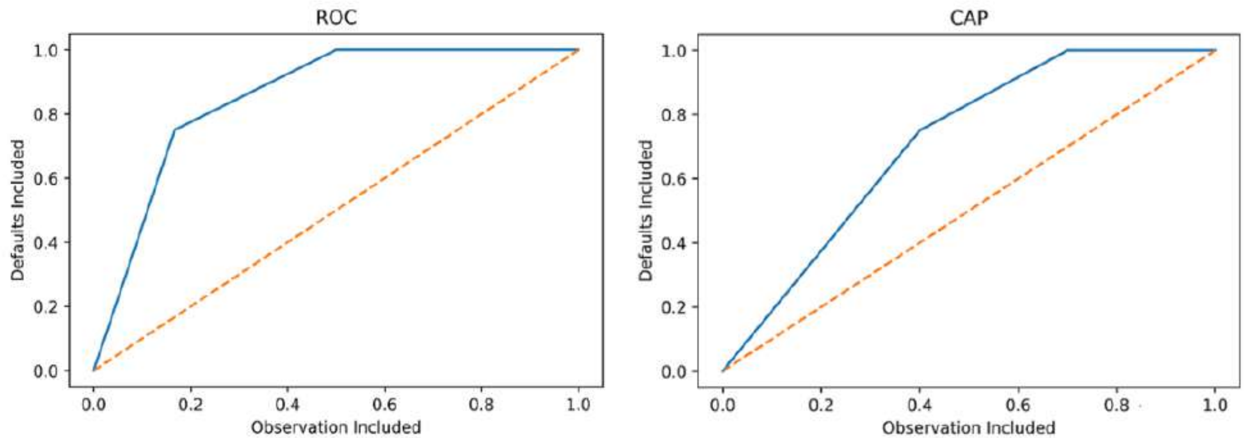

Figure VIII.35 The Receiver Operating Characteristic (ROC) and the Cumulative Accuracy Profile (CAP)

A measure that is able to test both discrimination and calibration is the **Brier score**. It translates the common principle of examining squared forecast errors to probability forecasts. It is defined as follows:

$$\text{Brier Score} = \frac{1}{N}\sum_{i=1}^{N}(d_i - PD_i)^2 \quad \textit{(Eq. VIII.148)}$$

where i indexes the N observations, d_i is an indicator variable that takes the value 1 if borrower i defaulted (0 otherwise), and PD_i is the estimated probability of default of borrower i.

To compute the Brier score, we then need probabilities of default, which we do not need for CAPs and ROCs. The Brier score lies between 0 and 1; better default probability forecasts are associated with lower score values.

Taking into consideration the values in the Table VIII.38, we can easily calculate the Brier score using the BrierScore(Dataset) function. It is equal to 0.3507.

i	1	2	3	4	5	6	7	8	9	10
PD	0.001	0.001	0.001	0.02	0.02	0.02	0.08	0.08	0.08	0.08
Default	0	0	0	1	0	0	1	1	1	0

Table VIII.38 The Brier score

In many rating systems used by financial institutions, obligors are grouped into rating categories. The default probability of a rating category can then be estimated in different ways. A credit analyst can use the historical default rate experience of obligors in a given rating grade; he can map his own rating into the categories of the rating agencies and use their published default rates; or he can average individual default probability estimates of obligors in the grade, and so on. Regardless of the way in which a default probability for a rating grade was estimated, the objective remains to test whether it is in line with observed default rates, and from the perspective of risk management and supervisors, it is often crucial to detect whether default probability estimates are too low.

Here we present one-sided tests for underestimation of default probabilities. In addition, the tests are conducted separately for each observation period (normally one year) and separately for each grade. We start by describing the methodology assuming that defaults are independent, i.e. default correlation ρ is zero. The number of defaults, D_{kt} in a given year t and grade k then follows a binomial distribution. The number of trials is N_{kt}, i.e. the number of obligors in grade k at the start of year t; the success probability is PD_{kt}, i.e. the default probability estimated at the start of year t.

At a significant level of α (for instance $\alpha = 1\%$), we can reject the hypothesis that the default probability is not underestimated if:

$$1 - \text{BINOM}(D_{kt} - 1, N_{kt}PD_{kt}) \leq \alpha \quad \textit{(Eq. VIII.149)}$$

Where $\text{BINOM}(x, N, q)$ denotes the binomial probability of observing x successes out of N trials with success probability q. If the last condition is true, we need to assume an unlikely scenario to explain the actual default count D_{kt} (or a higher one). This would lead us to conclude that PD has underestimated the true default probability. For a large N, the binomial distribution converges to the normal, so we can also use a normal approximation to the previous relation. If defaults follow a binomial distribution with default probability PD_{kt}, the default count D_{kt} has a standard deviation of $\sqrt{PD_{kt}(1 - PD_{kt})N_{kt}}$; the default count's mean is $PD_{kt}N_{kt}$. Mean and standard deviation of the approximating normal are set accordingly. Instead of the previous relation we can thus examine:

$$1 - \Phi\left(\frac{D_{kt} - 0.5 - PD_{kt}N_{kt}}{\sqrt{PD_{kt}(1 - PD_{kt})N_{kt}}}\right) \leq \alpha \quad \textit{(Eq. VIII.150)}$$

Where Φ denotes the cumulative standard normal distribution.

To adjust the test for the presence of default correlations, we can use the one-factor asset value model. There, we had modeled default correlation through correlations in asset values and we had assumed that the latter can be fully captured by only one factor Z.

In such a model, there are two possible reasons why the observed default rate in year t is larger than the underlying default probability:

- many obligors happened to have individual "bad luck".
- year t was generally a bad year for all credits.

In the binomial test and its normal approximation, we only allowed for the first reason. We would now like to allow for the two reasons at the same time. As it turns out, this is possible but complex to achieve. So. we only consider the second explanation in judging whether a PD is too low. The logic is as follows. We assess that a PD underestimated the default probability if we have to assume that the year was so extremely bad that it seems unlikely to be the right explanation. Technically, ignoring individual bad luck means assuming that the default rate in year t is identical to the default probability in year t. The crucial aspect to be noted is that the latter can vary. In the one-factor model, the probability of default in year t, p_{kt} depends on the factor realization Z_t, as well as on the average default probability p_k and the asset correlation ρ:

$$p_{kt} = \Phi\left[\frac{\Phi^{-1}(p_k)-\sqrt{\rho}Z_t}{\sqrt{1-\rho}}\right] \quad \textit{(Eq. VIII.151)}$$

Setting the average default probability to our estimate PD_{kt}, and the default probability equal to the default rate in year t we obtain:

$$\frac{D_{kt}}{N_{kt}} = \Phi\left[\frac{\Phi^{-1}(PD_{kt})-\sqrt{\rho}Z_t}{\sqrt{1-\rho}}\right] \quad \textit{(Eq. VIII.152)}$$

Solving this for factor Z_t tells us what kind of year we need in order to bring the PD in line with the default rate:

$$Z_t = \frac{\Phi^{-1}(PD_{kt})-\sqrt{1-\rho}\,\Phi^{-1}\left(\frac{D_{kt}}{N_{kt}}\right)}{\sqrt{\rho}} \quad \textit{(Eq. VIII.153)}$$

Let us note that a negative Z_t will push the default rate above the PD. In the one factor model, Z_t is standard normally distributed, so the probability of observing a year as bad as t or worse is $\Phi(Z_t)$. At a significant level α, we thus reject the PD if:

$$\Phi\left[\frac{\Phi^{-1}(PD_{kt})-\sqrt{1-\rho}\,\Phi^{-1}\left(\frac{D_{kt}}{N_{kt}}\right)}{\sqrt{\rho}} \leq \alpha\right] \quad \textit{(Eq. VIII.154)}$$

If this relation holds, the scenario Z_t that reconciles the default rate and the PD is too extreme by our standard of significance. Therefore, we conclude that the PD estimate was too low.

CalibrationTest(Dataset,rho) implements the three tests for default data coming from Standard & Poor's. Table VIII.39 shows the results of the tests performed on this sample.

Grade	AAA	AA	A	BBB	BB	B	CCC/C
Binomial	-	-	42.8871 (G)	0.004155 (R)	0.048299 (R)	1.0675 (Y)	2.76e-06 (R)
Normal	-	99.2014 (G)	53.196 (G)	2.04e-05 (R)	0.008213 (R)	0.833726 (R)	3.87e-07 (R)
One Factor	-	-	14.4642 (G)	1.74539 (Y)	6.5889 (G)	21.4833 (G)	2.04992 (Y)

Table VIII.39 Testing underestimation of default probabilities using Basel Traffic Light colors to the p-values of the tests

To obtain the default count from the observed default rates, we round the product of default rates and number of issuers. The asset correlation is $\rho = 7\%$. Running the script and examining the results, with the binomial test, we would classify three rating-specific PDs as underestimating the true default rate at a significance of 1%; and the number increases to four with the normal approximation.

Once we assume an asset correlation of 7%, however, the significance levels rise as we allow for the possibility that the year under scrutiny was a bad year in general. Now we can no longer reject a PD at a significance of 1%; we could, however reject two PDs at a significance of 5%. Let us note that the tests return error or null values if the realized default rate is zero. This makes sense because no evidence can be found for underestimating a default probability, if the realized default rate is at its minimum.

Clearly, decisions on significance levels are somewhat arbitrary. Therefore, in the Basel "Traffic lights" approach, two rather than one significance level must be chosen. If the p-value of a test is below α_{RED}, we assign an observation to the red zone (R), meaning that an underestimation of the default probability is very likely. If the p-value is above α_{RED}, but below α_{YELLOW}, we interpret the result as a warning that a PD might be underestimated, i.e. a yellow zone (Y). Otherwise, we assign it to the green zone (G).

Portfolio credit risk models produce a probability distribution for portfolio credit losses. To validate the quality of a given model, we can examine whether observed losses are consistent with the model's predictions. Certain analysts argue that portfolio models are difficult to validate empirically. Usually, such an opinion is justified by a comparison to market risk models. Market risk models produce loss forecasts for a portfolio (for instance the trading book of a bank) as well, but the underlying horizon is much shorter - often, it is restricted to a single day.

A standard validation procedure consists in checking the frequency with which actual losses exceeded the Value at Risk (VaR). In a market risk setting, risk managers usually examine 99% VaR. Over one year containing roughly 250 trading days, the expected number of exceedances of the 99% VaR is $250 \times (1 - 0.99) = 2.5$, provided that the VaR forecasts are correct.

When we observe the number of exceedances differing significantly from the expected number, we can conclude that the predictions were incorrect. Significance can be assessed with a traditional binomial test.

Obviously, such a test is not very useful for the validation of credit portfolio models, which mostly have a one-year horizon. There is a way out though: ifwe do not confine a test to the prediction of the extreme events but rather test the overall fit of the predicted loss distribution, we can make better use of information and possibly

learn valuable information about a model's validity with only 5 or 10 years of data.
We then introduce the Berkowitz test, which is a powerful method that has been used both for credit risk and for market risk. For each period (usually a length of one year), the information needed for applying the Berkowitz test is: a loss figure and a forecast of the loss distribution made at the start of the period. As a result, for a given loss L, we can compute the probability $F(L)$ with which this loss is not exceeded.
Obviously, the distribution can differ from year to year because of changes in portfolio composition or changes in the risk parameters of the portfolio constituents. The basic idea behind the Berkowitz (2001) test is to evaluate the entire distribution. The test involves a double transformation of observed losses, with the two transformations as follows:

- **1st transformation**: replace L_t, the loss in t, with the predicted probability of observing this loss or a smaller one. We obtain this probability by inserting the loss L_t into a cumulative distribution function $F(L_t)$: $p_t = F(L_t)$.

- **2nd transformation**: transform p_t by applying $\Phi^{-1}(x)$, the inverse cumulative standard normal distribution function. Formally: $z_t = \Phi^{-1}(p_t)$.

The 1st transformation produces numbers between 0 and 1. If the predicted distribution is correct, we have even more information: the numbers should be uniformly distributed between 0 and 1. To check this, we start by looking at the median of the distribution. If the model is correct, 50% of observed losses would be expected to end up below the median loss, which has F(median loss) = 0.5.
Thus, the transformed variable pt should be below 0.5 in 50% of all cases. We can continue in this way.
The 25th percentile, which has F(25% percentile) = 0.25, splits the first half into another pair of two halves, and again observations will be evenly spread on expectation. Similarly, we can conclude that there should be as many p_ts below 0.25 as there are p_ts above 0.75.
We can use finer partitions and still conclude that the p_t should be evenly spread across the intervals. Theoretically, we could stop after the 1st transformation and test whether the pts are actually uniformly distributed between 0 and 1. However, tests based on normally distributed numbers are more widespread in the scientific community and they are more powerful. This is the reason why we perform another transformation. If the model summarized by $F(L)$ is correct, transformed losses z_t will be normally distributed with zero mean and unit variance. The intuition behind this is similar to the 1st transformation. If p_t is uniform between 0 and 1, 2.5% of all observations will be below 2.5%, for example. Consequently, 2.5% of all z_t will be below -1.96, i.e. $\Phi^{-1}(0.025)$ and this is what we expect for a standard normal variable.
Berkowitz suggested the restriction of the test to the hypothesis that z_t have zero mean and unit variance. We could additionally test whether they are normally distributed, but tests of normality tend not to be very powerful if the number of observations is small, so we do not lose much information if we do not test for this property on z_t as well.
A convenient and powerful way of testing the joint hypothesis of zero mean and unit variance is a **likelihood ratio test**. The likelihood is the probability that we observe given data with a given model. With a likelihood ratio test, we test whether imposing a restriction (i.e. z_t have zero mean and unit variance) leads to a significant

loss in the likelihood. The test statistics is based on the log-likelihood function of the transformed series z_t. Since the z_t are normally distributed under the hypothesis that the model is correct, the likelihood is obtained through the normal density:

$$\text{Likelihood} = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(z_t-\mu)^2}{2\sigma^2}\right] \quad \textit{(Eq. VIII.155)}$$

That is, if we have T observations, we multiply the probabilities of having individual observations z_t to reach the likelihood to have the set of T observations. This is correct if unexpected losses, which are captured here by $z_t - \mu$, are independent across time. Although this assumption may be violated in some situations, it should be fulfilled if the loss forecasts make efficient use of information. It should be noted that this is not the same as assuming that losses themselves are independent across time.

Also, there is no need to abandon the concept of credit cycles, as long as the notion of credit cycles relates to losses, not unexpected losses. It is more convenient to work with $\ln L$, the logarithm of the likelihood:

$$\ln L = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln(\sigma^2) - \sum_{t=1}^{T} \frac{(z_t-\mu)^2}{2\sigma^2} \quad \textit{(Eq. VIII.156)}$$

To evaluate the log-likelihood, we calculate the maximum likelihood (ML) estimators for the mean and variance of the transformed variable z_t: $\hat{\mu}_{ML} = \frac{1}{T}\sum_{t=1}^{T} z_t$, $\hat{\sigma}^2_{ML} = \frac{1}{T}\sum_{t=1}^{T}(z_t - \hat{\mu}_{ML})^2$.

The likelihood ratio test is then structured to test the joint hypothesis that the z_t have zero mean and unit variance. It is given by: $\lambda = 2[\ln L\,(\mu = \hat{\mu}_{ML}, \sigma^2 = \hat{\sigma}^2_{ML}) - \ln L(\mu = 0, \sigma^2 = 1)]$.

If imposing the hypothesis $\mu = 0$ and $\sigma^2 = 1$ leads to large loss in likelihood, λ will be large. Therefore, the larger λ, the more evidence we have that the z_t do not have mean zero and unit variance. Under usual regularity conditions, the test statistics λ will be asymptotically distributed as a χ^2 variable with two degrees of freedom.

The CreditPortfolioValidation script allows the user to perform the Berkowitz test.

We assume in this example to have five years of loss data and for focusing on the core of the procedure we also assume that the predicted loss distribution was the same for every year and the specification of the loss distribution is such that we can immediately determine the exact probability of each loss.

The illustrative data are reported directly in the code. Running the script, we reach a p-value for the example equal to 0.329%, as a result we have to reject the hypothesis that the model which produced the outcome is correct.

FURTHER READINGS

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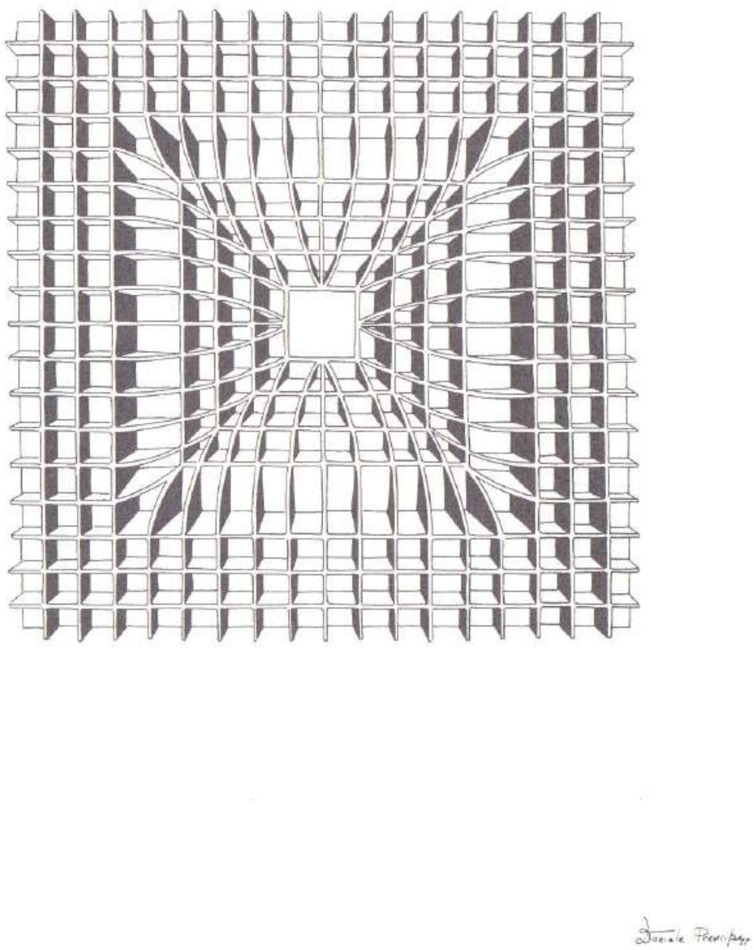

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ABOUT THE ARTIST

Daniele Prencipe is a Savonese intellectual whose knowledge spans from philosophical to mathematical subjects, from technical/computer science to artistic ones. A common factor with the author of the book is the passion for abstract disciplines and for grasping how they can, covertly yet so profoundly, influence our daily reality with elegance and lightness. The ink drawings placed at the end of the sections of this book are part of the works included in the "Man and Structure" collection of drawings. The first exhibition where Daniele exhibited these works was in Milan in Villa Litta in the summer of 1977. For further information, please refer to the single volume of this artistic exhibition.

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ISBN Print: 979-12-80245-26-7
ISBN Online: 979-12-80245-25-0